~~Tarea~~ Tarea Perturbaciones 3

Demostrar $\tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\partial_i L^i$

$$\tilde{B}_i = B_i + \partial_i T - \dot{L}_i$$

$$\tilde{E}_{ij} = E_{ij} - \partial_{\langle i} L_{j\rangle}$$

Nuestra métrica está dada por

$$ds^2 = a^2\{(1+2\psi)d\eta^2 - 2B_i dx^i d\eta - [(1-2\phi)\delta_{ij} + 2E_{ij}]dx^i dx^j\}$$

y la perturbación dada por:

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} - \delta_\mu^\alpha \delta_\nu^\beta\right)\bar{g}_{\alpha\beta}(\eta, x^i) - T\dot{\bar{g}}_{\mu\nu}(\eta, x^i)$$
$$- L^i \partial_i \bar{g}_{\mu\nu}(\eta, x^i)$$

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \begin{pmatrix} 1+\dot{T} & \partial_i T \\ \dot{L}_i & \delta_j^i + \partial_j L^i \end{pmatrix}$$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = \begin{pmatrix} 1-\dot{T} & -\partial_i T \\ -\dot{L}^i & \delta_j^i - \partial_j L^i \end{pmatrix}$$

Entonces

Podemos desarrollar las componentes espaciales de $\delta\tilde{g}_{\mu\nu}$ y despejar $\tilde{\phi}$

$$\Rightarrow \delta\tilde{g}_{ij} = \delta g_{ij} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^j} - \delta_i^\alpha \delta_j^\beta\right)\bar{g}_{\alpha\beta} - T\dot{\bar{g}}_{ij} - \cancel{L^i \partial_i \bar{g}_{ij}}$$

$$\delta \tilde{g}_{ij} = -2a^2(\tilde{\phi}\delta_{ij} + \tilde{E}_{ij})$$

$$\Rightarrow -2a^2(\phi\delta_{ij} + E_{ij}) = -2a^2(\phi\delta_{ij} + E_{ij}) + \frac{\partial x^0}{\partial \tilde{x}^i} g_{\alpha\beta} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial x^i} g_{j\beta} - \bar{g}_{ij} + 2\dot{\alpha}\alpha\delta_{ij}$$

$$= -2a^2(\phi\delta_{ij} + E_{ij}) + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^i}\bar{g}_{0\beta} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^i} g_{j\beta} - \bar{g}_{ij} + 2\dot{\alpha}\alpha\delta_{ij}$$

Además las derivadas temp y esp de las nuevas respecto a $\tilde{x}^i$ se anulan:

$$\frac{\partial x^0}{\partial \tilde{x}_j} = \frac{\partial x^i}{\partial \tilde{x}^j} = 0$$

$$= -2a^2(\phi\delta_{ij} + E_{ij}) + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j} g_{jl} - \bar{g}_{ij} + 2\dot{a}a\delta_{ij}$$

Sustituimos los valores de la métrica:

$$= -2a^2(\phi\delta_{ij} - E_{ij}) - a^2(\delta^j_i - \partial_i \zeta^j)(\delta^l_j - \partial_j \zeta^l)\tilde{g}_{il} - \tilde{g}_{ij} + 2\dot{\alpha}\alpha\delta_{ij}$$

$$= 2a^2(\phi\delta_{ij} + E_{ij}) - a^2(\delta^j_i \cancel{\delta^l_j})\tilde{g}_{il} \cancel{-\tilde{g}_{ij}} + 2\dot{\alpha}\alpha\delta_{ij} + a^2\partial_j \zeta^i$$

$l = j$

$$= -2a^2\left(\phi\,\delta_{ij} + E_{ij} + \frac{\dot{a}}{a}T\delta_{ij} + \partial_i \zeta^i\right)$$

$$= -2a^2(\phi\delta_{ij} + \mathcal{H}T\delta_{ij} + \partial_i \zeta^i) - 2a^2 E_{ij}$$

$$\Rightarrow \cancel{-2a^2}\tilde{\phi}\cancel{\delta_{ij}} - \cancel{2a^2}\tilde{E}_{ij} = \cancel{-2a^2}(\phi\cancel{\delta_{ij}} + \mathcal{H}T\cancel{\delta_{ij}} + \partial_i \zeta^i) - 2a^2\cancel{E_{ij}}$$

$$\Rightarrow \tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{\delta_{ij}}\partial_j \zeta^i$$

$$\boxed{\tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\partial_i \zeta^i}$$

Ahora, tomando $\mu = i$ espacial y $\nu = 0$ temp en la $\delta\tilde{g}_{\mu\nu}$:

$$\Rightarrow \delta\tilde{g}_{i0} = \underline{\underline{-a^2\tilde{B}_i}}$$

$$= \delta g_{i0} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^0} - \cancel{\delta_i^{\alpha}}\,\delta_0^\beta \right)\tilde{g}_{\alpha\beta} - T\tilde{g}_{i0} - l^i\partial_i\tilde{g}_{i0}$$

$\alpha = i$

$$= -a^2B_i + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^0}\tilde{g}_{0\beta} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^\beta}{\partial \tilde{x}^0}\tilde{g}_{j\beta} - \tilde{g}_{i0} - \cancel{T(-\tilde{B}_i) - l^i\partial_i(-\tilde{B}_i)}$$

Ahora expandimos β

$$= -a^2B_i + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^0}{\partial \tilde{x}^0}\tilde{g}_{00} + \frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^j}{\partial \tilde{x}^0}\tilde{g}_{0j} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^0}{\partial \tilde{x}^0}\tilde{g}_{j0} + \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial x^\ell}{\partial \tilde{x}^i}g_{j\ell}$$

$$= -a^2B_i(-\partial_iT)(1-\dot{T})[a^2(1+2\psi)] + (-\partial_iT)(\cancel{-\dot{l}^j})(-\tilde{B}_i)$$
$$+ (\delta_i^j - \partial_i l^j)(1-\dot{T})(-a^2\cancel{B_j}) + (\delta_i^j - \partial_i l^j)(-\dot{l}^\ell)a^2[-(1-2\phi)\delta_{j\ell} - E_{j\ell}]$$

$$= -a^2B_i - a^2(\partial_iT) + a^2\delta_i^j\delta_{j\ell}\dot{l}^\ell \quad \text{factorizamos } a^2$$

$$= -a^2(B_i + \partial_iT - \delta_i^j\dot{l}_j) \quad \text{Bajamos el índice y cambiamos } \underline{\underline{j \text{ por } i}}$$

Pero

$$\cancel{-a^2}\tilde{B}_i = \cancel{-a^2}(B_i + \partial_iT - \dot{l}_i)$$

$$\therefore \boxed{\tilde{B}_i = B_i + \partial_iT - \dot{l}_i} \checkmark$$

Tarea: mostrar Φ, Ψ, Φ_i no cambian bajo transformación de coordenadas

Tenemos: $\begin{cases} \Psi \equiv \psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E} \\ \tilde{\Psi} = \psi - \dot{T} - \mathcal{H}T \end{cases}$

$$\Rightarrow \tilde{\Psi} = \tilde{\varphi} + \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \dot{\tilde{B}} - \ddot{\tilde{E}}$$

$$= \Psi - \dot{T} - \mathcal{H}T + \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \dot{\tilde{B}} - \ddot{\tilde{E}}$$

Además $\tilde{B} = B + T + \dot{L}$, $\tilde{E} = E - L$

$$= \Psi - \cancel{\dot{T}} - \mathcal{H}T + \mathcal{H}(B + T - \dot{E}) + \dot{B} + \cancel{\dot{T}} - \ddot{E}$$

$$= \Psi + \mathcal{H}(B + \cancel{T} - \cancel{T} - \dot{E}) + \dot{B} - \ddot{E}$$

$$= \underbrace{\Psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E}}$$ pero esto es simplemente Ψ

$$\Rightarrow \underline{\underline{\tilde{\Psi} = \Psi}}$$

Ahora

$$\begin{cases} \Phi = \phi - \mathcal{H}(B - \dot{E}) + \frac{1}{3}\nabla^2 E \\ \tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L \end{cases}$$

Partimos de $\tilde{\Phi} = \tilde{\phi} - \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \frac{1}{3}\nabla^2 \tilde{E}$

$$= \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L - \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \frac{1}{3}\nabla^2 \tilde{E}$$

$$= \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L - \mathcal{H}(B + T - \dot{L}) - (\dot{E} - \dot{L}) + \frac{1}{3}\nabla^2 (E - L)$$

$$= \phi + \cancel{\mathcal{H}T} + \cancel{\frac{1}{3}\nabla^2 L} - \mathcal{H}(B + \cancel{T} - \dot{E}] + \frac{1}{3}\nabla^2 E - \cancel{\frac{1}{3}\nabla^2 L}$$

$$\tilde{\Phi} = \phi - \mathcal{H}(B - \dot{E}) + \frac{1}{3}\nabla^2 E$$

$$\therefore \tilde{\Phi} = \Phi$$

y
$$\begin{cases} \Phi_i = \dot{E}_i - B_i \\ \tilde{B}_i = B_i + \partial_i T - \dot{L}_i \end{cases}$$

$$\Rightarrow \tilde{\Phi}_i = \dot{\tilde{E}}_i - \tilde{B}_i = \dot{\tilde{E}}_i - B_i - \partial_i T + \dot{L}_i = \dot{E}_i - \cancel{\dot{L}_i} - B_i - \cancel{\partial_i T} + \cancel{\dot{L}_i}$$

$$= \dot{E}_i - B_i = \Phi_i$$

Tarea, Mostrar:

$$\left\{\begin{array}{ll} \delta \tilde{p} = \delta p - T \dot{\bar{p}} & ① \\ \tilde{q}^i = q^i + (\bar{\rho} + \bar{p}) \iota^i & ② \\ \tilde{\pi}^i_j = \pi^i_j & ③ \end{array}\right.$$

①

Tenemos la perturbación en $\tilde{T}^\mu_\nu$ como

$$\underline{\delta \tilde{T}^\mu_\nu = \delta T^\mu_\nu + \left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} - \delta^\mu_\alpha \delta^\beta_\nu\right)\bar{T}^\alpha_\beta - \dot{\bar{T}}^\mu_\nu T}$$

→ el caso espacial $\mu = \nu = i$

$$\Rightarrow \delta \tilde{T}^i_i = \delta T^i_i + \left(\frac{\partial \tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^i} - \delta^i_\alpha \delta^\beta_i\right)\bar{T}^\alpha_\beta - \dot{\bar{T}}^i_i T$$

Tenemos también

$$T^i_j = -(\bar{p} + \delta p)\delta_i^j + \pi_i^j$$

Por lo que $\dot{\bar{T}}^i_i = -\dot{\bar{p}}$, en orden coordenadas ind

sólo

$$= -\delta p - \cancel{\delta(\delta p)} + \left(\cancel{\frac{\partial x^\beta}{\partial x^\alpha}} - \delta_\alpha^\beta\right)\bar{T}_\beta^\alpha + \dot{\bar{p}} T$$

$$= -\delta p + \dot{\bar{p}} T$$

, pero por otro lado

$$\delta T^i_i = -\delta \tilde{p}$$

$$\Rightarrow \boxed{\delta \tilde{p} = \delta p + \dot{\bar{p}} T}$$

② Ahora tomamos $\mu = i$, $\nu = 0$, entonces $\delta \tilde{T}^i_0 = \delta \tilde{q}^i$

$$y \quad \delta T_0^i = \delta T_0^i + \left(\frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{\eta}} - \delta_\alpha^i \delta_0^\beta \right) \bar{T}_\beta^\alpha - \dot{\bar{T}}_0^i T$$

$$= \delta q^i + \frac{\partial \bar{x}^i}{\partial \eta} \frac{\partial x^\beta}{\partial \bar{\eta}} \bar{T}_\beta^0 + \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^\beta}{\partial \bar{\eta}} \bar{T}_\beta^j - \delta_\alpha^i \delta_0^\beta \bar{T}_\beta^\alpha - \cancel{\dot{\bar{q}}^i T}$$

$$= \delta q^i + \frac{\partial \bar{x}^i}{\partial \eta} \frac{\partial \eta}{\partial \bar{\eta}} \bar{T}_0^0 + \frac{\partial \bar{x}^i}{\partial \eta} \cancel{\frac{\partial x^j}{\partial \bar{\eta}}} \bar{T}_j^0 + \frac{\partial \bar{x}^i}{\partial x^j} \cancel{\frac{\partial \eta}{\partial \bar{\eta}}} \bar{T}_0^j + \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \bar{\eta}} \bar{T}_k^j$$
$$- \bar{T}_0^i$$

$$\tilde{q}_i = q^i + (l^i)(1 - \cancel{f}) \bar{\delta} + (\delta_j^i + \partial_j l^i)(-l^k) [-(\bar{p} + \delta p) \delta_k^j + \Pi_k^j]$$

$$\tilde{q}_i = q^i + l^i \bar{\delta} + \delta_j^i \delta_k^j l^k \bar{p}$$
$$\tilde{q}_i = q^i + l^i \bar{\delta} + \delta_k^i l^k \bar{p} \quad , \; i = k$$

$$\Rightarrow \boxed{\tilde{q}_i = q_i + l^i (\bar{\delta} + \bar{p})}$$

③ Tomamos ahora las comp espaciales de $\delta \tilde{T}_i^j$

$$\Rightarrow \quad \delta \tilde{T}_j^i = -\delta \bar{p}\, \delta_j^i + \Pi_j^i$$

Por otro lado

$$\delta T_j^i = \delta T_j^i + \left(\frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^j} - \delta_\alpha^i \delta_j^\beta \right) \bar{T}_\beta^\alpha - \dot{\bar{T}}_j^i T$$

$$= -\delta \bar{p}\, \delta_j^i + \Pi_i^j + \frac{\partial \bar{x}^i}{\partial x^0} \frac{\partial x^\beta}{\partial \bar{x}^j} \bar{T}_\beta^0 + \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^j} T_\beta^\alpha - \bar{T}_j^i - \cancel{\dot{\bar{T}}_0 T}$$

Abrimos los exp de α en 0 y espaciales

$$= -\delta \bar{p}\, \delta_j^i + \Pi_i^j + \frac{\partial \bar{x}^i}{\partial x^0} \frac{\partial x^0}{\partial \bar{x}^j} \bar{T}_0^0 + \cancel{\frac{\partial \bar{x}^i}{\partial x^0} \frac{\partial x^k}{\partial \bar{x}^j}} \bar{T}_k^0$$

$$+ \cancel{\frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^0}{\partial \bar{x}^j}} \bar{T}_0^\alpha + \frac{\partial \bar{x}^i}{\partial x^\ell} \frac{\partial x^\ell}{\partial \bar{x}^j} T_\ell^\ell$$

$$= -\delta\bar{p}\,\delta_j^{\,i} + \pi_j^{\,i} + (\cancel{i^{\,i})(-\partial_j T)}\,\tilde{\delta} + (\delta_k^{\,i} + \partial_k\, \iota^i)(\delta_j^{\,l} - \partial_l\, \iota^i)\,\bar{T}_l^{\,k}$$
$$-T_j^{\,i}$$

$l = j \quad , k = i$

$$= -\delta\bar{p}\,\delta_j^{\,i} + \pi_j^{\,i} \quad \cancel{+T_j^{\,i} - T_j^{\,i}}$$

$$\Rightarrow \cancel{-\delta\bar{p}\,\delta_j^{\,i}} + \tilde{\pi}_j^{\,i} = \cancel{-\delta\bar{p}\,\delta_j^{\,i}}\; \pi_j^{\,i}$$

$$\Rightarrow \boxed{\tilde{\pi}_j^{\,i} = \pi_j^{\,i}}$$

Tarea Mostrar Δ es norma - invariante

De nuevo tenemos la eq de continuidad
$$\dot{\bar{\rho}} + 3\mathcal{H}(\bar{\rho} + \bar{P}) = 0$$
y $\quad v = (\bar{P} + \bar{\rho})^{-1} q$

Tenemos que $\quad \bar{\rho}\Delta = \delta\rho + \dot{\bar{\rho}}\,(v + B) \quad$, sust. continuidad

$$\Rightarrow \quad \bar{\rho}\Delta = \delta\rho + (-3\mathcal{H}(\bar{\rho} + \bar{P}))\,(B + v)$$

$$\Rightarrow \quad \bar{\rho}\tilde{\Delta} = \delta\tilde{\rho} - 3\mathcal{H}(\tilde{\rho} + \tilde{P})\,(\tilde{B} + \tilde{v})$$

Además $\quad \tilde{v} = (\tilde{P} + \tilde{\rho})^{-1}\,\tilde{q}$

$$\Rightarrow \bar{P}\tilde{\Delta} = \delta\tilde{\rho} - 3\mathcal{H}(\tilde{\rho} + \tilde{P})\tilde{B} - 3\mathcal{H}\,(\cancel{\tilde{\rho} + \tilde{P}})\,(\cancel{\tilde{P} + \tilde{\rho}})^{-1}\,\tilde{q}$$

$$\text{Ahora}\begin{cases} \tilde{q} = (q + (\tilde{\rho} + \bar{P})\,\iota \\ \tilde{B} = B + T\,\iota \end{cases}$$

$$\Rightarrow \bar\rho\tilde\Delta = \delta\rho - 3H(\bar\rho+\bar P)(B+T-i) - 3H\left[q + (\bar\rho+\bar P)i\right]$$

$$= \delta p - T\left[\cancelto{0}{\dot{\tilde\rho} + 3H(\bar\rho+\bar P)}\right]T - 3H(\tilde\rho+\tilde p)\left[B - \cancel{i} + \cancel{(\bar p+\bar\rho)^{-1} q} \cancel{+i}\right]$$

$$= \delta P - 3H(\bar\rho+P)(B-V)$$

$$= \delta p + \dot{\bar\rho}(V+B)$$

$$\boxed{\bar\rho\tilde D = \bar\rho\Delta}$$

<u>invariante</u>