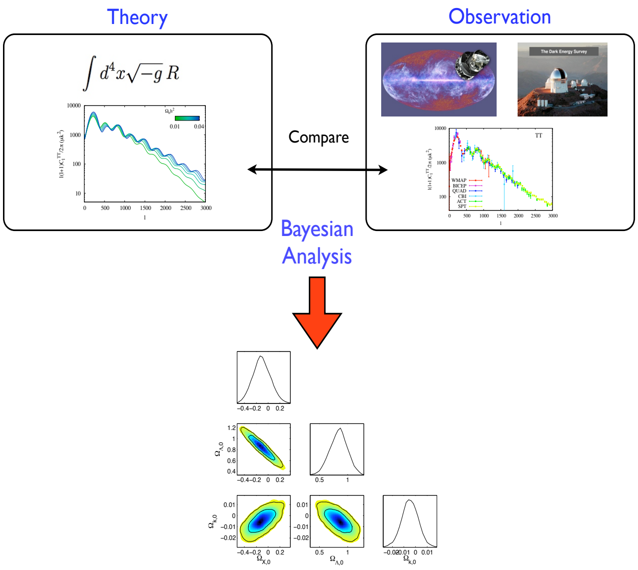
Numerical Methods



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1

Review of Calculus

Definition: A function f defined on a set X of real numbers has the **limit** L at x_0, written

$$\lim_{x \to x_0} f(x) = L, \tag{1.1}$$

if, given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$, whenever $x \in X$ and $0 < |x - x_0| < \delta$.

Definition: Let f be a function defined on a set X of real numbers and $x_0 \in X$. Then f is **continuos** at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0), \tag{1.2}$$

The function is continuous on the set X if it is continuous at each number in X.

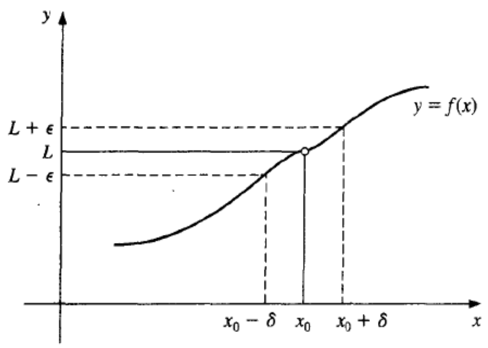
$C(X)$ denotes the set of all functions that are continuous in X.

Definition: Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real or complex numbers. The sequence $\{x_n\}_{n=1}^{\infty}$ has the **limit x (converges to x)** if, for any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $|x_n - x| < \epsilon$, whenever $n > N(\epsilon)$. The notation

$$\lim_{n \to \infty} x_n = x, \quad \text{or} \quad x_n \to x \quad \text{as} \quad n \to \infty, \tag{1.3}$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Theorem: If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

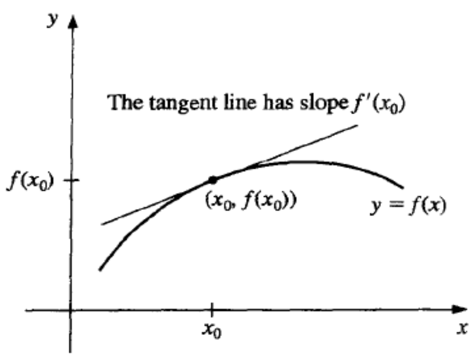


- f is continuous at x_0
- If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X converging to x_0, then $lim_{n\to\infty} f(x_n) = f(x_0)$.

Definition: Let f be a function defined in an open interval containing x_0. The function f is **differentiable** at x_0 if

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1.4}$$

exists. The number $f'(x_0)$ is called the **derivative** of f at x_0. A function that has a derivative at each number in a set X is **differentiable** on X. The derivative of f at x_0 is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$.

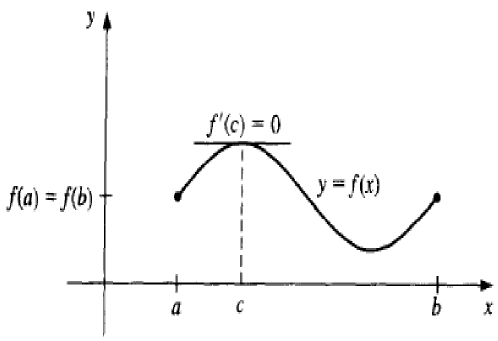


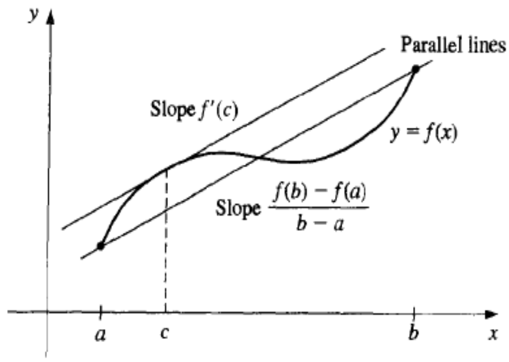
Theorem: If the function f is differentiable at x_0, then f is continuous at x_0.
The set of all function that have n continuous derivatives on X is denoted $C^n(X)$.

The next theorems are of fundamental importance in deriving methods for error estimation.

Rolle's Theorem

Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then a number c in (a,b) exists with $f'(x) = 0$.





Mean Value Theorem

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1.5}$$

Extreme Value Theorem

If $f \in C[a,b]$, then $c_1, c_2 \in [a,b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$, for all $x \in [a,b]$. In addition, if f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the endpoints of $[a,b]$ of where f' is zero.

Definition: The **Riemann integral** of the function f on the interval $[a,b]$ is the following limit, provided it exists:

$$\int_a^b f(x)dx = \lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} f(z_i)\Delta x_i \tag{1.6}$$

where the numbers $x_0, x_1, ..., x_n$ satisfy $a = x_0 \leq x_1 \leq ... \leq x_n = b$, and where $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, 2, ..., n$ and z_i is arbitrarily chosen in the interval $[x_{i-1}, x_i]$.

Every continuous function f on $[a,b]$ is Riemann integrable on $[a,b]$. This permits us to choose, for computational convenience, the points x_i to be equally spaced in $[a,b]$, and for each

$i = 1, 2, ..., n$, to choose $z_i = x_i$. In this case,

$$\int_a^b f(x)dx = \lim_{n \to 0} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \tag{1.7}$$

where the numbers are $x_i = a + i(b-a)/n$.

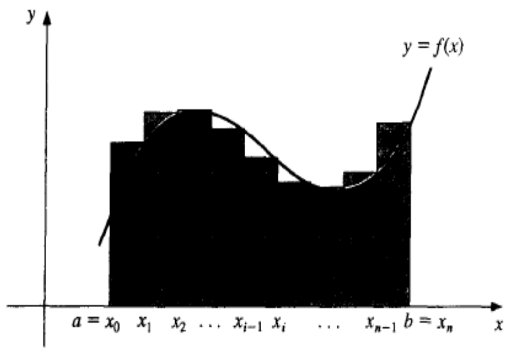
Weighted Mean Value Theorem for Integrals

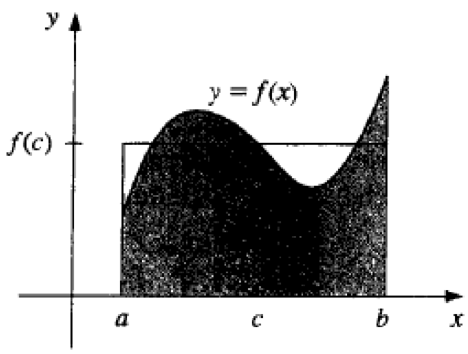
Suppose $f \in C[a,b]$, the Riemann integral of g exists on $[a,b]$, and $g(x)$ does not change sign on $[a,b]$. Then there exists a number c in (a,b) with

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx. \tag{1.8}$$

When $g(x) = 1$, is the usual Mean Value Theorem for Integrals. It gives the **average value** of the function f over the interval $[a,b]$ as

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx. \tag{1.9}$$





Intermediate Value Theorem

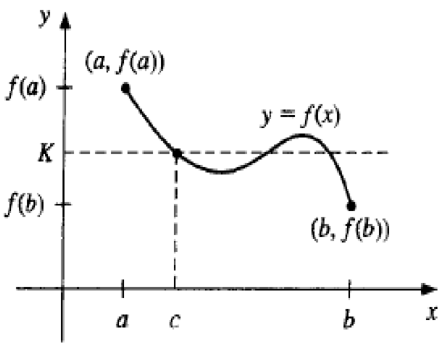
If $f \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number in (a,b) for which $f(c)$= K.

Example: $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval $[0,1]$?

Consider

$$f(0) = -1 < 0 < 1 = f(1) \tag{1.10}$$

and f is continuous, then the Intermediate Value Theorem implies that a number x exists with $0 < x < 1$, for which $x^5 - 2x^3 + 3x^2 - 1 = 0$.



Taylor's Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x), \tag{1.11}$$

where

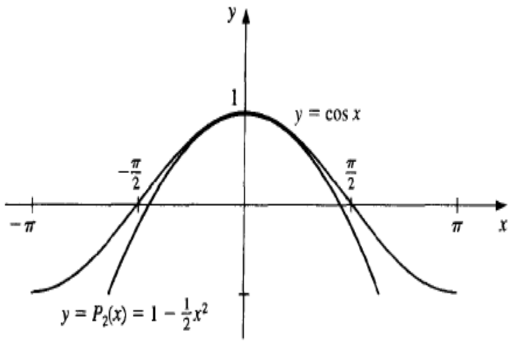
$$P_n(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \tag{1.12}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} \tag{1.13}$$

Here $P_n(x)$ is called the **nth Taylor polynomial** for f about x_0, and $R_n(x)$ is called the **remainder term** (or **truncation error**) associated with P_nx. In the case $x_0 = 0$, the Taylor polynomial is often called a **Maclaurin polynomial**.

The term **truncation error** refers to the error involved in using a truncated, or finite, summation to approximate the sum of an infinite series.


y
1
y = cos x
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$-\pi$
π
x
$y = P_2(x) = 1 - \frac{1}{2}x^2$