Tarea

$$\boxed{g_{\mu\nu}\,(E_i)^\mu (E_j)^\nu = -\delta_{ij}}$$

Tenemos la métrica dada por el intervalo ds^2

$$ds^2 = a^2(\eta)\{(1+2\psi)\,d\eta^2 - 2B_i\,dx^i\,d\eta - [(1-2\phi)\delta_{ij} + 2E_{ij}]dx^i dx^j\}$$

Para el tipo espacialoide tenemos:

$$(E_i)^\mu = \frac{1}{a}[B_i\,\delta_0^{\ \mu} + (1+\phi)\delta_i^{\ \mu} - E_i^{\ j}\delta_j^{\ \mu}]$$

De la métrica tenemos

$$g_{00} = (1+2\psi)\,a^2$$
$$g_{ij} = [(1+2\phi)\delta_{ij} + 2E_{ij}]\,a^2$$
$$g_{i0} = -2B_i\,a^2$$

Expandiendo la suma

$$\Rightarrow g_{\mu\nu}(E_i)^\mu (E_i)^\nu = g_{00}(E_i)^0(E_j)^0 + g_{i0}(E_i)^i(E_j)^0 + g_{ij}(E_i)^i(E_j)^j$$

$\mu=0$

$$(E_i)^0 = \frac{1}{a}\left(B_i + \cancel{(1+\phi)\delta_i^{\ 0}} - \cancel{E_i^{\ j}\delta_j^{\ 0}}\right)$$

$$= \underline{\frac{1}{a}B_i}$$

$$(E_j)^0 = \underline{\underline{\frac{1}{a}B_j}}$$

luego

$$(E_i)^i = \frac{1}{a}\left(\cancel{B_i\delta_0^{\ i}} + (1+\phi)\delta_i^{\ i} - E_i^{\ j}\delta_j^{\ i}\right)$$

$$= \underline{\frac{1}{a}(1+\alpha) - E_i^{\ j}\delta_i^{\ j}}$$

$$(E_i)^j = \frac{1}{a}\left((1+\phi) - E_j{}^i\, \delta_i{}^j\right)$$

$$\Rightarrow g_{\mu\nu}(E_i)^\mu (E_j)^\nu = \frac{\cancel{a^2}}{\cancel{a^2}}\Big\{(1+2\psi)B_iB_j - 2B_{i,j}((1+\phi) - E_j{}^i)$$

$$-((1-2\phi)\delta_{ij} + 2E_{ij}][(1+\phi)E_i{}^j \delta_j{}^i][(1+\phi) - E_j{}^i)\Big\}$$

$$= \{-(1-2\phi)(1+\phi)\delta_i{}^j - (1-2\phi)\delta_{ij}E_i{}^j\delta_j{}^i + 2E_{ij}(1+\phi) - 2E_j{}^i \cancel{E_j}\delta_i{}^i)$$
$$((1+\phi) - E_j{}^i)\}$$

$$= [-(1-\phi)\delta_{ij} - \delta_{ij}E_i{}^j\delta_j{}^i + 2E_i{}^j][(1+\phi) - E_j{}^i]$$

$$= -(1-\cancel{\phi})^2\delta_{ij} + (1-\cancel{\phi})\delta_{ij}E_j{}^i + \delta_{ij}E_i{}^j\delta_i{}^j\cancel{(1+\phi)} - 2E_{ij}\cancel{(1+\phi)}$$

$$= -\delta_{ij} + \underline{\delta_{ij}E_j{}^i + \cancel{\delta_{ij}E_j{}^j\delta_j{}^i}} - \cancel{2E_{ij}}$$

$$= -\delta_{ij}$$

$$\Rightarrow \boxed{g_{\mu\nu}(E_i)^\mu (E_j)^\nu = -\delta_{ij}}$$

(2) Ahora tenemos que demostrar:

$$T_0{}^i = q^i$$

$$T_j{}^i = -(\bar{p} + \delta p)\delta_i{}^j + \pi_j{}^i$$

Tomemos los mismos componentes de la métrica

En clase vimos que $T^{0i} = \frac{1}{a^2}(q^i + \bar{p}B^i)$

Bajamos el índice multiplicando por la métrica

$$\Rightarrow \quad g_{\mu 0} T^{\mu i} = g_{00} T^{0i} + g_{j0} T^{ij} = \frac{a^2}{a^2}(1+2\Psi)(q^i + \bar{p} B^i)$$

$$= (1+2\Psi)(q^i + pB^i) = q_i + \cancel{\bar{p} B^i} + (2\Psi \cancel{\bar{p} B^i}) + \cancel{2\Psi q^i}$$

$$\underline{T_0{}^i = q_i}$$

Lo mismo para $\underline{T_j{}^i}$ Consideramos sólo términos de primer nivel

$$\Rightarrow \quad g_{j\mu} T^{i\mu} = g_{i0} T^{i0} + g_{ik} T^{ik}$$

$$= -\frac{1}{a^2}\left((1+2\phi)\delta_{ik} + 2E_{jk}\right] a^2 \left[\bar{p}\delta^{ik} + 2(\bar{p}\phi + \delta p)\delta^{ik} - 2\bar{p}E^{ik} - \Pi^{ik}\right]$$

$$= -\left[\bar{p}\delta_{ik}\delta^{ik} - 2\phi\bar{p}\cancel{\delta_{ik}\delta^{ik}} + 2E_{jk}\cancel{\bar{p}\delta^{ik}} + \cancel{2\bar{p}\phi\delta_{jk}\delta^{ik}} + \delta p\,\delta_{jk}\delta^{ik} - \cancel{2\bar{p}E^{ik}\delta_{ik}} - \Pi^{ik}\delta_{jk}\right.$$

$$k = j$$

$$= \underline{-(\bar{p} + \delta p)}\,\delta_j^i + \Pi_j^i$$

$$\therefore \quad \boxed{T_j{}^i = -(\bar{p} + \delta p)\,\delta_i^j + \Pi_j^i}$$