# Tarea perturbaciones 3

## Cosmología

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$$\tilde\phi = \phi + \mathcal{H}T + \frac{1}{3}\partial_i L^i$$
$$\tilde B_i = B_i + \partial_i T - \dot L_i$$
$$\tilde E_{ij} = E_{ij} - \partial_{<i}L_{j>}$$

- $\delta\tilde g_{i0} = \delta g_{i0} + \left(\frac{\partial x^\alpha}{\partial \tilde x^i}\frac{\partial x^\beta}{\partial\tilde\eta} - \delta^\alpha_i\delta^\beta_0\right)\bar g_{\alpha\beta} - \cancel{T\dot{\bar g}_{i0}} - \cancel{L^i\partial_i \bar g_{i0}}$

$$-a^2\tilde B_i = -a^2 B_i + \left(\frac{\partial\eta}{\partial\tilde x^i}\frac{\partial\eta}{\partial\tilde\eta}\right)\bar g_{00} + \cancel{\left(\frac{\partial\eta}{\partial\tilde x^i}\frac{\partial x^i}{\partial\tilde\eta} + \frac{\partial x^i}{\partial \tilde x^i}\frac{\partial\eta}{\partial\tilde\eta}\right)\bar g_{i0}} + \left(\frac{\partial x^i}{\partial\tilde x^i}\frac{\partial x^j}{\partial\tilde\eta}\right)\bar g_{ij}$$

$$\cancel{a^2}\tilde B_i = \cancel{a^2}B_i - [(-\partial_i T)(1-\dot T)]\cancel{a^2} - [(1-\partial_i L^i)(-\dot L^j)](-\delta_{ij})\cancel{a^2}$$

$$\tilde B_i = B_i + \partial_i T - \dot L^j\delta_{ij} \to \tilde B_i = B_i+\partial_i T - \dot L_i$$

- $\delta\tilde g_{ii} = \delta g_{ii} + \left(\frac{\partial x^\alpha}{\partial\tilde x^i}\frac{\partial x^\beta}{\partial\tilde x^i} - \delta^\alpha_i\delta^\beta_i\right)\bar g_{\alpha\beta} - T\dot{\bar g}_{ii} - \cancel{L^i\partial_i\bar g_{ii}}$

$$a^2(2\tilde\phi) = a^2(2\phi) + \left(\frac{\partial\eta}{\partial\tilde x^i}\frac{\partial\eta}{\partial\tilde x^i}\right)\bar g_{00} + \left(\frac{\partial x^i}{\partial\tilde x^i}\frac{\partial x^j}{\partial\tilde x^i} - \delta^j_i\right)\bar g_{ij} - T\partial_\eta(-a^2)$$

$$\cancel{2a^2}\tilde\phi = \cancel{2a^2}\phi + \cancel{[(-\partial_i T)(-\partial_i T)]}a^2 + [(1-\partial_i L^i)(\delta^j_i - \partial_i L^j) - \delta^j_i](\cancel{-a^2}) + \cancel{2a}\partial_\eta a T$$

$$\tilde\phi = \phi + \frac13\partial_i L^i + T\mathcal{H}$$

$-\partial_i L^i\delta^j_i - \partial_i L^j = \frac13 \partial_i L^i$ ?

- $\delta\tilde g_{ij} = \delta g_{ij} + \left(\frac{\partial x^\alpha}{\partial\tilde x^i}\frac{\partial x^\beta}{\partial\tilde x^j} - \delta^\alpha_i\delta^\beta_j\right)\bar g_{\alpha\beta} - T\dot{\bar g}_{ij} - \cancel{L^i\partial_i\bar g_{ij}}$

$$-a^2[-2\tilde\phi\delta_{ij} + 2\tilde E_{ij}] = -a^2[-2\phi\delta_{ij}+2E_{ij}] + \left(\frac{\partial\eta}{\partial\tilde x^i}\frac{\partial\eta}{\partial\tilde x^j}\right)\bar g_{00}$$
$$+\left(\frac{\partial x^i}{\partial\tilde x^i}\frac{\partial x^j}{\partial\tilde x^j}-1\right)\bar g_{ij} - T\partial_\eta(-a^2\delta_{ij})$$

$$\cancel{-2a^2}[-\tilde\phi\delta_{ij}+\tilde E_{ij}] = \cancel{-2a^2}[-\phi\delta_{ij}+E_{ij}] + \cancel{[(-\partial_i T)(-\partial_j T)]}a^2$$
$$+ [(1-\partial_i L^i)(1-\partial_j L^j)-1](\cancel{-a^2}\delta_{ij}) + \cancel{2a}T\partial_\eta a\,\delta_{ij}$$

$$-\tilde\phi\delta_{ij} + \tilde E_{ij} = -\phi\delta_{ij} + E_{ij} - \partial_i L^i\delta_{ij} - T\mathcal{H}\delta_{ij}$$

$$\tilde E_{ij} = E_{ij} + \cancel{\delta_{ij}(\tilde\phi - \phi - T\mathcal{H} - \tfrac13\partial_i L^i)} - \frac23\partial_i L^i\delta_{ij}$$

$$\tilde E_{ij} = E_{ij} - \partial_{<i}L_{j>}$$

$\frac23\partial_i L^i\delta_{ij} = \partial_{<i}L_{j>}$ ?

$$
\begin{aligned}
\delta\tilde{P} &= \delta P - T\dot{\bar{P}} \\
\tilde{q}^i &= q^i + (\bar{\rho} + \bar{P})\dot{L}^i \\
\tilde{\Pi}^i_j &= \Pi^i_j
\end{aligned}
$$

- $\delta\tilde{T}^i{}_0 = \delta T^i{}_0 + \left(\frac{\partial \tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{\eta}} - \delta^i_\alpha \delta^\beta_0\right)\bar{T}^\alpha_\beta \cancel{- \dot{\bar{T}}^i{}_0 (T)} \cancel{- \partial_i \bar{T}^i{}_0 L^i}$

$$\tilde{q}^i = q^i + \left(\frac{\partial \tilde{x}^i}{\partial \eta}\frac{\partial \eta}{\partial \tilde{\eta}}\right)\bar{T}^0_0 + \cancel{\left(\frac{\partial \tilde{x}^i}{\partial \eta}\frac{\partial x^i}{\partial \tilde{\eta}}\right)}\bar{T}^0{}_i + \cancel{\left(\frac{\partial \tilde{x}^i}{\partial x^i}\frac{\partial \eta}{\partial \tilde{\eta}}\right)}\bar{T}^i{}_0 + \left(\frac{\partial \tilde{x}^i}{\partial x^i}\frac{\partial x^i}{\partial \tilde{\eta}}\right)\bar{T}^i{}_i$$

$$\tilde{q}^i = q^i + [(\dot{L}^i)(1-\dot{T})](\bar{\rho}) + [(1+\partial_i L_i)(-\dot{L}^i)](-\bar{P})$$

$$= q^i + \dot{L}^i\bar{\rho} + \dot{L}^i\bar{P} \longrightarrow \tilde{q}^i = q^i + (\bar{\rho} + \bar{P})\dot{L}^i$$

- $\delta\tilde{T}^i{}_i = \delta T^i{}_i + \left(\frac{\partial \tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^i} - \delta^i_\alpha \delta^\beta_i\right)\bar{T}^\alpha_\beta - \dot{\bar{T}}^i{}_i (T) - \partial_i \bar{T}^i{}_i L^i$

$$-\delta\tilde{P} = -\delta P + \left(\frac{\partial \tilde{x}^i}{\partial \eta}\frac{\partial \eta}{\partial \tilde{x}^i}\right)\bar{T}^0_0 + \left(\frac{\partial \tilde{x}^i}{\partial x^i}\frac{\partial x^i}{\partial \tilde{x}^i} - 1\right)\bar{T}^i{}_i - (T)(-\dot{\bar{P}}) \cancel{- \partial_i(-\bar{P})L^i}$$

$$-\delta\tilde{P} = -\delta P + \cancel{[(L^i)(-\partial_i T)]\bar{\rho}} + \cancel{[(1+\partial_i L_i)(1-\partial_i L^i)-1]}(-\bar{P}) + T\dot{\bar{P}}$$

$$\delta\tilde{P} = \delta P - T\dot{\bar{P}}$$

- $\delta\tilde{T}^i{}_j = \delta T^i{}_j + \left(\frac{\partial \tilde{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^j} - \delta^i_\alpha \delta^\beta_j\right)\bar{T}^\alpha_\beta - \dot{\bar{T}}^i{}_j (T) \cancel{- \partial_i \bar{T}^i{}_j L^i}$

$$-\delta\tilde{P}\delta^i{}_j + \tilde{\Pi}^i{}_j = -\delta P\delta^i{}_j + \Pi^i{}_j + \left(\frac{\partial \tilde{x}^i}{\partial \eta}\frac{\partial \eta}{\partial \tilde{x}^j}\right)\bar{T}^0_0 + \left(\frac{\partial \tilde{x}^i}{\partial x^i}\frac{\partial x^i}{\partial \tilde{x}^j} - 1\right)\bar{T}^i{}_i + \dot{\bar{P}}T\delta^i_j$$

$$-\delta\tilde{P}\delta^i{}_j + \tilde{\Pi}^i{}_j = -\delta P\delta^i{}_j + \dot{\bar{P}}T\delta^i{}_j + \Pi^i{}_j + \cancel{[(L^i)(-\partial_j T)]}\bar{\rho} + \cancel{[(1+\partial_i L_i)(1-\partial_j L^i)-1]}$$

$$\tilde{\Pi}^i{}_j = \cancel{(\delta\tilde{P} - \delta P + T\dot{\bar{P}})}\delta^i{}_j + \Pi^i{}_j \longrightarrow \tilde{\Pi}^i{}_j = \Pi^i{}_j$$

**Tarea**: show that $\Phi$, $\Psi$, $\Phi_i$ don't change under a coordinate transformation

$\tilde{\psi} = \psi - \dot{T} - \mathcal{H}T$

$\tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L$

- $\tilde{B} = B + T - \dot{L}$
- $\tilde{E} = E - L$

- $\Psi = \psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E}$
- $\Phi = \phi - \mathcal{H}(B - \dot{E}) + \frac{1}{3}\nabla^2 E$
- $\Phi_i = \dot{E}_i - B_i$

- $\tilde{\Psi} = \tilde{\psi} + \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \dot{\tilde{B}} - \ddot{\tilde{E}}$

$$= \psi - \dot{T} - \mathcal{H}T + \mathcal{H}(B + T - \cancel{\dot{L}} - \dot{E} + \cancel{\dot{L}}) + \dot{B} + \dot{T} - \cancel{\ddot{L}} - \ddot{E} + \cancel{\ddot{L}}$$

$$= \psi - \cancel{\dot{T}} - \cancel{\mathcal{H}T} + \mathcal{H}B + \cancel{\mathcal{H}T} - \mathcal{H}\dot{E} + \dot{B} + \cancel{\dot{T}} - \ddot{E}$$

$$= \psi + \mathcal{H}B + \dot{B} - \mathcal{H}\dot{E} - \ddot{E} = \psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E} = \Psi$$

- $\tilde{\Phi} = \tilde{\phi} - \mathcal{H}(\tilde{B} - \dot{\tilde{E}}) + \frac{1}{3}\nabla^2 \tilde{E}$

$$= \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L - \mathcal{H}(B + T - \cancel{\dot{L}} - \dot{E} + \cancel{\dot{L}}) + \frac{1}{3}\nabla^2(E - L)$$

$$= \phi + \cancel{\mathcal{H}T} + \cancel{\frac{1}{3}\nabla^2 L} - \mathcal{H}(B - \dot{E}) - \cancel{\mathcal{H}T} + \frac{1}{3}\nabla^2 E - \cancel{\frac{1}{3}\nabla^2 L}$$

$$= \phi - \mathcal{H}(B - \dot{E}) + \frac{1}{3}\nabla^2 E = \Phi$$

- $\tilde{\Phi}_i = \dot{\tilde{E}}_i - \tilde{B}_i = \dot{E}_i - \cancel{\dot{L}_i} - B_i + \cancel{\dot{L}_i} = \dot{E}_i - B_i = \Phi_i$

## Exercise $\Delta$ is gauge-invariant

$$\bar\rho\Delta \equiv \delta\rho + \dot{\bar\rho}(v+B) = \delta\rho - 3\mathcal{H}(\bar\rho+\bar P)(B+v)$$

- $$\bar\rho\tilde\Delta = \delta\tilde\rho - 3\mathcal{H}(\tilde{\bar\rho}+\tilde{\bar P})(\tilde B+\tilde v)$$

$$= \delta\tilde\rho - 3\mathcal{H}(\bar\rho+\bar P)\tilde B - 3\mathcal{H}\cancel{(\bar\rho+\bar P)(\bar\rho+\bar P)^{-1}}\tilde q$$

$$= \delta\rho - \underline{T\dot{\bar\rho}} - 3\mathcal{H}(\bar\rho+\bar P)(B+\underline{T}-\dot L) - 3\mathcal{H}[q+(\bar\rho+\bar P)\dot L]$$

$$= \delta\rho - T\overset{\text{Continuity eq}}{\cancel{[\dot{\bar\rho}+3\mathcal{H}(\bar\rho+\bar P)]}}T - 3\mathcal{H}(\bar\rho+\bar P)[B-\cancel{\dot L}+\underbrace{(\bar\rho+\bar P)^{-1}q}_{v}+\cancel{\dot L}]$$

$$= \delta\rho - 3\mathcal{H}(\bar\rho+\bar P)(B+v) = \bar\rho\Delta$$