Demostrar $k$ constante

$$-k = a^2\left(H^2 - \frac{8\pi G}{3}\rho - \frac{1}{3}\Lambda\right) \quad (1)$$

Hence

$$-\frac{\partial k}{\partial t} = 2a^2 H\left(H^2 - \frac{8\pi G}{3}\rho - \tfrac{1}{3}\Lambda\right) + a^2\left(2H\frac{\partial H}{\partial t} - \frac{8\pi G}{3}\frac{\partial \rho}{\partial t}\right)$$

$$= a^2\left[2H^3 - \frac{16\pi G}{3}H\rho - \tfrac{2}{3}H\rho - \frac{2}{3}H\Lambda + 2H\left(-H^2 - \frac{4\pi G}{3}\rho + \frac{\Lambda}{3}\right) + 8\pi G\rho H\right.$$

$$= 0 \quad (2)$$

Vamos a partir de la ecuación (1)

$$\Rightarrow -\frac{\partial k}{\partial t} = 2a^2 H\left(H^2 - \frac{8\pi G}{3}\rho - \tfrac{1}{3}\Lambda\right) + a^2\left(2H\frac{\partial H}{\partial t} - \frac{8\pi G}{3}\frac{\partial \rho}{\partial t}\right)$$

Factoricemos $a^2$

$$= a^2\left(2H^3 - \frac{2H\kappa_0}{3}\rho - \frac{2}{3}H\Lambda + 2H\frac{\partial H}{\partial t} - \frac{\kappa_0}{3}\frac{\partial \rho}{\partial t}\right)$$

por otro lado $\frac{\partial H}{\partial t} = -H^2 + \tfrac{1}{3}\Lambda - \frac{4\pi G}{3}\rho$ y $\frac{\partial \rho}{\partial t} = -3H\rho$

$$\Rightarrow = a^2\left(\cancel{2H^3} - \cancel{\frac{2H\kappa_0}{3}}\rho - \cancel{\frac{2}{3}H\Lambda} + 2H\left(\cancel{-H^2} - \cancel{\frac{\kappa_0}{6}}\rho + \cancel{\tfrac{1}{3}\Lambda}\right) + \cancel{\kappa_0 H\rho}\right)$$

$$\Rightarrow -\frac{\partial k}{\partial t} = a^2(0) = 0$$

$$\Rightarrow \underline{\underline{k = cte}}$$

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Demostrar : $\partial_t \delta = -\frac{1}{a}(\nabla\cdot\bar{v}) = 0$

Partimos de la ec de continuidad : $\frac{\partial \rho}{\partial t} + \bar{\nabla}_r \cdot (\rho\bar{u}) = 0$

Background

Entonces tenemos que $\rho = \bar{\rho}(1+\delta)$ , $\bar{u} = Ha\bar{x} + \bar{v}$

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}_r\cdot(\rho\bar{u}) = \left(\frac{\partial}{\partial t} - H\bar{x}\cdot\nabla\right)\left(\bar{\rho}(1+\delta)\right) + \frac{1}{a}\nabla\cdot\left(\bar{\rho}(1+\delta)\,\bar{u}\right) = 0$$

$$-\left(\partial_t - H\bar{x}\cdot\nabla\right)\left(\bar{\rho}(1+\delta)\right) + \tfrac{1}{a}\nabla\cdot\left(\bar{\rho}(1+\delta)\right)\left(Ha\bar{x}+\bar{v}\right)=0$$

Distribuyendo los operadores:

$$\partial_t\bar{\rho} + \partial_t(\bar{\rho}\delta) - H\bar{x}\cdot\cancel{\nabla\bar{\rho}}^{\,0} - H\bar{x}\cdot\nabla(\bar{\rho}\delta) + \frac{1}{a}\bar{\rho}\nabla\cdot\underbrace{\left(Ha\bar{x}+\bar{v}+Ha\bar{x}\delta+\bar{v}\delta\right)}=0$$

Hacemos que $\nabla$ actue sobre las coordenadas espaciales

$$\Rightarrow \partial_t\bar{\rho} + \partial_t(\bar{\rho}\delta) - H\bar{x}\cdot\nabla(\bar{\rho}\delta) + \frac{1}{a}\bar{\rho}\left[Ha(\nabla\cdot\bar{x}) + Ha\nabla\cdot(\bar{x}\delta) + \nabla\cdot(\bar{v}\delta)\right]=0$$

Tenemos que, debido a la Homogeneidad e isotropía, las derivadas espaciales de $\bar{P}$ y $\rho$ son cero

Podemos asociar términos, para simplificar, recordando

$$\underline{\dot{\bar{\rho}} + 3H\bar{\rho} = 0}$$

$$\left(\partial_t\bar{\rho} + H\bar{\rho}(\nabla\cdot\bar{x})\right) + \partial_t(\bar{\rho}\delta) - H\bar{x}\cdot\nabla(\bar{\rho}\delta) + \frac{1}{a}\bar{\rho}(\nabla\cdot\bar{v}) + H\bar{\rho}\,\nabla\cdot(\bar{x}\delta)$$

Como $\bar{x}$ es el vector de coordenadas espaciales $\nabla\cdot\bar{x} = 1+1+1 = 3$

$$\Rightarrow \cancel{\left(\dot{\bar{\rho}} + 3H\bar{\rho}\right)}^{\,0} + \dot{\bar{\rho}}\delta + \dot{\delta}\bar{\rho} - H\bar{x}\cdot\left(\cancel{\nabla\bar{\rho}}^{\,0}\delta + \bar{\rho}\nabla\delta\right) + \tfrac{1}{a}\bar{\rho}(\nabla\cdot\bar{v}) + H\bar{\rho}\left(\nabla\cdot\bar{x}\,\delta + \bar{x}\cdot\nabla\delta\right)$$

$$= \underline{\dot{\bar{\rho}}\delta} + \dot{\delta}\bar{\rho} - H\bar{x}\cdot(\bar{\rho}\nabla\delta) + \tfrac{1}{a}\bar{\rho}(\nabla\cdot\bar{v}) + \underline{H\bar{\rho}(3\delta} + \bar{x}\cdot\nabla\delta) = 0$$

$$\delta\cancel{(\dot{\bar{\rho}} + 3H\bar{\rho})}^{\,0} + \dot{\delta}\bar{\rho} - \cancel{H\bar{x}\cdot(\bar{\rho}\nabla\delta)} + \tfrac{1}{a}\bar{\rho}(\nabla\cdot\bar{v}) + \cancel{H\bar{\rho}(\bar{x}\cdot\nabla\delta)} = 0$$

$$\Rightarrow \dot{\delta}\bar{\rho} + \tfrac{1}{a}\bar{\rho}(\nabla\cdot\bar{v}) = 0$$ , dividiendo entre $\bar{\rho}$

$$\Rightarrow \boxed{\dot{\delta} = -\tfrac{1}{a}(\nabla\cdot\bar{v})}$$

② Demostrar $\partial_t \bar{v} + H\bar{v} = -\frac{1}{a\bar{\rho}} \nabla \delta\rho - \frac{1}{a}\nabla\phi$

Aquí al igual que en el inciso anterior, vamos a despreciar las perturbaciones de orden superior y tomamos de nuevo resultados conocidos, como el de la eq de continuidad de grado cero.

$$\Rightarrow \left(\partial_t - H\bar{x}\cdot\nabla + 1/a\ \bar{u}\cdot\nabla\right)\bar{u} = \frac{1}{\rho(1+\delta)a}\nabla[p+\delta p] - \frac{1}{a}\nabla(\Phi\,\phi)$$

$$\Rightarrow \partial_t(Ha\bar{x}+\bar{v}) - H\bar{x}\cdot\nabla(Ha\bar{x}+\bar{v}) + 1/a\,(Ha\bar{x}+\bar{v})\cdot\nabla(Ha\bar{x}+\bar{v}) = -\frac{\nabla(p+\delta p)}{a\bar{\rho}(1+\delta)}$$
$$-\frac{1}{a}(\nabla\Phi - \nabla\phi)$$

Primero tratamos el término de la izquierda:

$$= \underline{a\bar{x}\partial_t H + H^2 a\bar{x}} + H\bar{v} + \partial_t\bar{v} - H^2 a\cancel{(\bar{x}\cdot\nabla)\bar{x}} - H\cancel{(\bar{x}\cdot\nabla)\bar{v}} + \cancel{H^2 a(\bar{x}\cdot\nabla)\bar{x}} + \cancel{H(\bar{x}\cdot\nabla)\bar{v}}$$
$$+ H(\bar{v}\cdot\nabla)\bar{x} + \frac{1}{a}(\bar{v}\cdot\nabla)\bar{v}$$

tomado del otro lado de la eq

$$= a\bar{x}\left(\underline{\partial_t H + H^2 + 1/3\,(4\pi G\bar{\rho}}\,)\right) + (\partial_t\bar{v} + H\bar{v}) + (\bar{v}\cdot\nabla)\left(\bar{H}x + \frac{1}{a}\bar{v}\right)$$

Tenemos también la ecuación de Fiedler dada por:

$$\frac{\partial}{\partial t}H + H^2 = 1/3\,(4\pi G\bar{\rho} + \cancel{\Lambda})$$

$$\Rightarrow \underline{\partial_t\bar{v} + H\bar{v}\ \ H(\bar{v}\cdot\nabla)[H\bar{x} + 1/a\ \bar{v}]}$$

Ahora desarrollamos el otro lado de la ecuación

$$= \frac{-1}{a\bar{\rho}(1+\delta)}\nabla(p+\delta p) - \frac{1}{a}\nabla\phi$$

Como dijimos despreciamos $\nabla p$, por homogeneidad e Iso

$$= \frac{-1}{a\bar{\rho}(\delta+1)}\nabla\delta P - \frac{1}{a}\nabla\phi$$

$$= \frac{-1}{a\bar{\rho}} \left( \frac{1}{\delta+1} \right) \nabla \delta P - \frac{1}{a} \nabla \phi$$ , como $\delta$ muy pequeño $\frac{1}{\delta+1} \to 1$

$$= \frac{-1}{a\rho} \nabla \delta P - \frac{1}{a} \nabla \phi$$

Juntando ambos términos de nuevo

$$\boxed{\partial_t \bar{v} + H\bar{v} + (\bar{v} \cdot \nabla) [H\bar{x} + \frac{1}{a} \bar{v}] = \frac{-1}{a\rho} \nabla \delta P - \frac{1}{a} \nabla \phi}$$

Finalmente vamos a mostrar la ecuación de Poisson

$$\Rightarrow \left(\frac{1}{a} \nabla\right)\left(\frac{1}{a} \nabla\right)(\phi + \Phi) = 4\pi G \bar{\rho}(1+\delta)$$

$$\frac{1}{a^2} \nabla^2 (\phi + \Phi) = 4\pi G \bar{\rho} + 4\pi G \bar{\rho} \delta$$

$$\Rightarrow \underbrace{\frac{1}{a^2} \nabla \Phi - 4\pi G \bar{\rho}}_{\text{Poisson orden } 0} = \frac{-1}{a^2} \nabla^2 \phi + 4\pi G \bar{\rho} \delta$$

$$\Rightarrow \boxed{-\frac{1}{a^2} \nabla^2 \phi + 4\pi G \bar{\rho} \delta = 0}$$