1.- Compute the following integral

$$t = \frac{1}{H_0} \int_0^a \left[\frac{x}{\sqrt{\Omega_{r,0} + (1 - \Omega_{r,0}) x^2}} \right] dx, \tag{1}$$

for $\Omega_{r,0} < 1$ ($k = -1$) & $\Omega_{r,0} > 1$ ($k = 1$), to get

$$a(t) = (2H_0 \Omega_{r,0}^{1/2} t)^{1/2} \left(1 + \frac{1 - \Omega_{r,0}}{2\Omega_{r,0}^{1/2}} H_0 t \right)^{1/2}. \tag{2}$$

2.- Compute 3

$$t = \frac{1}{H_0} \int_0^a \left[\frac{x}{\sqrt{\Omega_{m,0} x + \Omega_{r,0}}} \right] dx, \tag{3}$$

to get

$$H_0 t = \frac{2}{3\Omega_{m,0}^2} \left[(\Omega_{m,0} a + \Omega_{r,0})^{1/2} (\Omega_{m,0} a - 2\Omega_{r,0}) + 2\Omega_{r,0}^{3/2} \right]. \tag{4}$$

3.- Compute

$$t = \frac{1}{H_0} \int_0^a \sqrt{\frac{x}{(1 - \Omega_{\Lambda,0}) + \Omega_{\Lambda,0} x^3}} dx, \tag{5}$$

to get

$$H_0 t = \frac{2}{3\sqrt{|\Omega_{\Lambda,0}|}} f(x) = \begin{cases} \sinh^{-1}[\sqrt{a^3 |\Omega_{\Lambda,0}| (1 - \Omega_{\Lambda,0})}], & \Omega_{\Lambda,0} > 0. \\ . \\ \sin^{-1}[\sqrt{a^3 |\Omega_{\Lambda,0}| (1 - \Omega_{\Lambda,0})}], & \Omega_{\Lambda,0} < 0. \end{cases} \tag{6}$$

4.- We start by assuming the components of the Universe behave as perfect fluids and hence described by a barotropic equation of state $p_i = (\gamma_i - 1)\rho_i c^2$, where γ_i describes each fluid: radiation ($\gamma_r = 4/3$), baryonic and dark matter ($\gamma_m = 1$), and dark energy in the form of cosmological constant ($\gamma_\Lambda = 0$). Once we introduce the dimensionless *density parameters*, defined as

$$\Omega_i = \frac{\kappa_0}{3H^2} \rho_i, \tag{7}$$

a) Show that the continuity eqns. can be written as a dynamical system with the following form:

$$\Omega_i' = 3(\Pi - \gamma_i)\Omega_i, \tag{8}$$

with $\Pi = \sum_i \gamma_i \Omega_i$, and prime notation means derivative with respect to the e-fold parameter $N = \ln(a)$.

b) Also, show that the Friedmann equation becomes a constraint for the density parameters at all time $\sum_i \Omega_i = 1$.

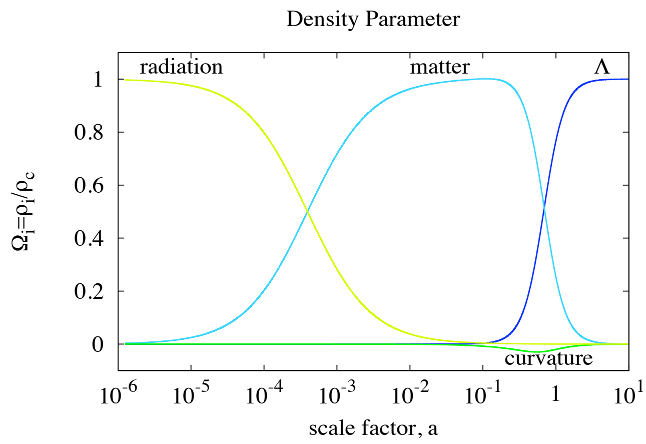

Figure 1: The evolution of the density parameters $\Omega_i(a)$.

c) Considering the initial conditions ($a = 1$) $\Omega_{r,0} = 10^{-4}$, $\Omega_{m,0} = 0.3$, $\Omega_{k,0} = -0.01$, $H_0 = 68\text{kms}^{-1}\text{Mpc}$, with cosmological constant, solve the dynamical system (8), along with the Friedmann constraint to get the following plot.

d) The deceleration parameter is computed in terms of the contents of the universe, as $q = \frac{1}{2}\sum_i \Omega_i(1 + 3w_i)$. Use the solutions from above to plot $q(z)$, where $1 + z = 1/a$.

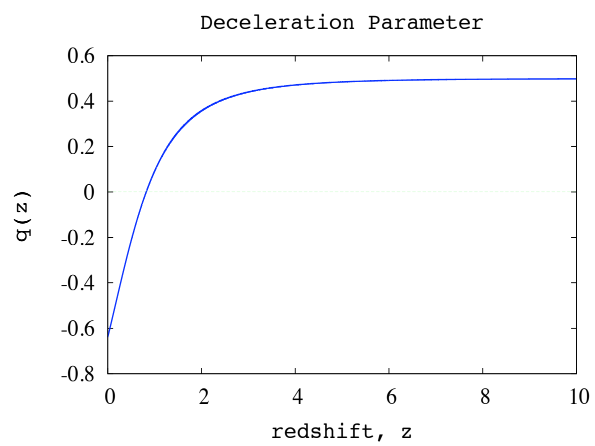

Figure 2: Deceleration parameter $q(z)$ as a function of redshift z for a multi-fluid universe. Notice that the universe is currently accelerating ($q(z = 0) < 0$).

5.- Consider the Universe from the previous exercise.

The *comoving distance* d_c is defined as

$$\chi_e = c \int_t^{t_0} \frac{dt}{R(t)} = \frac{c}{R_0} \int_0^z \frac{dz}{H(z)}. \tag{9}$$

The *luminosity distance* d_L is given by

$$d_L \equiv (1+z) R_0 S_k(\chi). \tag{10}$$

The angular distance is given by

$$d_A \equiv \frac{R_0 S_k(\chi)}{(1+z)}. \tag{11}$$

where

$$R_0 = h_0^{-1} \sqrt{-k/\Omega_{k,0}} = \frac{H_0^{-1}}{\sqrt{|\Omega_{k,0}|}}. \tag{12}$$

.

Plot these three distances

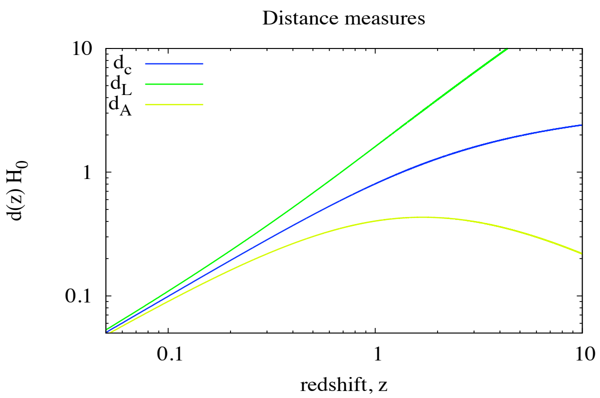

Figure 3: Comoving distance d_c, luminosity distance d_L, and angular distance d_A for a universe filled with the same constituents as in Figure 1.