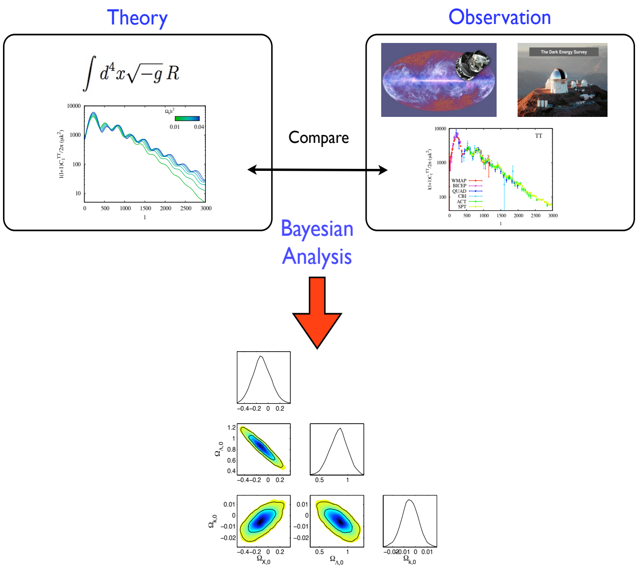
Updated Cosmology

with Python



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Homework

The Two-sphere

Let us consider the two-dimensional geometry of the surface of a sphere in terms of its radius ($\rho \ \epsilon \ [0, a]$) and zenith angle ($\phi \ \epsilon \ [0, 2\pi]$), assuming it is embedded in a three-dimensional Euclidean space (see Figure 1).

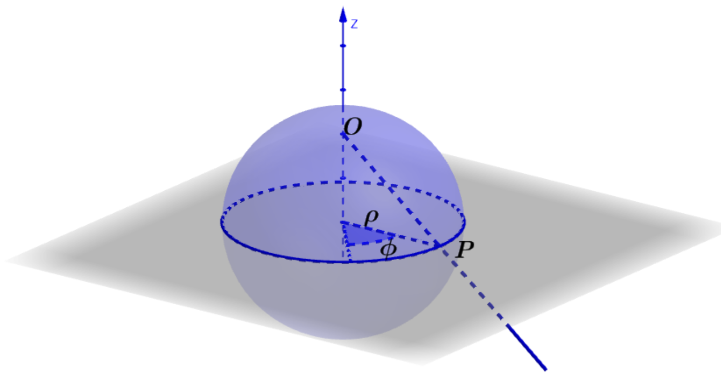

Figure 1: 2D - Sphere.

with some algebra we obtain

$$ds^2 = \frac{a^2}{a^2 - \rho^2} d\rho^2 + \rho^2 d\phi^2. \tag{1}$$

Note that the line element contains a '*hidden symmetry*', namely our freedom to choose an arbitrary point on the sphere as the origin $\rho = 0$. There is also a *coordinate singularity*, which has resulted simply from choosing coordinates with a restricted domain of validity. From Eqn. (1) we see that this coordinate system is orthogonal, with $g_{\rho\rho} = a^2/(a^2 - \rho^2)$ and $g_{\phi\phi} = \rho^2$.

a) Compute the distance *in the surface* D from the centre to a particular radius $\rho = R$, at constant ϕ, is

$$D = \int_0^R \frac{a}{(a^2 - \rho^2)^{1/2}} d\rho = a \sin^{-1}\left(\frac{R}{a}\right). \tag{2}$$

b) Compute the circumference of the circle, given by

$$C = \int_0^{2\pi} R d\phi = 2\pi R. \tag{3}$$

c) Similarly, compute the area within the circumference C

$$A = \int_0^{2\pi} \int_0^R \frac{a}{(a^2 - \rho^2)^{1/2}} \rho d\rho d\phi = 2\pi a^2 \left[1 - \left(1 - \frac{R^2}{a^2}\right)^{1/2}\right]. \tag{4}$$

d) For completeness, show the total area, by symmetry, is given by

$$A_{\text{tot}} = 2 \int_0^{2\pi} \int_0^a \frac{a}{(a^2 - r^2)^{1/2}} r dr d\phi = 4\pi a^2. \tag{5}$$

e)

$$ds^2 = \frac{a^2}{a^2 - r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{6}$$

Compute the Volume of the 3D sphere. Hint:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{ar^2 \sin\theta}{(a^2 - r^2)^{1/2}} dr d\theta d\phi,$$

[the total volume V_{tot} is found when $R = a$].