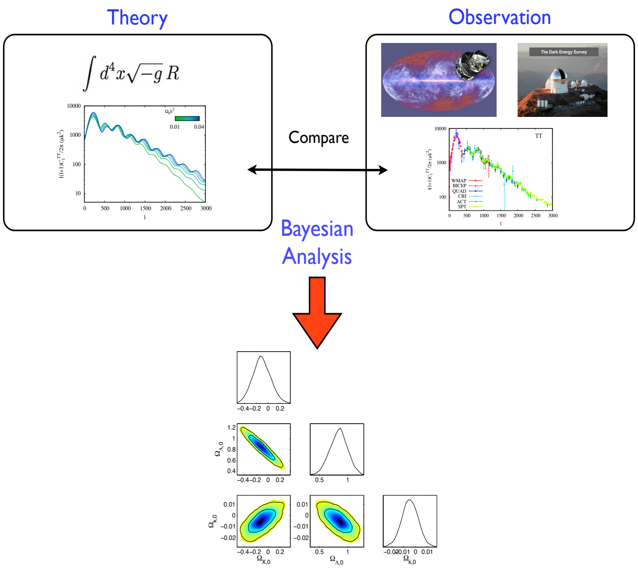
Updated Cosmology

with Python



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In progress

August 12, 2017

0.1 World Models

https://www.youtube.com/watch?v=Y-dMtbHQVI0

Let us take the equation for the total density $\Omega_T + \Omega_k = 1$ and the equation for acceleration $q = \frac{1}{2}\sum_i \Omega_i(1+3w_i)$.

- *open-closed line* ($k = 0$)

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0} \tag{1}$$

- *accelerating-decelerating line* ($q = 0$)

$$\Omega_{\Lambda,0} = \frac{1}{2}\Omega_{m,0} \tag{2}$$

- *expand-forever-recollapse* & *big bang- no big bang* requires a little more work

In general

$$\dot{a}^2 = a^2 H_0^2(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}) \tag{3}$$

with the condition that $1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda$ for all time, in particular for $a = 1$. FRW Universes dominated by matter and vacuum energy are named as *Lemaitre models*. In conclusion from the plot we have that (considering present data): we live in a nearly flat accelerating universe that presents big bang in the past and will expand forever in the future,

Cosmological models with a zero cosmological constant ($\Omega_{\Lambda,0} = 0$), and strictly non-zero matter or radiation density, are known as the *Friedmann models*.

Dust only Friedmann models ($\Omega_{r,0} = 0$, $\Omega_{k,0} = 1 - \Omega_{m,0}$)

$$\dot{a}^2 = H_0^2(\Omega_{m,0}a^{-1} + 1 - \Omega_{m,0}) \qquad \rightarrow \qquad t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\Omega_{m,0} + (1-\Omega_{m,0})x}\right]^{1/2} dx. \tag{4}$$

- $\Omega_{m,0} = 1$ ($k = 0$) - Einstein de-Sitter model

$$a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3}. \tag{5}$$

- $\Omega_{m,0} > 1$ ($k = 1$), we write

$$x = \left[\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\sin^2\psi/2\right], \qquad \psi = [0, \pi], \tag{6}$$

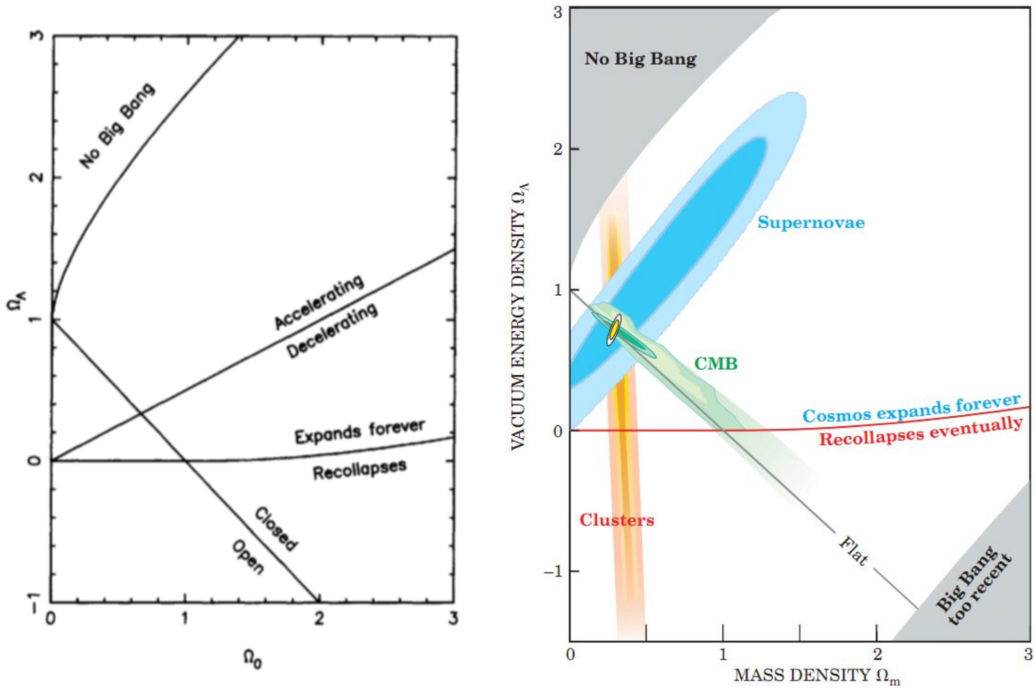

Figure 1: (jav: I'll do it later)

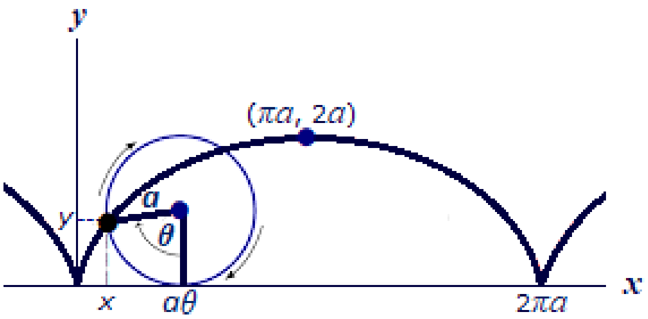

Figure 2: (jav: Figure of a cycloid)

and we have

$$a(t) = \frac{\Omega_{m,0}}{2(\Omega_{m,0}-1)}(1-\cos\psi), \qquad t = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0}-1)^{3/2}}(\psi - \sin\psi) \tag{7}$$

where the first term represents the expression for a cycloid.

- $\Omega_{m,0} < 1$ $(k = -1)$, we write

$$x = \left[\frac{\Omega_{m,0}}{1-\Omega_{m,0}} \sinh^2 \psi/2\right], \qquad \psi = [0, \pi]. \tag{8}$$

and we have

$$a(t) = \frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}(\cosh\psi - 1), \qquad t = \frac{\Omega_{m,0}}{2H_0(1-\Omega_{m,0})^{3/2}}(\sinh\psi - \psi), \tag{9}$$

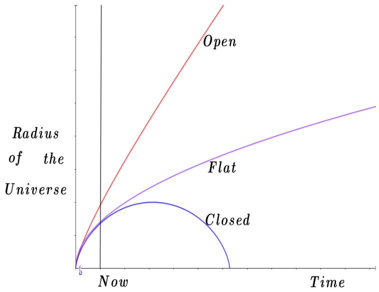

Figure 3: (jav: Figure of universes)

Radiation only $(\Omega_{m,0} = 0,\ \Omega_{k,0} = 1 - \Omega_{r,0})$

$$\dot{a}^2 = H_0^2(\Omega_{r,0}a^{-2} + 1 - \Omega_{r,0}) \qquad \rightarrow \qquad t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\sqrt{\Omega_{r,0} + (1-\Omega_{r,0})x^2}}\right] dx. \tag{10}$$

- $\Omega_{r,0} = 1$ $(k = 0)$

$$a(t) = (2H_0 t)^{1/2}. \tag{11}$$

- $\Omega_{r,0} < 1$ $(k = -1)$ & $\Omega_{r,0} > 1$ $(k = 1)$

$$a(t) = (2H_0\Omega_{r,0}^{1/2}t)^{1/2}\left(1 + \frac{1-\Omega_{r,0}}{2\Omega_{r,0}^{1/2}}H_0 t\right)^{1/2}. \tag{12}$$

Spatially flat $(\Omega_{k,0} = 0,\ \Omega_{m,0} + \Omega_{r,0} = 1)$

$$\dot{a}^2 = H_0^2(\Omega_{m,0}a^{-1} + \Omega_{r,0}a^{-2}) \qquad \rightarrow \qquad t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\sqrt{\Omega_{m,0}x + \Omega_{r,0}}}\right] dx, \tag{13}$$

haciendo $y = \Omega_{m,0}x + \Omega_{r,0}$

$$H_0 t = \frac{2}{3\Omega_{m,0}^2}\left[(\Omega_{m,0}a + \Omega_{r,0})^{1/2}(\Omega_{m,0}a - 2\Omega_{r,0}) + 2\Omega_{r,0}^{3/2}\right]. \tag{14}$$

Cannot be easily inverted to give a(t). Nevertheless $t = \frac{2}{3}a^{3/2}$ for matter only, and $t = \frac{1}{2}a^2$ for radiation as expected.

Lemaitre models ($\Omega_{\Lambda,0} \neq 0$) but $\Omega_{r,0} = 0$

- Spatially flat ($\Omega_{m,0} + \Omega_{\Lambda,0} = 1$)

$$\dot{a}^2 = H_0^2[(1-\Omega_{\Lambda,0})a^{-1} + \Omega_{\Lambda,0}a^2] \qquad \to t = \frac{1}{H_0}\int_0^a \sqrt{\frac{x}{(1-\Omega_{\Lambda,0}) + \Omega_{\Lambda,0}x^3}}dx, \tag{15}$$

writing $y^2 = x^3|\Omega_{\Lambda,0}|/(1-\Omega_{\Lambda,0})$, we have then

$$H_0 t = \frac{2}{3\sqrt{|\Omega_{\Lambda,0}|}}\int_0^{\sqrt{a^3|\Omega_{\Lambda,0}|/(1-\Omega_{\Lambda,0})}} \frac{dy}{\sqrt{1 \pm y^2}} \tag{16}$$

with solutions

$$H_0 t = \frac{2}{3\sqrt{|\Omega_{\Lambda,0}|}} f(x) = \begin{cases} \sinh^{-1}[\sqrt{a^3|\Omega_{\Lambda,0}|(1-\Omega_{\Lambda,0})}], & \Omega_{\Lambda,0} > 0. \\ . \\ \sin^{-1}[\sqrt{a^3|\Omega_{\Lambda,0}|(1-\Omega_{\Lambda,0})}], & \Omega_{\Lambda,0} < 0. \end{cases} \tag{17}$$

- Arbitrary spatial curvature ($\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{\Lambda,0}$)

$$\dot{a}^2 = H_0^2(\Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + \Omega_{k,k}). \tag{18}$$

Quite complicated to solve using elliptical functions to get

$$a(t) = \left(\frac{3}{2}H_0\sqrt{\Omega_{m,0}t}\right)^{3/2} \qquad \text{small t, radiation domination.} \tag{19}$$

$$a(t) \propto \exp(H_0\sqrt{\Omega_{\Lambda,0}}t) \qquad \text{large t, } \Lambda \text{ domination.} \tag{20}$$

De-Sitter model ($\Omega_{m,0} = 0,\ \Omega_{r,0} = 0,\ \Omega_{\Lambda,0} = 1 \quad \to \quad k = 0$)

Not a true model but interesting to study, specially during inflation, and as we shall see in Dark Energy domination

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2, \tag{21}$$

wit solutions of the form

$$a(t) = \exp[H_0(t-t_0)] = \exp[\sqrt{\Lambda/3}c(t-t_0)]. \tag{22}$$

Anti de-Sitter space (negative cosmological constant?)

Einstein static Universe

Before the discovery of the expansion, Einstein introduced the cosmological constant Λ to get $\dot{a} = \ddot{a} = 0$ which has the following implications

$$4\pi G\rho_{m,0} = \Lambda c^2 = \frac{c^2 k}{a^2}, \tag{23}$$

from the first equality (from the acceleration equation) we have that $\rho_{m,0} = 2\rho_{\Lambda,0}$ and $\Lambda > 0$, and from the second (from the Friedmann equation) that $k = 1$. However this type of universe is an unstable one.

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