Calcular los símbolos de Christoffel y verificar el valor de R.

Tenemos el intervalo dado por , usamos notación en t, para:

$$ds^2 = -(1+2\phi)\,dt^2 + a^2(1-2\psi)\,\delta_{ij}\,dx^i dx^j$$

Lo que nos da las componentes de nuestro tensor métrico

$g_{00} = a^2(2\psi+1)$ $\qquad g^{00} = \frac{1}{a^2}\frac{1}{2\psi+1} \sim \frac{1}{a^2}\,1-2\psi$

$g_{ij} = a^2(2\phi-1)\,\delta_{ij}$ $\qquad g^{ij} = \frac{1}{a^2}\frac{1}{2\phi-1} \sim \frac{-1}{a^2}(1+2\phi)\,\delta^{ij}$

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)$$

$\Gamma^0_{00} = \frac{1}{2} g^{0\beta}\left(g_{\beta\mu,\nu} + g_{\beta 0,0} - g_{00,\beta}\right)$, debe $\beta=0$

$= \frac{1}{2} g^{00}\left(g_{00,0} + g_{00,0}\right) = \frac{1}{2} g^{00}\, 2\left(\partial_0\; a^2(2\psi+1)\right)$

$= g^{00}\left(2a\dot{a}\,(2\psi+1) + a^2(2\dot{\psi}+1)\right)$

$= \frac{(1-2\psi)}{2a^2}\left(2a\dot{a}\,(2\psi+1) + a^2\,(2\dot{\psi}+1)\right)$

$= (1-2\psi)\,\frac{\dot{a}}{a}\,(2\psi+1) + \frac{1}{2}\,(2\dot{\psi}+1) = (1-2\psi)\left[\mathcal{H}\,(2\psi+1) + \frac{1}{2}(2\dot{\psi}+1)\right]$

Eliminamos los términos con ψ y sólo queda el de $\mathcal{H}$ y $\mathcal{H}\psi$

$= \mathcal{H} - \cancel{2\psi\mathcal{H}} + \cancel{2\psi\mathcal{H}} + \frac{1}{2}\,2\dot{\psi} = \boxed{\mathcal{H} + \dot{\psi}}$

$$\Gamma^0_{0i} = \tfrac{1}{2} g^{00} \left(g_{00,i} + \cancel{g_{0i,0}} - \cancel{g_{0i,0}} \right)$$

$$= \tfrac{1}{2} g^{00} \left(\partial_i \left[a^2 (2\psi+1) \right] \right) \quad , \partial_i a^2 = 0$$

$$= \cancel{\tfrac{1}{2}} g^{00} \cancel{2} (a^2 \partial_i \psi) = g^{00} \partial_i \psi \, a^2 = \frac{1}{\cancel{a^2}} (1-2\psi) \partial_i \psi \cancel{a^2}$$

$$= \partial_i \psi - \cancel{2\psi \partial_i \psi} \xrightarrow{} O(2)$$

$$= \boxed{\partial_i \psi}$$

$$\Gamma^i_{00} = \tfrac{1}{2} g^{i\beta} \left(g_{\beta 0,0} + g_{\beta 0,0} - g_{00,\beta} \right) \quad ; \text{pero } i = \beta$$

$$= \tfrac{1}{2} g^{ii} \left(-g_{00,\beta} \right)$$

$$= \tfrac{1}{2} \left(-\tfrac{1}{\cancel{a^2}} (2\phi+1) \right) \left(-\partial_i \left(\cancel{a^2} (2\psi+1) \right) \right) \quad , \partial_i a^2 = 0$$

$$= \tfrac{1}{2} (2\phi+1) \, \partial_i (2\psi+1)$$

$$= \tfrac{1}{2} (2\phi+1) (2\partial_i \psi) \quad \text{— los términos cruzados los eliminamos}$$

$$= \tfrac{1}{2} \, 2 \partial_i \psi = \underline{\partial_i \psi}$$

$$= \boxed{\delta^{ij} \partial_j \psi}$$

Ahora

$$\Gamma^0_{ij} = \frac{1}{2} g^{0\beta} \left(\cancel{g_{\beta i,j}} + \cancel{g_{\beta j,i}} - g_{ij,\beta} \right) \quad , \beta = 0$$

$$= \frac{1}{2} g^{00} \left(g_{ij,0} \right)$$

$$= \tfrac{1}{2} g^{00} \left[\partial_t \left(a^2 (2\phi - 1) \delta_{ij} \right] \right.$$

$$= \tfrac{1}{2} g^{00} \left(2a\dot{a}(2\phi-1) + 2a^2\dot{\phi} \right) \delta_{ij}$$

$$= \tfrac{1}{2} \frac{1}{a^2}(1-2\psi)\left(2a\dot{a}(2\phi-1) + 2a^2\dot{\phi}\right) \delta_{ij}$$

$$= (1-2\psi)\left(\mathcal{H}(2\phi-1) + \dot{\phi}\right) \delta_{ij}$$

$$= \left(\mathcal{H} + 2\mathcal{H}\phi - 2\psi\mathcal{H} - \dot{\phi}\right) \delta_{ij}$$

$$= \boxed{\mathcal{H}\delta_{ij} - [\dot{\phi} + 2\mathcal{H}(\psi+\phi)]\delta_{ij}}$$

Ahora $\Gamma^{i}_{j0} = \tfrac{1}{2} g^{i\beta}(g_{\beta j,0} + g_{\beta 0,j} - g_{j0,\beta})$ $\beta = i$

$$= \tfrac{1}{2} g^{ii}(g_{ij,0} + \cancel{g_{i0,j}} - \cancel{g_{j0,\beta}})$$

$$= \tfrac{1}{2} g^{ii}(g_{ij,0}) = \tfrac{1}{2} - \frac{(1+2\phi)}{a^2}\left(\partial_t a^2(2\phi-1)\delta_{ij}\right)$$

$$= -\frac{(1+\phi)}{2a^2}\left(2a\dot{a}(2\phi-1) + 2a^2\dot{\phi}\right)\delta_{ij}$$

$$= -(1+\phi)\left(\mathcal{H}(2\phi-1) + \dot{\phi}\right)\delta_{ij}$$, solo dejamos $\mathcal{O}(1)$

$$= \boxed{(\mathcal{H} - \dot{\phi})\delta_{ij}}$$

Por último

$\Gamma^{k}_{ij} = \tfrac{1}{2} g^{k\beta}(g_{\beta i,j} + g_{\beta j,i} + g_{ij,\beta})$, $k = \beta$

$$= \tfrac{1}{2} g^{kk}(g_{ki,i} + g_{kj,i} + g_{ij,k})$$

$$= -\frac{(1+2\phi)}{a^2}\left[\delta_{ij}\,a^2(\partial_k(2\phi-1)) + \delta_{ik}\,a^2\partial_j(2\phi-1) - \delta_{jk}\,a^2\,\partial_i(2\phi-1)\right]$$

$$= (1+\cancel{2\phi})(\delta_{ij}\,\phi_{,k} - \delta_{ik}\,\phi_{,j} + \delta_{jk}\,\phi_{,i})$$

$$= -\delta_{ij}\,\phi_{,k} - \delta_{ik}\,\phi_{,j} + \delta_{jk,}\phi_{,i}$$

agregando una δ^{il} y agrupando queda igual que en la tarea

Para calcular R, tenemos

$$R = g^{\mu\nu} R_{\mu\nu}$$

y podemos escribirlo como parte temp + parte esp

$$R = g^{00} R_{00} + g^{ij} R_{ij}$$

Ya tenemos g^{00} y g^{ij}, vamos a calcular R_{00}

$$\Rightarrow \quad R_{00} = \Gamma^{\mu}_{00,\mu} - \Gamma^{\mu}_{0\mu,0} + \Gamma^{\mu}_{\rho\mu}\Gamma^{\rho}_{00} - \Gamma^{\mu}_{\rho 0}\Gamma^{\rho}_{0\mu}$$

$$= \Gamma^{i}_{00,i} - \Gamma^{i}_{0i,0} + \Gamma^{i}_{0i}\Gamma^{0}_{00} - \cancel{\Gamma^{i}_{00}\Gamma^{0}_{0i}} + \cancel{\Gamma^{i}_{ji}\Gamma^{j}_{00}} - \Gamma^{i}_{j0}\Gamma^{j}_{0i}$$

ambos contienen perturbación $\mathcal{O}(1)$

$$= \delta^{ij}\partial_i\partial_j\psi - \partial_t(\mathcal{H}(\delta_j{}^i - \dot\phi\,\delta_j{}^i)) + \partial_i\psi(\mathcal{H} - \dot\phi) - (\mathcal{H} - \dot\phi)\,\delta_i{}^j\delta_j{}^i$$

$$= \cancel{\partial_{ii}}^{\nabla^2} \Psi - 3\partial_t(\mathcal{H} - \dot\phi) - 3(\mathcal{H} + \dot\psi)(\mathcal{H} - \dot\phi) - (\mathcal{H} - \dot\phi)^2 \delta_i^j \delta_j^i$$

$$= \nabla^2\psi - 3\dot{\mathcal{H}} + 3\ddot\phi \cancel{- 3\mathcal{H}^2 + 3\mathcal{H}^2} + 3\mathcal{H}(\dot\phi + \dot\psi)$$

$$= \nabla^2\psi - 3\dot{\mathcal{H}} + 3\ddot\phi + 3\mathcal{H}(\dot\phi + \dot\psi)$$

Ahora $R_{ij} = \Gamma^\mu_{ij,\mu} - \Gamma^\mu_{i\mu,j} + \Gamma^\mu_{\ell\mu}\Gamma^\ell_{ij} - \Gamma^\mu_{\ell j}\Gamma^\ell_{i\mu}$

$$= \Gamma^0_{ij,0} + \Gamma^k_{ij,k} - \Gamma^0_{i0,j} - \Gamma^k_{ik,j} + \Gamma^0_{00}\Gamma^0_{ij} + \Gamma^k_{0k}\Gamma^0_{ij} + \Gamma^0_{kj}\Gamma^k_{i0}$$
$$- \Gamma^k_{0j}\Gamma^0_{ik}$$

$$= \delta_{ij} a^2 \Big(\mathcal{H}^2 - 2\mathcal{H}^2(\phi + \psi) - 4\mathcal{H}\dot\phi + \frac{\ddot a}{a} - 2\frac{\ddot a}{a}(\phi + \psi) - \mathcal{H}(\dot\psi) - \ddot\phi \Big)$$
$$- 2\partial_i\partial_j\phi + \delta_{ij}\partial_k\partial_k\phi - \partial_i\partial_j\psi + \delta_i^i \partial_i\partial_j\phi$$
$$+ \delta_{ij} a^2 H\dot\phi + \delta_{ij} a^2 [\mathcal{H}^2 - 2\mathcal{H}^2(\phi + \psi) - 2\mathcal{H}(\dot\phi)]$$

Agrupamos con las δ y las parciales

$$= \delta_{ij} a^2 \Big\{ \frac{1}{a^2}\nabla^2\phi + \frac{\ddot a}{a}[1 - 2(\psi + \phi)] - H\dot\psi - \ddot\phi + 2\mathcal{H}^2$$
$$- 4\mathcal{H}^2(\psi + \phi) - 6\mathcal{H}\dot\phi \Big\} + (\phi - \psi)_{,ij}$$

$$= \delta_{ij} a^2 \Big\{ \frac{1}{a^2}\nabla^2\phi + \dot{\mathcal{H}}[1 - 2(\psi + \phi)] - \mathcal{H}\dot\psi - \ddot\phi + 2\mathcal{H}\dot\phi$$
$$+ 2\mathcal{H}^2 - 4\mathcal{H}^2(\psi + \phi) + 6\mathcal{H}\dot\phi \Big\} + (\phi - \psi)_{,ij}$$

$$R_{ij} = \delta_{ij}\Big\{ \nabla^2\phi - \ddot\phi + [(1 - 2(\phi + \psi)](\dot{\mathcal{H}} + 2\mathcal{H}^2) - \mathcal{H}(\dot\phi + 5\dot\phi) \Big\}$$
$$- (\phi - \psi)_{,ij}$$

$\Rightarrow$ Ahora podemos hacer

$$R = g^{00} R_{00} + g^{ij} R_{ij}$$

$$= \underline{\underline{(1-2\psi) R_{00}}} - (1+2\phi)\delta^{ij} R_{ij}$$

$$= (1-2\psi)\left[\nabla^2\psi - 3\dot{H} + 3\ddot{\phi} + 3\mathcal{H}(\dot{\phi}+\dot{\psi})\right]$$

$$= \nabla^2\psi - 3\dot{H} + 3\ddot{\phi} + 3H(\dot{\phi}+\dot{\psi}) - 6\dot{H}\psi$$

$$= \underline{\underline{\nabla^2\psi - 3\dot{H}(1+2\dot{H}\psi) + 3\ddot{\phi} + 3H(\dot{\phi}+\dot{\psi})}}$$

Ahora $-(1+2\phi)\delta^{ij} R_{ij}$

$$= -(1+2\phi)\delta^{ij}\left[\delta_{ij}\left\{\nabla^2\phi - \ddot{\phi} + [1-2(\phi+\psi)](\dot{\mathcal{H}} + 2\mathcal{H}^2) - \mathcal{H}(\dot{\psi}+5\dot{\phi})\right\} - (\phi-\psi)_{,ij}\right]$$

$$= -(1+2\phi)\left[3\left\{\nabla^2\phi - \ddot{\phi} + [1-2(\phi+\psi)](\dot{\mathcal{H}} + 2\mathcal{H}^2) - \mathcal{H}(\dot{\psi}+5\dot{\phi})\right\} - (\phi-\psi)_{,ii}\right]$$

$$= 3\left\{\nabla^2\phi - \ddot{\phi} + [1-2(\phi+\psi)](\dot{\mathcal{H}} + 2\mathcal{H}^2) - \mathcal{H}(\dot{\psi}+5\dot{\phi})\right\} - \nabla^2(\phi-\psi)$$
$$+ 2\phi\, 3\{(\dot{H} + 2H^2)$$

$$R = \nabla^2\psi - 3\dot{H}(1+2\dot{H}\psi) + 3\ddot{\phi} + 3H(\dot{\phi}+\dot{\psi}) + 3\nabla^2\phi - \ddot{\phi} + [1-2(\phi+\psi)](\dot{\mathcal{H}} + 2\mathcal{H}^2) - \mathcal{H}(\dot{\psi}+5\dot{\phi})\} - \nabla^2(\phi-\psi)$$
$$+ 2\phi\, 3\{(\dot{H} + 2H^2)$$