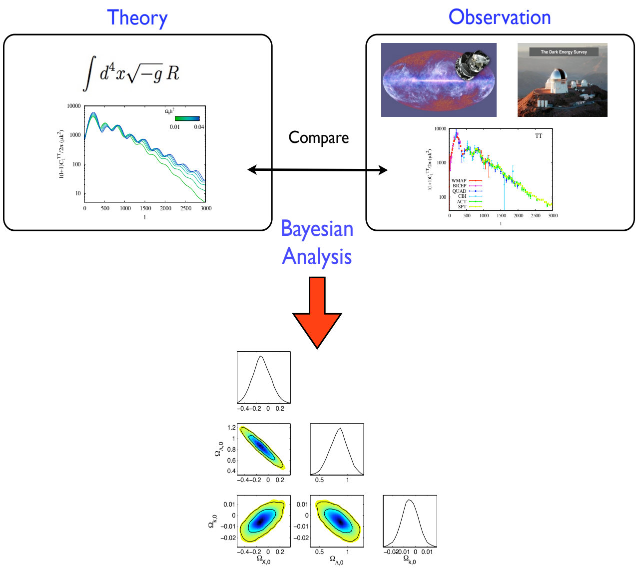
Updated Cosmology

with Python



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Homework

Dust only Friedmann model ($\Omega_{\mathrm{r},0} = 0$, $\Omega_{k,0} = 1 - \Omega_{\mathrm{m},0}$)

1. Compute the integral

$$t = \frac{1}{H_0} \int_0^a \left[\frac{x}{\Omega_{\mathrm{m},0} + (1 - \Omega_{\mathrm{m},0})x} \right]^{1/2} dx. \tag{7}$$

and show that:

- Flat Universe ($k = 0$) $\Omega_{\mathrm{m},0} = 1$: This type of Universe is called the **Einstein de-Sitter model**, and we have seen the behaviour before:

$$a(t) = \left(\frac{3}{2} H_0 t \right)^{2/3}. \tag{8}$$

- For spherical Universe $\Omega_{\mathrm{m},0} > 1$ ($k = 1$), we write

$$x = \left[\frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0} - 1} \sin^2 \psi/2 \right], \qquad \psi = [0, \pi], \tag{9}$$

  and have

$$a(t) = \frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0} - 1)} (1 - \cos\psi), \qquad t = \frac{\Omega_{\mathrm{m},0}}{2H_0(\Omega_{\mathrm{m},0} - 1)^{3/2}} (\psi - \sin\psi), \tag{10}$$

  where the first term represents the expression for a cycloid, see Figure **??**.

- For hyperbolic Universe $\Omega_{\mathrm{m},0} < 1$ ($k = -1$), we write

$$x = \left[\frac{\Omega_{\mathrm{m},0}}{1 - \Omega_{\mathrm{m},0}} \sinh^2 \psi/2 \right], \qquad \psi = [0, \pi]. \tag{11}$$

  and have

$$a(t) = \frac{\Omega_{\mathrm{m},0}}{2(1 - \Omega_{\mathrm{m},0})} (\cosh\psi - 1), \qquad t = \frac{\Omega_{\mathrm{m},0}}{2H_0(1 - \Omega_{\mathrm{m},0})^{3/2}} (\sinh\psi - \psi). \tag{12}$$

Radiation domination ($\Omega_{\mathrm{m},0} = 0$, $\Omega_{k,0} = 1 - \Omega_{\mathrm{r},0}$)

2. Compute the integral

$$t = \frac{1}{H_0} \int_0^a \left[\frac{x}{\sqrt{\Omega_{\mathrm{r},0} + (1 - \Omega_{\mathrm{r},0})x^2}} \right] dx. \tag{13}$$

and show that:

- Flat Universe $\Omega_{\mathrm{r},0} = 1$ ($k = 0$):

$$a(t) = (2H_0 t)^{1/2}. \tag{14}$$

- Spherical $\Omega_{\mathrm{r},0} < 1$ ($k = -1$) or Hyperbolic $\Omega_{\mathrm{r},0} > 1$ ($k = 1$):

$$a(t) = (2H_0 \Omega_{\mathrm{r},0}^{1/2} t)^{1/2} \left(1 + \frac{1 - \Omega_{\mathrm{r},0}}{2\Omega_{\mathrm{r},0}^{1/2}} H_0 t \right)^{1/2}. \tag{15}$$

Spatially flat ($\Omega_{k,0} = 0$, $\Omega_{\mathrm{m},0} + \Omega_{\mathrm{r},0} = 1$)

3. Compute the integral

$$t = \frac{1}{H_0} \int_0^a \left[\frac{x}{\sqrt{\Omega_{\mathrm{m},0} x + \Omega_{\mathrm{r},0}}} \right] dx, \tag{16}$$

doing $y = \Omega_{\mathrm{m},0} x + \Omega_{\mathrm{r},0}$, to get

$$H_0 t = \frac{2}{3\Omega_{\mathrm{m},0}^2} \left[(\Omega_{\mathrm{m},0} a + \Omega_{\mathrm{r},0})^{1/2} (\Omega_{\mathrm{m},0} a - 2\Omega_{\mathrm{r},0}) + 2\Omega_{\mathrm{r},0}^{3/2} \right]. \tag{17}$$

Cannot be easily inverted to give $a(t)$. Nevertheless $t = \frac{2}{3} a^{3/2}$ for matter only, and $t = \frac{1}{2} a^2$ for radiation, as expected.

Lemaitre models ($\Omega_{\Lambda,0} \neq 0$) but $\Omega_{\mathrm{r},0} = 0$

4. Compute the integral

- Spatially flat ($\Omega_{\mathrm{m},0} + \Omega_{\Lambda,0} = 1$):

$$t = \frac{1}{H_0} \int_0^a \sqrt{\frac{x}{(1 - \Omega_{\Lambda,0}) + \Omega_{\Lambda,0} x^3}} dx, \tag{18}$$

writing $y^2 = x^3 |\Omega_{\Lambda,0}| / (1 - \Omega_{\Lambda,0})$, we have then

$$H_0 t = \frac{2}{3\sqrt{|\Omega_{\Lambda,0}|}} \int_0^{\sqrt{a^3 |\Omega_{\Lambda,0}|/(1-\Omega_{\Lambda,0})}} \frac{dy}{\sqrt{1 \pm y^2}}, \tag{19}$$

to show the solutions

$$H_0 t = \frac{2}{3\sqrt{|\Omega_{\Lambda,0}|}} f(x) = \begin{cases} \sinh^{-1}[\sqrt{a^3 |\Omega_{\Lambda,0}|(1-\Omega_{\Lambda,0})}], & \Omega_{\Lambda,0} > 0. \\ . \\ \sin^{-1}[\sqrt{a^3 |\Omega_{\Lambda,0}|(1-\Omega_{\Lambda,0})}], & \Omega_{\Lambda,0} < 0. \end{cases} \tag{20}$$