Problema 5. Con calculo variacional:

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]$$
$$=\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\,\eta^{\mu\alpha}\partial_\alpha\phi - V(\phi)\right]$$

Variamos la acción

$$\delta S=\int d^4x\sqrt{-g}\;\delta\left[\frac{1}{2}\partial_\mu\phi\,\eta^{\mu\alpha}\partial_\alpha\phi - V(\phi)\right]$$

Veamos que el término variado del lado derecho se desarrolla como sigue

$$\delta\left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]=\frac{1}{2}\left[(\delta\partial_\mu\phi)\eta^{\mu\alpha}\partial_\alpha\phi+\partial_\mu\phi\,\eta^{\mu\alpha}(\delta\partial_\alpha\phi)\right]-\delta V(\phi)$$
$$=\frac{1}{2}\left[(\partial_\mu\delta\phi)\eta^{\mu\alpha}\partial_\alpha\phi+\partial_\mu\phi\,\eta^{\mu\alpha}(\partial_\alpha\delta\phi)\right]-\frac{dV}{d\phi}\delta\phi$$
$$=\frac{1}{2}\Big[\cancel{\partial_\mu(\eta^{\mu\alpha}\partial_\alpha\phi\,\delta\phi)}-\partial_\mu(\eta^{\mu\alpha}\partial_\alpha\phi)\delta\phi$$
$$+\cancel{\partial_\alpha(\partial_\mu\phi\,\eta^{\alpha\alpha}\delta\phi)}-\partial_\alpha(\partial_\mu\phi\,\eta^{\mu\alpha})\delta\phi\Big]-\frac{dV}{d\phi}\delta\phi$$
$$=-\frac{1}{2}\left[\partial_\mu(\eta^{\mu\alpha}\partial_\alpha\phi)\delta\phi+\partial_\alpha(\partial_\mu\phi\,\eta^{\mu\alpha})\delta\phi\right]-\frac{dV}{d\phi}\delta\phi$$
$$=-\left[\frac{1}{2}\left\{\partial_\mu(\eta^{\mu\alpha}\partial_\alpha\phi)+\partial_\alpha(\partial_\mu\phi\,\eta^{\mu\alpha})\right\}+\frac{dV}{d\phi}\right]\delta\phi$$

Realizando el cambio $\mu\leftrightarrow\alpha$ en el segundo término

$$=-\left[\frac{1}{2}\left\{\partial_\mu(\eta^{\mu\alpha}\partial_\alpha\phi)+\partial_\mu(\partial_\alpha\phi\,\eta^{\mu\alpha})\right\}+\frac{dV}{d\phi}\right]\delta\phi$$
$$=-\left[\partial_\mu(\eta^{\mu\alpha}\partial_\alpha\phi)+\frac{dV}{d\phi}\right]\delta\phi$$
$$=-\left[\partial_\mu\partial^\mu\phi+\frac{dV}{d\phi}\right]\delta\phi=-\left[\Box^2\phi+\frac{dV}{d\phi}\right]\delta\phi$$

Por lo que la acción variada es

$$\delta S=-\int d^4x\sqrt{-g}\left[\Box^2\phi+\frac{dV}{d\phi}\right]\delta\phi$$

Aplicando mínima acción obtenemos que

$$-\int d^4x\sqrt{-g}\left[\Box^2\phi+\frac{dV}{d\phi}\right]\delta\phi=0\implies\Box^2\phi+\frac{dV}{d\phi}=0$$