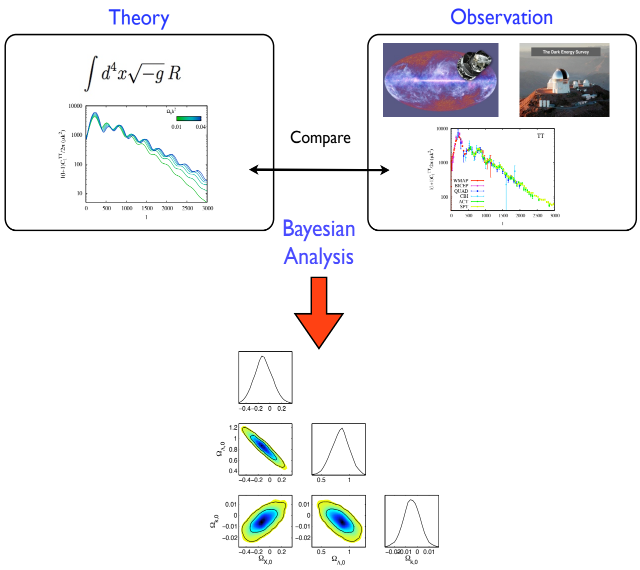
Updated Cosmology

with Python



José-Alberto Vázquez

ICF-UNAM / Kavli-Cambridge

In progress

August 12, 2017

1) The most general spherically symmetric metric can be written as

$$ds^2 = -e^{2F(r,t)}dt^2 + e^{2H(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta dphi^2). \tag{13}$$

This metric is very important as it underlies the theory of both homogeneous cosmological models and of spherically symmetric fluid models for massive stars and black holes. The functions $F(r,t)$ and $H(r,t)$ are determined by the material content of the space-time as described by the energy–momentum tensor and by the boundary conditions defining the problem.
Compute the components of the Einstein tensor.

$$\begin{aligned}
G_{00} &= e^{-2H}\left(\frac{2}{r}H' - \frac{1}{r^2}\right) + \frac{1}{r^2},\\
G_{11} &= e^{-2H}\left(\frac{2}{r}F' + \frac{1}{r^2}\right) - \frac{1}{r^2},\\
G_{22} = G_{33} &= e^{-2H}\left(F'' + F'^2 - H'F' + \frac{1}{r}(F' - H')\right)\\
&\quad - e^{-2F}\left(\ddot{H} + \dot{H}^2 - \dot{H}\dot{F}\right),\\
G_{01} = G_{10} &= \frac{2}{r}\dot{H}\, e^{-(F+H)}.
\end{aligned}$$

2) The Bianchi models are a large family of homogeneous but anisotropic cosmological models. We consider the homogenous and anisotropic space-time described by Bianchi type-III metric in the form

$$ds^2 = dt^2 - A(t)^2dx^2 - B(t)^2e^{-2\alpha x}dy^2 - C(t)^2dz^2, \tag{14}$$

where $A(t), B(t) and C(t)$ are the scale factors (metric tensors) and functions of the cosmic time t, and $\alpha = 0$ is a constant. (Bianchi type-I metric can be recovered by choosing $\alpha = 0$). Here, we assume an anisotropic fluid whose energy-momentum tensor is in diagonal form:

$${T_\nu}^\mu = \mathrm{diag}[1, -w_x, -w_y, -w_z] = \mathrm{diag}[1, -w, -(w+\gamma), -(w+\delta)]\rho, \tag{15}$$

where ρ is the energy density of the fluid, w_x, w_y and w_z are the directional EoS parameters on the x, y and z axes respectively; w is the deviation-free EoS parameter of the fluid. δ and γ are not necessarily constants and can be functions of the cosmic time t.
a) Compute the components of the Einstein's field equations

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{C}}{C}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}-\frac{\alpha^2}{A^2}=\rho, \tag{5}$$

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}=-w\rho, \tag{6}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\frac{\dot{C}}{C}=-(w+\delta)\rho, \tag{7}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{\alpha^2}{A^2}=-(w+\gamma)\rho, \tag{8}$$

$$\alpha\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0 \tag{9}$$

where the over dot denotes derivation with respect to the cosmic time t.

b) Show the solution of Eqn. (4) gives $B = c_1 A$, where c_1 is the positive constant of integration.

c) Substitute this solution into (7), and subtract the result from (6), to show that the the skewness parameter on the y axis is null, i.e. $\delta = 0$, which means that the directional EoS parameters, hence the pressures, on the x and y axes are equal.

The directional Hubble parameters in the directions of x, y and z for the Bianchi type-III metric may be defined as follows,

$$H_x \equiv \frac{\dot{A}}{A}, \quad H_y \equiv \frac{\dot{B}}{b}, \quad H_z \equiv \frac{\dot{C}}{C}. \tag{16}$$

and the mean Hubble parameter is given as

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right) \tag{17}$$

By solving the set of field equations, the difference between the expansion rates on x and z axes could be found $H_x - H_z$, [see Gen Relativ Gravit (2010) 42:763–775].

3) A step further to the standard model is to consider the dark energy being dynamic, where the evolution of its EoS is usually parameterised. A commonly used form of $w(z)$ is to take into account the next contribution of a Taylor expansion in terms of the scale factor $w(a) = w_0 + (1-a)w_a$ or in terms of redshift $w(z) = w_0 + \frac{z}{1+z}w_a$; we refer to this model as CPL. The parameters w_0 and w_a are real numbers such that at the present epoch $w|_{z=0} = w_0$ and $dw/dz|_{z=0} = -w_a$; we recover ΛCDM when $w_0 = -1$ and $w_a = 0$.

a) Show the Friedmann equation for the CPL parameterisation turns out to be:

$$\frac{H(z)^2}{H_0^2} = \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + (1 - \Omega_{m,0} - \Omega_{k,0})(1+z)^{3(1+w_0+w_a)} e^{-\frac{3w_a z}{1+z}}.$$

By using the initial conditions from the previous homework, plot the Comoving distance d_c, luminosity distance d_L, and angular distance d_A for $[w_0 = 0.9, w_a = 0.5]$ and $[w_0 = -1.1, w_a = -0.5]$.

b) Repeat the same process in a), but now use the equation of state $w(z) = w_0 + w_a ln(1+z)$

4) As part of some models that allow deviations from ΛCDM we also use the polynomial-CDM model, that can be thought as a parameterisation of the Hubble function. This model has the following Friedmann equation:

$$\frac{H(z)^2}{H_0^2} = \Omega_{m,0}(1+z)^3 + (\Omega_{1,0} + \Omega_{k,0})(1+z)^2 + \Omega_{2,0}(1+z)^1 + (1 - \Omega_{m,0} - \Omega_{1,0} - \Omega_{2,0} - \Omega_{k,0}),$$

where $\Omega_{1,0}$ and $\Omega_{2,0}$ are two additional parameters, which within the ΛCDM both of them remain absent ($\Omega_{1,0} = 0$ and $\Omega_{2,0} = 0$). Nevertheless, $\Omega_{2,0}$ could be interpreted as a 'missing matter' component introduced to allow a symmetry that relates the big bang to the future conformal singularity [see JCAP09(2012)020].

By using the initial conditions from the previous homework, plot the Comoving distance d_c, luminosity distance d_L, and angular distance d_A for $[\Omega_{1,0} = 0.2, \Omega_{2,0} = -0.2]$ and $[\Omega_{1,0} = -0.2, \Omega_{2,0} = 0.2]$.

5) The action of a massive scalar field, with potential $V(\phi)$, is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]. \tag{18}$$

a) Show that the corresponding field equation for ϕ, obtained from the Euler-Lagrange equations, reads as

$$\Box^2 \phi + \frac{dV}{d\phi} = 0. \tag{19}$$

where the $d'Alembertian$ is : $\Box^2 = \partial_a \partial^a$.