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Mostrar que
$$\tilde{\phi} = \phi + \mathcal{H} T + \frac{1}{3}\partial_i L^i$$

Recordemos que
$$ds^2 = a^2\{(1+2\Psi)\,d\eta^2 - 2B_i\,dx^i d\eta - [(1-2\phi)\delta_{ij} + 2E_{ij}]\,dx^i dx^j\}$$

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} - \delta^\alpha{}_\mu \delta^\beta{}_\nu\right)\bar{g}_{\alpha\beta} - T\dot{\bar{g}}_{\mu\nu} - L^i\partial_i \bar{g}_{\mu\nu} \quad (9.28)$$

Entonces, para obtener $\tilde{\phi}$ hacemos $\mu=i$, $\nu=j$ en la ecuación (9.28)

$$\delta\tilde{g}_{ii} = -2a^2(\tilde{\phi}\delta_{ij} + \tilde{E}_{ij}) = \delta g_{ij} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^j} - \delta^\alpha{}_i \delta^\beta{}_j\right)\bar{g}_{\alpha\beta} - T\dot{\bar{g}}_{ij} - L^i\cancelto{0}{\partial_i \bar{g}_{ij}}$$

$$= -2a^2(\phi\delta_{ij} + E_{ij}) + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^j}\bar{g}_{0\beta} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^j} g_{j\beta} - \bar{g}_{ij}$$

$$+ 2\dot{a}a\delta_{ij}$$

$$= -2a^2(\phi\delta_{ij} + E_{ij}) + \frac{\partial x^0}{\partial \tilde{x}^i}\cancelto{0}{\frac{\partial x^0}{\partial \tilde{x}^j}}\bar{g}_{00} + \frac{\partial x^0}{\partial \tilde{x}^i}\cancelto{0}{\frac{\partial x^j}{\partial \tilde{x}^j}}\bar{g}_{0j}$$

$$+ \cancelto{0}{\frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^0}{\partial \tilde{x}^j}} g_{j0} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}\bar{g}_{jl} - \bar{g}_{ij} + 2\dot{a}a\delta_{ij}$$

$$= -2a^2(\phi\delta_{ij} + E_{ij}) - a^2(\delta^j{}_i - \partial_i L^j)(\delta^l{}_j - \partial_j L^l)\bar{g}_{jl} - \bar{g}_{ij} + 2\dot{a}a\delta_{ij}$$

$$= -2a^2(\phi\delta_{ij} + E_{ij}) - a^2\cancel{(\delta^j{}_i\delta^l{}_j)\bar{g}_{jl} - \bar{g}_{ij}} + 2\dot{a}a\delta_{ij} + a^2\partial_j L^i$$

$$= -2a^2\left(\phi\delta_{ij} + E_{ij} + \cancelto{\mathcal{H}}{\frac{\dot{a}}{a}} T\delta_{ij} + \partial_i L^i\right)$$

$$-2a^2(\tilde{\phi}\delta_{ij} + \cancel{\tilde{E}_{ij}}) = -2a^2\left(\phi\delta_{ij} + \cancel{E_{ij}} + \mathcal{H}T\delta_{ij} + \partial_i L^i\right)$$

$$3\tilde{\phi} = 3\phi + 3\mathcal{H}T + \partial_i L^i$$

$$\Longrightarrow \boxed{\tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\partial_i L^i}$$

Para el segundo componente, $\tilde{\phi}$ hacemos $\mu=i$, $\nu=0$ en la ecuación (9.28)

$$\delta\tilde{g}_{io} = -a^2\tilde{B}_i = \delta g_{io} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^0} - \delta^\alpha{}_i\,\delta^\beta{}_0\right)\bar{g}_{\alpha\beta} - T\dot{\bar{g}}_{io} - L^i\partial_i\bar{g}_{io}$$

$$= -a^2B_i + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^0}\bar{g}_{0\beta} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^0}\bar{g}_{j\beta} - \bar{g}_{io} - \cancelto{0}{T(-\dot{\bar{B}}_i)} - L^i\cancelto{0}{\partial_i(-\bar{B}_i)}$$

$$= -a^2B_i + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^0}{\partial \tilde{x}^0}\bar{g}_{00} + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^j}{\partial \tilde{x}^0}\bar{g}_{0i} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^0}{\partial \tilde{x}^0}\bar{g}_{j0} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^0}\bar{g}_{jl}$$

$$= -a^2B_i + (-\partial_i T)(1-\dot{T})\left[a^2(1+2\psi)\right] + (-\partial_i T)\cancelto{0}{(-\dot{L}^j)}(-\bar{B}_i)$$

$$+ (\delta^j{}_i - \partial_i L^j)(1-\dot{T})(-a^2\bar{B}_j) + (\delta^j{}_i - \partial_i L^j)(-\dot{L}^l)a^2\left[-(1-2\phi)\delta_{jl} - E_{jl}\right]$$

$$= -a^2B_i - a^2(\partial_i T) - a^2\bar{B}_j\,\delta^j{}_i + a^2\delta^j{}_i\,\delta_{jl}\dot{L}^l$$

$$= a^2\left[-B_i - (\partial_i T) - \bar{B}_i + \delta_{il}\dot{L}^l\right]$$

$$= a^2\left[-B_i - (\partial_i T) - \underbrace{\bar{B}_i}_{\text{Sobra}} + \dot{L}^i\right]$$

$$\Rightarrow \boxed{\tilde{B}_i = B_i + \partial_i T - \dot{L}^i + \bar{B}_i}$$

Mostrar que Φ, Ψ, Φ_i no cambian bajo una transformación de coordenadas.

Estos potenciales se definen como

$$\Psi \equiv \psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E}$$
$$\Phi \equiv \phi - \mathcal{H}(B - \dot{E}) + \tfrac{1}{3}\nabla^2 E$$
$$\Phi_i \equiv \dot{E}_i - B_i$$

Además, otras variables importantes a definir, son

$$\tilde{\psi} = \psi - \dot{T} - \mathcal{H}T \qquad \tilde{E} = E - L$$
$$\tilde{\phi} = \phi + \mathcal{H}T + \tfrac{1}{3}\nabla^2 L \qquad \tilde{B}_i = B_i + \partial_i T - \dot{L}_i$$
$$\tilde{B} = B + T - \dot{L}$$

Entonces, el primer caso es $\tilde{\Psi}$

$$\begin{aligned}
\tilde{\Psi} &= \tilde{\psi} + \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \dot{\tilde{B}} - \ddot{\tilde{E}} \\
&= \psi - \dot{T} - \mathcal{H}T + \mathcal{H}[(B + T - \dot{L}) - (\dot{E} - \dot{L})] + \dot{B} + \dot{T} - \ddot{L} - (\ddot{E} - \ddot{L}) \\
&= \psi - \dot{T} - \mathcal{H}T + \mathcal{H}(B + T - \dot{E}) + \dot{B} + \dot{T} - \ddot{E} \\
&= \psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E}
\end{aligned}$$

Ψ no cambia!

Continuamos con $\tilde{\Phi}$

$$\begin{aligned}
\tilde{\Phi} &= \tilde{\phi} - \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \tfrac{1}{3}\nabla^2 \tilde{E} \\
&= \phi + \mathcal{H}T + \tfrac{1}{3}\nabla^3 L - \mathcal{H}[(B + T - \dot{L}) - (\dot{E} - \dot{L})] + \tfrac{1}{3}\nabla^2(E - L) \\
&= \phi + \mathcal{H}T + \tfrac{1}{3}\nabla^3 L - \mathcal{H}[B + T - \dot{E}] + \tfrac{1}{3}\nabla^2 E - \tfrac{1}{3}\nabla^2 L \\
&= \phi - \mathcal{H}(B - \dot{E}) + \tfrac{1}{3}\nabla^2 E
\end{aligned}$$

Φ no cambia!

Finalmente, para $\tilde{\Phi}_i$

$$\begin{aligned}
\tilde{\Phi}_i = \dot{\tilde{E}}_i - \tilde{B}_i &= \dot{\tilde{E}}_i - B_i - \partial_i T + \dot{L}_i \\
&= \dot{E}_i - \dot{L}_i - B_i - \partial_i T + \dot{L}_i \\
&= \dot{E}_i - B_i \underline{- \partial_i T}
\end{aligned}$$

Sobra

Mostrar que

$$\delta\tilde{\varphi} = \delta\varphi - T\dot{\varphi} \quad , \quad \tilde{q}^i = q^i + (\bar{\rho} + \bar{p})L_i \quad , \quad \tilde{\Pi}^i_j = \Pi^i_j$$

Primero, recordemos que

$$\delta\tilde{T}^\mu{}_\nu = \delta T^\mu{}_\nu + \left(\frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} - \delta^\mu{}_\alpha\,\delta^\beta{}_\nu\right)\bar{T}^\alpha{}_\beta - \dot{\bar{T}}^\mu{}_\nu\,T \qquad (9.42)$$

$$T^0{}_0 = \bar{\rho}(1+\delta) \quad , \quad T^i{}_0 = q^i \quad , \quad T^i{}_j = -(\bar{p} + \delta p)\,\delta^i_j + \Pi^i_j$$

Para el primer caso, hacemos $\mu=\nu=i$ en la ecuación (9.42)

$$\delta\tilde{T}^i{}_i = -\delta\tilde{p} = \delta T^i{}_i + \left(\frac{\partial\tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^i} - \delta^i{}_\alpha\,\delta^\beta{}_i\right)\bar{T}^\alpha{}_\beta - \dot{\bar{T}}^i{}_i\,T$$

$$= -\delta\bar{p} - \overset{\mathcal{O}(2)}{\cancel{\delta(\delta p)}} + \underbrace{\left(\overset{\to\,\delta^\beta_\alpha}{\cancel{\frac{\partial x^\beta}{\partial x^\alpha}}} - \delta_\alpha{}^\beta\right)}_{0}\bar{T}^\alpha{}_\beta + \dot{\bar{p}}\,T$$

$$= -\delta\bar{p} + \dot{\bar{p}}\,T$$

$$\Longrightarrow \boxed{\delta\tilde{p} = \delta\bar{p} - \dot{\bar{p}}\,T}$$

Para el segundo caso, hacemos $\mu=i$, $\nu=0$ en la ecuación (9.42)

$$\delta\tilde{T}^i{}_0 = \delta\tilde{q}^i = \delta T^i{}_0 + \left(\frac{\partial\tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{\eta}} - \delta^i{}_\alpha\,\delta^\beta{}_0\right)\bar{T}^\alpha{}_\beta - \dot{\bar{T}}^i{}_0\,T$$

$$= \delta q^i + \frac{\partial\tilde{x}^i}{\partial\eta}\frac{\partial x^\beta}{\partial\tilde{\eta}}\bar{T}^0{}_\beta + \frac{\partial\tilde{x}^i}{\partial x^j}\frac{\partial x^\beta}{\partial\tilde{\eta}}\bar{T}^j{}_\beta - \delta^i{}_\alpha\,\delta^\beta{}_0\,\bar{T}^\alpha{}_\beta - \overset{\mathcal{O}(2)}{\cancel{\dot{q}^i\,T}}$$

$$= \delta q^i + \frac{\partial\tilde{x}^i}{\partial\eta}\frac{\partial\eta}{\partial\tilde{\eta}}\bar{T}^0{}_0 + \overset{\mathcal{O}(2)}{\cancel{\frac{\partial\tilde{x}^i}{\partial\eta}\frac{\partial x^j}{\partial\tilde{\eta}}\bar{T}^0{}_j}} + \overset{\mathcal{O}(2)}{\cancel{\frac{\partial\tilde{x}^i}{\partial x^j}\frac{\partial\eta}{\partial\tilde{\eta}}\bar{T}^j{}_0}} + \frac{\partial\tilde{x}^i}{\partial x^j}\frac{\partial x^k}{\partial\tilde{\eta}}\bar{T}^j{}_k$$

$$- \bar{T}^i{}_0$$

$$\tilde{q}^i = q^i + (L^i)(1-\dot{T})\bar{\rho} + (\delta^i_j + \partial_j L^i)(-L^k)\left[-(\bar{p} + \delta p)\,\delta^j_k + \Pi^j_k\right]$$

$$= q^i + L^i\bar{\rho} + \delta^i_j\,\delta^j_k\,L^k\,\bar{p} = q^i + L^i\bar{\rho} + \delta^i_k\,L^k\,\bar{p}$$

$$\boxed{\tilde{q}^i = q^i + L^i(\bar{\rho} + \bar{p})}$$

Para el tercer término, escribimos $\mu = i$, $\nu = j$

$$\delta \tilde{T}^i{}_j = -\cancel{\delta\bar{\rho}}\,\delta^i{}_j + \tilde{\Pi}^i{}_j = \delta T^i{}_j + \left(\frac{\partial \tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^j} - \delta^i_\alpha\,\delta^\beta_j\right)\bar{T}^\alpha{}_\beta - \dot{\bar{T}}^i{}_j T$$

$$= -\cancel{\delta\bar{\rho}}\,\delta^i{}_j + \Pi^i{}_j + \frac{\partial \tilde{x}^i}{\partial x^0}\frac{\partial x^\beta}{\partial \tilde{x}^j}\bar{T}^0{}_\beta + \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial x^\beta}{\partial \tilde{x}^j}\bar{T}^k{}_\beta - \bar{T}^i{}_j$$

$$-\cancel{\dot{\bar{T}}^i{}_j}^{\,\to 0}\, T$$

$$= \Pi^i{}_j + \frac{\partial \tilde{x}^i}{\partial x^0}\frac{\partial x^0}{\partial \tilde{x}^j}\bar{T}^0{}_0 + \cancel{\frac{\partial \tilde{x}^i}{\partial x^0}\frac{\partial x^k}{\partial \tilde{x}^j}}^{\,\to 0}\bar{T}^0{}_k + \cancel{\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial x^0}{\partial \tilde{x}^j}}^{\,\to 0}\bar{T}^k{}_0$$

$$+ \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \tilde{x}^j}\bar{T}^k{}_l - \bar{T}^i{}_j$$

$$= \Pi^i{}_j + \cancel{(L^i)}^{\,\to 0}(-\partial_j T)\bar{\rho} + (\delta^i{}_k + \partial_k L^i)(\delta^l_j - \partial_l L^j)\bar{T}^k{}_l$$

$$- \bar{T}^i{}_j$$

$$= \Pi^i{}_j + \delta^i{}_k\,\delta^l_j\,\bar{T}^k{}_l - \bar{T}^i{}_j$$

$$= \Pi^i{}_j + \bar{T}^i{}_j - \bar{T}^i{}_j$$

$$\Longrightarrow \boxed{\tilde{\Pi}^i{}_j = \Pi^i{}_j}$$