Deterministic Ratchets, Circle Maps, and Current Reversals

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In this work we reformulate the deterministic dynamics of an overdamped tilting ratchet into a discrete dynamical map by looking stroboscopically at the continuous motion originally ruled by differential equations. We show that, for the simple and widely used case of periodic dichotomous driving forces, the resulting discrete map belongs to the class of circle homeomorphisms. This approach allows us to apply the well-known properties of such maps to derive the necessary and sufficient conditions that the ratchet potential must satisfy in order to have a vanishing current. Furthermore, as a consequence of the above, we show (i) that there is a class of periodic potentials which do not exhibit the rectification phenomenon in spite of their asymmetry and (ii) that current reversals occur in the deterministic case for a large class of ratchet potentials.

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Ratchet models have recently been studied in several different contexts, ranging from biological molecular motors to transport in microelectronic devices and particle separation [1]. One of the main characteristics of these systems is that they are able to rectify the motion of particles from unbiased correlated fluctuations. The archetypal model used to show such a phenomenon is a particle in an asymmetric periodic potential (a ratchet potential) driven by a time periodic force and subjected to Gaussian white noise [2]. Many other models stem from this one when considering colored noise, stochastic dichotomous forces, asymmetric unbiased time-dependent forces, etc. A classification of the main types of ratchets can be found in Ref. [3].

In the absence of noise such systems are called deterministic ratchets and play a central role in the explanation of the origin of the current from a microscopic point of view. Furthermore, these systems can be explored experimentally, e.g., in Josephson junction arrays [4]. Deterministic ratchets can be divided into three categories: Hamiltonian (dissipationless), inertia (underdamped), and overdamped ratchets. Numeric [5–7] and analytic [5,7,8] studies of overdamped deterministic ratchets, for which the inertia term is negligible, reveal that the direction of the current depends upon the asymmetry of both the potential and the external driving force. The current shows an intricate and rich behavior as a function of the model parameters such as “current quantization,” current reversal, and devil’s staircase phenomena [6,7]. Though some explanations have been forwarded for such behaviors a theory describing these and other phenomena in ratchet systems has not yet been fully formulated [3].

In this work we reformulate the problem of overdamped deterministic tilting ratchets within the theory of circle maps. Using a periodic dichotomous driving force, we show that the discrete dynamics generated by looking stroboscopically at the continuous motion is ruled by a discrete map belonging to a class of circle homeomorphisms [9]. This approach enables us to use well-established results of circle maps to derive the necessary and sufficient conditions that the potential must satisfy in order to have a vanishing current for all the parameter space values. As a consequence, we find that there is a set of periodic potential functions which do not exhibit current rectification even though they are not symmetric. Furthermore, we show that some ratchets exhibit current reversal in the (overdamped) deterministic limit, even when the driving force is fully symmetric.

We set up our model considering a particle moving in a one-dimensional periodic potential $V(x)$ with period $L$. In the overdamped regime, the motion of the particle subjected to an external driving force $F(t)$ is governed by the differential equation

$$\gamma \frac{dx}{dt} = f(x) + F(t),$$

(1)

where $\gamma$ is the friction coefficient (which will be set to 1 in the following) and $f(x) = -dV(x)/dx$. We will assume the driving force $F(t)$ is a periodic function with period $T$. Given an initial condition $x(0) = x_0$, Eq. (1) has a unique solution $x(t)$. If we look stroboscopically at the trajectory $x(t)$ every time period $T$, we generate the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$ where $x_n = x(nT)$. This sequence can be obtained from a recursive mapping $R_T(x)$ as

$$x_{n+1} = R_T(x_n),$$

(2)

where $R_T(x)$ has the following properties: (i) For every fixed $T$, $R_T(x)$ is a continuous and monotonic increasing function of $x$ with continuous inverse. This follows from the fact that any two different solutions of (1) do not cross each other. (ii) The group property, $R_{2T}(x) = R_T[R_T(x)]$. (iii) The lift property, $R_T(x + L) = R_T(x) + L$. The last two properties follow from the fact that $x(t + T)$ and $x(t) + L$ are solutions of Eq. (1) whenever $x(t)$ is a solution. Note also that property (ii) ensures the validity of...
Eq. (2), and by properties (i) and (iii), $R_T(x)$ is called a lift of a circle homeomorphism.

From Eq. (2) and the three properties above, it follows that the mean velocity

$$\bar{v} = \lim_{t \to \infty} \frac{x(t) - x(0)}{t} = \frac{1}{T} \lim_{n \to \infty} \frac{x_n - x_0}{n} = \frac{\rho(R_T)}{T},$$

(3)

where $\rho(R_T)$ is the rotation number of the map $R_T(x)$. This link between deterministic ratchets and circle maps lets us assert some statements for the mean velocity. For instance, all of the properties of the mean velocity defined in Eq. (3) are identical to those of the rotation number. Therefore, existence, uniqueness, and independence of $\bar{v}$ on the initial condition can be considered as proven. In particular, the independence of $\bar{v}$ on the initial condition makes it possible to identify it, up to a constant factor, with the ratchet’s current $J$, which is defined as $L^{-1}$ times the average of $\bar{v}$ over all the possible initial conditions. Analogously, complete or incomplete devil’s staircases, which appear naturally for the rotation number in circle homeomorphisms, can also be present in ratchet systems [5,7,10].

Our goal is to determine properties of $\bar{v}$ by means of the map $R_T(x)$ and its rotation number $\rho[R_T(x)]$. For general $F(t), R_T(x)$ cannot be derived analytically unless the solutions of Eq. (1) are known, consider a particle in a periodic potential subjected to a periodic dichotomous driving force of the form

$$F(t) = \begin{cases} F_0 & \text{if } t \mod [T] < T/2 \\ -F_0 & \text{if } t \mod [T] \geq T/2 \end{cases}.$$  

(4)

To establish the main result of this Letter, let us define the functions $\tau_+(x)$ and $\tau_-(x)$ as

$$\tau_+(x) = \int_{x}^{x_n} dx' / f(x') + F_0, \quad \tau_-(x) = \int_{0}^{x} dx' / f(x') - F_0.$$  

(5)

Note that $\tau_+(L)$ ($\tau_-(L)$) is the time it would take the particle to travel a spatial period $L$ if it were driven only by the force $F_0$ ($-F_0$). Intuitively, a sufficient condition for the vanishing current is

$$\tau_+(L) + \tau_-(L) = 0.$$  

(6)

The vanishing current is guaranteed by the above condition in the “adiabatic” limit $T \to \infty$, or when the ratchet potential is symmetric or supersymmetric. Here we show that, even outside the adiabatic limit and for a much wider class of potentials, condition (6) is necessary and sufficient in order to have vanishing current for all values of the parameters $T$ and $F_0$.

Since the force $F(t)$ changes sign every half period $T/2$, it is convenient to define the sequence $\chi_0, \chi_1, \dots, \chi_m, \dots$ as the position of the particle every half period, namely, $\chi_m = x(mT/2)$. Evidently, the sequences $\{x_n\}$ and $\{\chi_m\}$ are related through $x_n = \chi_{2n}$. Thus, starting out from an initial condition $x(0) = x_0 = \chi_0$, after a time $T/2$ the particle will be at position $x(T/2) = \chi_1$, which is implicitly given by

$$\int_{0}^{\chi_1} dx' / f(x') + F_0 = \frac{T}{2}.$$  

(7)

It is convenient to define the functions $h_+(x)$ and $h_-(x)$ as

$$h_+(x) = \frac{L}{\tau_+(L)} \tau_+(x), \quad h_-(x) = \frac{L}{\tau_-(L)} \tau_-(x).$$  

(8)

Note that for $h_+(x)$ ($h_-(x)$) to be well defined for all $x \in [0, L]$, it is necessary that $F_0 > \min[f(x)]$ ($F_0 > \max[f(x)]$) [11]. Under such circumstances, both $h_+(x)$ and $h_-(x)$ are continuous increasing functions with continuous inverses and such that $h_+(x + L) = h_+(x) + L$. Denoting as $h_+^{-1}$ the inverse of $h_+$, Eq. (7) can be solved for $\chi_1$, which gives

$$\chi_1 = h_+^{-1}[h_+(\chi_0) + \alpha_+]$$

where $\alpha_+ = LT/2\tau_+(L)$.

At time $t = T/2$ the particle’s position is $\chi_1$, and the driving force changes to $-F_0$. Therefore, the position $\chi_2$ of the particle at $t = T$ is given by the implicit equation

$$\int_{\chi_1}^{\chi_2} dx / f(x) - F_0 = \frac{T}{2}.$$  

(10)

Using the function $h_-(x)$, defined above, and its inverse, Eq. (10) can be formally solved for $\chi_2$, obtaining

$$\chi_2 = h_+^{-1}[h_-(\chi_1) + \alpha_-].$$  

(11)

where $\alpha_- = LT/2\tau_-(L)$.

Defining the functions $M_+(x)$ and $M_-(x)$ as

$$M_+(x) = h_+^{-1}[h_+(x) + \alpha_+]$$

and

$$M_-(x) = h_+^{-1}[h_-(x) + \alpha_-].$$

(12)

Equations (9) and (11) can be written as $\chi_1 = M_+(\chi_0)$ and $\chi_2 = M_-(\chi_1)$, respectively [11]. Taking into account that $x_n = \chi_{2n}$, the above is equivalent to $x_1 = (M_- \circ M_+)(\chi_0)$. Iterating this process over time, we obtain that the sequence $x_0, x_1, \ldots, x_n, \ldots$ is generated by

$$x_{n+1} = (M_- \circ M_+)(x_n) \equiv R_T(x_n).$$  

(13)

From Eq. (3) it is clear that vanishing current is equivalent to having a zero rotation number corresponding to a fixed point of $R_T(x)$. Therefore, there will be no current in the ratchet system whenever the fixed point equation $R_T(x) = x$ has solution. Taking into account that topological conjugacy leaves the rotation number invariant [10]; the above mathematical condition can be expressed more conveniently as $R_T(x) = x$, where $R_T = h_- \circ R_T \circ h_+^{-1}$ is a topological conjugacy of $R_T(x)$. Defining $h = h_+ \circ h_+^{-1}$, the null current condition $R_T(x) = x$ can be expressed as

$$h^{-1}[h(x) + \alpha_+] + \alpha_- = x,$$

(14)

where $h(x)$ is a homeomorphism also satisfying the lift property: $h(x + L) = h(x) + L$. Therefore, $h(x)$ can always be written as $h(x) = x + g(x)$, where $g(x)$ is a continuous bounded periodic function that satisfies $g(x + L) = g(x)$ and $|g(x) - g(y)| < L, \forall x, y \in \mathbb{R}$. The same holds for $h^{-1}$, which can be written as $h^{-1}(x) = x + \tilde{g}(x)$. The functions $g(x)$ and $\tilde{g}(x)$ are related through
g(x) = -\bar{g}[h(x)]. Using these expressions, the null current condition Eq. (14) can be written as

\[ \alpha_- + \alpha_+ + \bar{g}[h(x) + \alpha_+] - \bar{g}[h(x)] = 0. \]  

(15)

If \(|\alpha_- + \alpha_+| > L\), then, by boundedness of \(g(x)\), the null current condition Eq. (15) has no solution on \(R\) and therefore a finite current arises. Hence, a necessary condition for vanishing rotation number (and current) is \(|\alpha_- + \alpha_+| \leq L\). In particular, this condition is satisfied for \(\alpha_- + \alpha_+ = 0\). But in this case Eq. (15) always has a solution since \(g(x)\) is a continuous periodic function. Therefore, using the definition of \(\alpha_-\) and \(\alpha_+\), we have a sufficient condition for vanishing rotation number:

\[ \alpha_+ + \alpha_- = \frac{LT \tau_+(L) + \tau_-(L)}{2 \tau_-(L) \tau_+(L)} = 0. \]  

(16)

Furthermore, the later condition is necessary and sufficient for the current to be zero in all of the \(\{F_0, T\}\) parameter space. Indeed, since neither \(\tau_-(L)\) nor \(\tau_+(L)\) depend on \(T\), if \(\tau_+(L) + \tau_-(L) \neq 0\) then \(|\alpha_- + \alpha_+|\) can be made arbitrarily large by choosing \(T\) sufficiently large. For such large values of \(T\), the condition \(|\alpha_- + \alpha_+| \leq L\) is not fulfilled and, consequently, Eq. (15) has no solution.

From Eq. (16), \(\alpha_- + \alpha_+ = 0\) is equivalent to \(\tau_-(L) + \tau_+(L) = 0\), hence, from the definition of \(\tau_+(L)\) and \(\tau_-(L)\) [see Eq. (5)] we obtain that the null current condition can be written as

\[ \tau_-(L) + \tau_+(L) = \int_0^L \left( \frac{1}{F_0 + f(x)} - \frac{1}{F_0 - f(x)} \right) dx = -2 \int_0^L \int_0^{\infty} e^{-u F_0} \sinh[u f(x)] dudx = 0. \]

Therefore, the rotation number will always be zero if and only if the equality

\[ \int_0^L \sinh[u f(x)] dx = 0, \]  

(17)

holds for all \(u\).

Note that ratchets with symmetric potentials for which \(f(x + \xi) = -f(-x)\) for some \(\xi\), fulfill Eq. (17) as well as the supersymmetric potentials [12] with \(f(x) = -f(x + L/2)\). Our approach highlights that there are other sets of periodic functions that are neither symmetric nor supersymmetric which do not exhibit current rectification. For example, consider the symmetric potential of Fig. 1(a), for which \(f(x)\) is a step function with a finite number of bounded discontinuities, as in Fig. 1(b) [13]. By performing two-step permutations on this potential we obtain the new potential and corresponding force shown in Figs. 1(c) and 1(d). The compliance with Eq. (17) follows since integration is not altered by changes in the step order. As one of the possible generalizations of the above we consider the even \(N\) step function \(f_N(x) = f_i\) for \(x \mod[L] \in \mathcal{L}_i\), where \(\{\mathcal{L}_i\}\) is a partition of the interval \(\mathcal{L} = [0, L]\) and \(i = 0, 1, 2, \ldots, N\). In particular, condition (17) will be satisfied by \(f_N(x)\) if for every subinterval \(\mathcal{L}_i\) there exists a unique \(\mathcal{L}_j\) such that their lengths are equal to each other and \(f(\mathcal{L}_i) = -f(\mathcal{L}_j)\).

Our results allow us to investigate one of the most intriguing phenomena present in ratchet systems, which is the existence of current reversals. Though this has been extensively studied for noise driven ratchets [3,5], deterministic models with symmetric forcing have only been considered for specific cases [14]. Here we show that there is a large class of ratchet potentials with such a forcing that are able to rectify motion in both directions depending upon model parameters.

Since \(\rho(R_T) = \rho(\bar{R}_T)\) by topological conjugacy, then from Eq. (3) we obtain for the mean velocity

\[ \bar{v} = \frac{1}{T} \rho(R_T) = \lim_{N \to \infty} \frac{1}{TN} \sum_{n=0}^{N-1} \bar{R}_T(x_n) - x_n \]

\[ = \frac{\alpha_+ + \alpha_-}{T} + \lim_{N \to \infty} \frac{1}{TN} \sum_{n=0}^{N-1} \bar{g}[h(x_n) + \alpha_+] - \bar{g}[h(x_n)]. \]
Recalling that $|\bar{g}[h(x_n) + \alpha_+ + \bar{g}[h(x_n)]| < L$ we have that in the adiabatic limit $T \to \infty$ the mean velocity is given by

$$\lim_{T \to \infty} \bar{v} = \frac{L}{2} \frac{\tau_-(L) + \tau_+(L)}{\tau_-(L) \tau_+(L)}.$$

The sign of the current in this limit is then dictated by $\tau_+(L) + \tau_-(L)$ since $\tau_-(L) < 0$ and $\tau_+(L) > 0$ in all of the parameter space region where they are real. Because the integral $G(u) = \int_0^L \sinh[u f(x)] dx$ is related to this quantity, up to a constant factor, by a Laplace transform, it follows that the sign of $-G(u)$ is always equal to that of the mean velocity. Therefore, whenever $G(u)$ changes sign $\bar{v}$ changes sign and a current reversal occurs as a function of the forcing amplitude $F_0$. For example, consider the step function depicted in Fig. 2 with $L = 1$, $l_1 = 3/5$, $l_2 = 3/10$, $l_3 = 1/50$, $l_4 = 4/50$, $f_1 = -5/3$, $f_2 = 10/3$, $f_3 = -4$, and $f_4 = 1$. The corresponding $G(u)$ is shown in Fig. 3(b) and the current as a function of $F_0$ for fixed $T = 5$ is shown in Fig. 3(a).

Summing up, we have shown that the discrete dynamics defined by looking stroboscopically at a continuous deterministic ratchet motion is ruled by a circle map. From this connection we find necessary and sufficient conditions for the potential in order to have vanishing current in all the $\{F_0, T\}$ parameter space. Symmetry considerations related to these conditions lead us to a set of periodic potentials different from symmetric or supersymmetric ratches which do not exhibit the rectification phenomena. Based on these results we have also found a criterion for current reversal. Though our findings are for periodic dichotomous external forcing, our study can be generalized (straightforwardly and laboriously) to periodic multivalued forces, giving criteria for null current and current reversals. Continuous forcing can also be dealt with by taking a zero width step limit. In particular, for sine function forces Eq. (17) remains as a null current requirement.

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[11] If $F_0 \leq \max|f(x)|$ or $F_0 \leq \min|f(x)|$, the functions $M(x)$ can be defined as circle homeomorphisms with at least one fixed point by an appropriate extension of the above arguments.
[13] For piecewise linear potentials, though $f(x)$ is not Lipschitz continuous, the map $R(x, T)$ can be defined if $\lim_{x \to 0} f(x) < M$ for some $M > 0$ at the discontinuities $\{x_i\}$ of $f$.