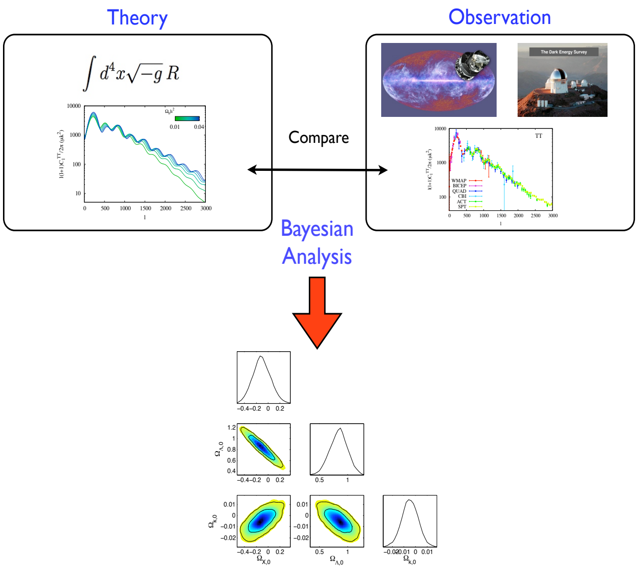
Updated Cosmology

with Python



José-Alberto Vázquez

ICF-UNAM / Kavli-Cambridge

In progress

August 12, 2017

0.1 The Friedmann-Lemaître equations

Once we have specified the metric that describes the homogeneous and isotropic expanding universe, the evolution of both the scale factor and matter density follows from Einstein's equations :

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_0 \, T_{\mu\nu}. \tag{1}$$

As we have seen previously $G_{\mu\nu}$ is the *Einstein metric tensor*, the *Ricci tensor* $R_{\mu\nu}$ is a combination of first and second derivatives of the metric $g_{\mu\nu}$, and its trace is defined by the *Ricci scalar* $R \equiv g^{\mu\nu} R_{\mu\nu}$; G is the Newton's constant and $\kappa_0 = 8\pi G/c^4$. On the right hand side, the energy-momentum tensor $T_{\mu\nu}$ contains the constituents of the universe. An acceptable modification to Einstein's equations is the introduction of a Lorentz-invariant constant-term $\Lambda g_{\mu\nu}$ into the field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa_0 \, T_{\mu\nu}, \tag{2}$$

where Λ is called the *cosmological constant* and its value, according to astrophysical observations, is $\Lambda \sim 10^{-52}\mathrm{m}^{-2}$ [3, 9]; we will see more about this component in subsequent sections. Equation (2) is in general a complicated set of coupled quasilinear second-order partial differential equations for the ten elements of the metric $g_{\mu\nu}$. Nevertheless, they may exhibit simple analytical solutions in the presence of generic symmetries, for instance, under the assumption of the FRW metric. Considering $g_{\mu\nu}$ in the form of (**??**), the Christoffel symbols are then given by

$$\begin{aligned}
\Gamma^0{}_{ij} &= R^2 H g_{ij}, \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = H\delta^i_j, \\
\Gamma^1{}_{11} &= \frac{kr}{1-kr^2}, \qquad \Gamma^1{}_{22} = -r(1-kr^2), \qquad \Gamma^1{}_{33} = -r(1-kr^2)\sin\theta, \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta, \qquad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r}, \qquad \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta.
\end{aligned}$$

and the non-vanishing curvature terms are given by

$$R_{00} = -3\frac{\ddot{R}}{R} = -3(\dot{H} + H^2), \tag{3}$$

$$R_{ij} = \left[\frac{\ddot{R}}{R} + 2\left(\frac{\dot{R}}{R}\right)^2 + \frac{2c^2k}{R^2}\right] g_{ij}, \tag{4}$$

$$R = 6\left[\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{c^2k}{R^2}\right], \tag{5}$$

and, finally, the Einstein tensor

$$G_0^0 = -3\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{c^2 k}{R^2}\right], \tag{6}$$

$$G_j^i = -\left[2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{c^2 k}{R^2}\right]\delta_j^i. \tag{7}$$

where an overdot indicates again derivative with respect to cosmic time t ($^{\cdot} \equiv d/dt$).

HW: Compute R and G

HW: Compute the Kretschmann scalar for a FRW spacetime.

The geometry of the space-time is determined by equations (6)-(7), then to solve Einstein's equations we just need to specify the matter content under consideration.

0.2 The Energy-momentum tensor

The energy-momentum is divided in the following components:

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & \vdots & T^{i0} \\ \cdots & \vdots & \cdots\cdots\cdots \\ T^{0j} & \vdots & T^{ij} \end{bmatrix}.$$

- T^{00}- total energy density.
- T^{i0} - energy flux ($\times c^{-1}$) in the i-direction.
- T^{0j} - momentum density ($\times c$) in the j-direction.
- T^{ij} - momentum flow (random thermal motions).
  - T^{ii} - Isotropic pressure in the i-direction.
  - T^{ij} ($i \neq j$) viscous stresses.

Because we seek a $T_{\mu\nu}$ consistent with the requirements of homogeneity and isotropy, we need the following to be satisfied:

- Isotropy requires that the mean values of the 3-vector vanish, i.e. $T^{0i} = T^{0j} = 0$.

- T^{ij} at any point (more specifically at $x = 0$) be proportional to δ_{ij} and hence to g_{ij} ($= R^2\delta_{ij}$ at $x = 0$), then

$$T_{ij}(x = 0) \propto \delta_{ij} \propto g_{ij}(x = 0). \tag{8}$$

- Homogeneity requires that the proportionality coefficient be only a function of time

$$T_{00} = \rho(t), \quad \pi_i \equiv T_{i0} = 0, \quad T_{ij} = p(t)g_{ij}(t, \vec{x}). \tag{9}$$

$$T^{\mu}_{\nu} = g^{\mu\lambda}T_{\lambda\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}, \tag{10}$$

where ρ is the energy density and p the isotropic pressure of the fluid, both of them measured by an observer in a local inertial frame in which the fluid is at rest. This is the stress-energy tensor of a *perfect-fluid* as seen by a comoving observer.
More generally

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^{\mu}u^{\nu} - pg^{\mu\nu}, \tag{11}$$

in the rest frame, where the 4-velocity of the fluid between the fluid and the observer is given by u^{μ}. For a comoving observer $u^{\mu} = (1, 0, 0, 0)$, the energy-momentum tensor hence reduces to $T^{\mu}_{\nu} = \mathrm{diag}(\rho, g^{ii}p)$.

Thus, Einstein's equations for a perfect fluid in a FRW background provide two independent expressions (time-time and space-space components), which together yield to the *Friedmann* and *acceleration* equations:

$$G^0_0 \quad \Rightarrow \quad \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{1}{3}\Lambda c^2, \tag{12}$$

$$G^i_j \quad \Rightarrow \quad \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2. \tag{13}$$

The cosmological equations, in which $R(t)$ is computed under the aforementioned conditions, are known as the Friedmann-Lemaître-Robertson-Walker equations; we simply refer to them as **Friedmann equations**.

Another equation of interest is the conservation of the energy-momentum tensor, $\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda = 0$, which leads ($\nu = 0$) to the *continuity equation*:

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{14}$$

In order to solve the full set of cosmological equations, we still need to specify an extra condition, for instance the pressure for every kind of material the universe is filled up with. The usual, and well-founded, assumption is that there is a pressure contribution associated to each energy density, so that $p \equiv p(\rho)$. Such a relationship is known as the *equation-of-state*. The Friedmann equation (12), the energy-momentum conservation (14), and the equation-of-state $p = p(\rho)$ are therefore the fundamental expressions that describe the dynamics of a homogeneous and isotropic universe.

Á la Newton

The Friedmann equation describes the expansion of the Universe, and is therefore the most important equation in cosmology. Let us derive the equations by using Newton's theory.

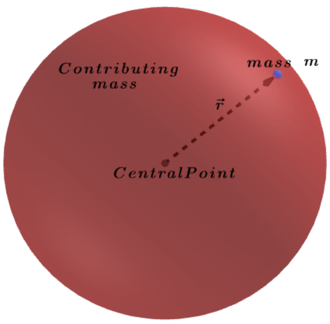


Figure 1: Figure of a mass in a solid gravity sphere.

Example 0.2.1: The Friedmann equation

Let us consider an observer in a uniform expanding medium with mass m and density ρ. Then, the total mass $M = 4\pi\rho r^3/3$ contributes to a force, see Figure 1

$$F = \frac{GMm}{r^2} = \frac{4\pi G\rho rm}{3}.$$

Our particle has a gravitational potential energy

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2 m}{3},$$

with the velocity of the particle $\dot{r}$ giving the kinetic energy

$$T = \frac{1}{2}m\dot{r}^2.$$

Energy conservation for that particle $U = T + V$, where U is a constant

$$U = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m,$$

using comoving coordinates $\vec{r} = R(t)\vec{x}$ (r the physical coordinates and x the fix location), then

$$U = \frac{1}{2}m\dot{R}^2x^2 - \frac{4\pi}{3}G\rho R^2x^2m,$$

if we multiply for $2/mR^2x^2$ and rearranging, we have

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2},$$

where $kc^2 = -2U/mx^2$. k is just a constant unchanging with either space or time, often called the curvature.

- $k > 0$- the expansion will at some time, halt and reverse itself.
- $k < 0$- the expansion will continue forever.
- $k = 0$- the expansion of the universe will slow down, but only halt at $t = \infty$.

Example 0.2.2: Evolution of the density

From thermodynamics second law $dE + PdV = TdS$ and using $E = mc^2$,

$$\begin{aligned} E &= \frac{4\pi}{3}R^3\rho c^2, \\ \rightarrow & \frac{dE}{dt} = 4\pi R^2\rho c^2\frac{da}{dt} + \frac{4\pi}{3}R^3\frac{d\rho}{dt}c^2, \end{aligned}$$

and for the volume change

$$\frac{dV}{dt} = 4\pi R^2\frac{da}{dt},$$

assuming a reversible expansion $dS = 0$, we get the fluid equation

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0.$$

The first term in the parenthesis corresponds to the dilution because the volume has increased, while the second is the loss of energy because the pressure has done work as the Universe's volume increased (gravitational potential energy).

Example 0.2.3: The accelerated equation

Differentiating the Friedmann equation, we have

$$2\frac{\dot{R}}{R}\frac{R\ddot{R} - \dot{R}^2}{R^2} = \frac{8\pi G}{3}\dot{\rho} + \frac{2kc^2\dot{R}}{R^3},$$

using $\dot{\rho}$

$$\begin{aligned} \frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 &= -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{R^2}, \\ \frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \end{aligned}$$

independent of the constant k.

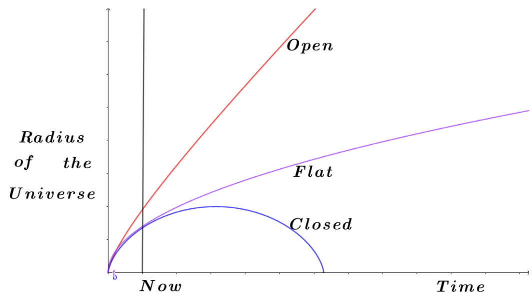


Figure 2: Figures for curve spaces. [redo this figure]

0.2.1 Cosmic Inventory

In order to understand the dynamical properties of the universe, we first need to bear in mind the whole content of it. Let us focus on single-barotropic perfect-fluids that satisfy, in general, a time-dependent equation-of-state $w(z)$, of the form

$$p = c^2 w(z)\rho. \tag{15}$$

For any component, with constant w, the continuity equation (14) can be easily integrated to give [1]

$$\rho \propto R^{-3(1+w)}. \tag{17}$$

Moreover, in a universe dominated by the energy density ρ, the Friedmann equation leads to the time evolution of the scale factor:

$$R(t) \propto t^{2/3(1+w)}, \qquad \forall \quad w \neq -1, \tag{18}$$

or, in conformal time (Eqn. **??**):

$$R(\eta) = \eta^{2/(1+3w)}, \qquad \forall \quad w \neq -1. \tag{19}$$

That is, the evolution of a universe filled with a given perfect fluid is known once its equation-of-state is specified. The standard Λ-Cold Dark Matter model (ΛCDM) relies upon four main ingredients, described by radiation (photons, massless neutrinos), matter (baryons), the inclusion of a dark matter component (DM) and vacuum energy (Λ). The behaviour of each of these components is summarised as follows:

[1] For a more general description of $w(a)$, the evolution of the energy density is given by

$$\rho \propto \exp[-3X(a)], \qquad \text{with} \qquad X(a) = \int_1^a [1 + w(a')] d\ln a'. \tag{16}$$

Radiation

This relativistic component dominates during the earliest stages of the universe. Radiation is characterised by an associated pressure $p_{\rm r} = \rho_{\rm r}/3$[1], with an associated equation-of-state $w_{\rm r} = 1/3$. The evolution of its energy-density and scale factor are thus given by

$$\rho_{\rm r}(t) = \rho_{\rm r,0}\left[\frac{R(t)}{R_0}\right]^{-4}, \qquad \text{and} \qquad R(t) \propto t^{1/2}. \tag{20}$$

On the other hand, the energy density of the blackbody radiation

$$\rho_r(T) = \sigma T^4, \tag{21}$$

where σ is the Stefan-Boltzmann constant. If the present day temperature is pin-up to be $T_0 = 2.726K$ then the number density of radiation today is $n_{\gamma,0} \simeq 4 \times 10^8 m^{-3}$ (see more about them in the next section). On the other hand, comparing eqns. (20) and (21) we have that the universe was denser and hotter in the past

$$T \propto R^{-1}. \tag{22}$$

The big bang is somehow explain by:

$$t \to 0, \qquad R \to 0, \qquad \rho \to \infty, \qquad T \to \infty. \tag{23}$$

Extrapolating our assumptions, at the beginning of the universe ($t \to 0$), the universe was denser and hotter concentrated in a minusculous tiny region.

The total radiation energy-density ρ_r in the universe may be written as the sum of two main contributions: photons (γ) and massless neutrinos (ν):

$$\rho_{\rm r}(t) = \rho_\gamma(t) + \rho_\nu(t). \tag{24}$$

Photons - Primordial photons play a key role in observational cosmology as they constitute the cosmic microwave background radiation we nowadays observe, as we shall see in more detail in the following Sections.

Massless Neutrinos - Neutrinos are very weakly interacting leptons, which come in three types or 'flavours': electron, muon, and tau; all of them with an associated antiparticle. The

[1] we'll obtain this result with statistical mechanics in the following sections.

amount of massless neutrinos in the cosmic background (estimated from theoretical arguments) is given by

$$\rho_\nu = N_{\rm eff} \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma, \tag{25}$$

where $N_{\rm eff}$ is the effective number of neutrino species; note that $N_{\rm eff} = 3.046$ for the standard neutrino species [8]. Nevertheless, several experiments suggest they do have a very small mass. For instance experiments detecting atmospheric neutrinos, solar neutrinos, also reactor neutrinos and accelerator beam neutrinos. Cosmological observations have also provided limits on the neutrino mass; some reviews in the subject can be found in: Dolgov [4], Elgarøy and Lahav [5], Hannestad [6], Lesgourgues and Pastor [7].

Matter

Any type of material with negligible pressure is often referred as 'dust'. It is represented by an equation-of-state $w_{\rm m} = 0$, with energy-density given by

$$\rho_{\rm m}(t) \propto R^{-3}, \qquad \text{and} \qquad R(t) \propto t^{2/3}. \tag{26}$$

that is, the dilution of the energy-density is because the expansion of the volume $V \propto R^3$.

Combining expressions in (26) we have

$$H = \frac{\dot{R}}{R} = \frac{2}{3t}. \tag{27}$$

that is, the universe expands forever but with a decreasing rate. Notice that $t_0 = \frac{2}{3}\frac{1}{H_0}$ and using the Hubble parameter we have that the age of the Universe with content purely giving in the form of matter is $t_0 = 9.3$ Gyrs. This type of universe is called *Einstein de-Sitter Universe*. The total matter content of the universe comes in several forms. In addition to the familiar baryonic matter, observations of the Large-Scale Structure (LSS) suggest that most of the galactic content is in the form of non-baryonic matter, called *Dark-Matter*. The total matter density may be expressed as the sum of baryonic (b) and dark-matter (dm) contributions:

$$\rho_{\rm m}(t) = \rho_{\rm b}(t) + \rho_{\rm dm}(t). \tag{28}$$

Baryons - make up the familiar matter of our universe (protons and neutrons). Since the charge of the universe is neutral, there must be an equal number of protons than electrons (charged leptons). An elaborated review of Big Bang Nucleosynthesis (BBN) is given by Steigman [12] (see also next section).

Dark matter - The existence of non-baryonic dark matter has been inferred from its gravitational manifestations through the flat rotation curves of galaxies, the mass-to-light ratios in clusters of galaxies, and gravitational lensing of background sources. An extended discussion of the current status of particle dark matter, including experimental evidence and theoretical motivations, is presented by Bertone et al. [2], Sellwood and Kosowsky [11].

Vacuum

If the cosmological constant term is moved to the right-hand-side on Einstein's equations, it can be associated to the *vacuum energy-density*, given by

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}. \tag{29}$$

At future cosmic times, while the matter and radiation density dilute away, the vacuum energy-density remains with the same constant value ρ_Λ. The vacuum energy can be modelled as a perfect fluid with negative pressure equal to $p_\Lambda = -\rho_\Lambda$, which corresponds to an equation-of-state $w_\Lambda = -1$: a *de-Sitter Universe*. For a review about the cosmological constant term see e.g. Carroll [3], Padmanabhan [9], Peebles and Ratra [10]. The cosmological constant is also seen as the simplest form of a more generic *'dark energy'* component, commonly considered as the main candidate to explain the current acceleration of the universe. We shall see in Chapter **??** that $w_{\rm DE}(a)$ evolving in time provides a slightly better description for the current observational data.

Curvature

The contribution of the spatial curvature can be considered as any other energy component by defining a fictitious energy density:

$$\rho_k \equiv -\frac{3kc^2}{8\pi G}R^{-2}. \tag{30}$$

This energy-density is described by an equation-of-state $w_k = -1/3$, for which the scale factor evolves proportionally to the cosmic time $a \propto t$. The general behaviour of the curvature term is easily understood if we rewrite the Friedmann equation, with a vanished cosmological constant, in the following way

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}(\rho + \rho_k). \tag{31}$$

For a positive density contribution ρ, the universal expansion can only be stopped if the universe is closed $k > 0$ ($\rho_k < 0$), otherwise it will expand forever.

component	Ω_i	w_i	$\rho(R)$	$R(t)$	$H(t)$
radiation	$\Omega_{\rm r}$	1/3	$\propto R^{-4}$	$\propto t^{1/2}$	1/2t
matter	$\Omega_{\rm m}$	0	$\propto R^{-3}$	$\propto t^{2/3}$	2/3t
curvature	Ω_k	-1/3	$\propto R^{-2}$	$\propto t$	1/t
missing matter	Ω_X	-2/3	$\propto R^{-1}$	$\propto t^2$	2/t
cosmological constant	Ω_Λ	-1	$\propto R^0$	$\propto \exp(\sqrt{\frac{\Lambda}{3}}t)$	const

Table 1: Constituents of the universe and their cosmological parameters: density parameter Ω_i, equation-of-state parameter w_i; and their behaviour: density evolution $\rho(R)$, scale factor $R(t)$, Hubble parameter $H(t)$. [comoving quantities, $r : \eta, dm : \eta^2, \Lambda : -\eta^{-1}$]

Missing matter

If the Friedmann equation is written in terms of the present energy-density components, we have

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left[\rho_{\rm r,0}\left(\frac{R}{R_0}\right)^{-4} + \rho_{\rm m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{k,0}\left(\frac{R}{R_0}\right)^{-2} + \rho_{\Lambda,0}\left(\frac{R}{R_0}\right)^{-0}\right]. \tag{32}$$

Notice that the right-hand-side can be seen as a power series expansion, however with a missing component with contribution R^{-1}. To complete the series, we include this term and named it as the *missing-energy* component [13], for which its energy-density satisfies

$$\rho_X(t) = \rho_{X,0}\left[\frac{R}{R_0}\right]^{-1}, \qquad \text{and} \qquad R \propto t^2. \tag{33}$$

The missing-energy component has therefore an equation-of-state $w_X = -2/3$, and behaves similarly to domain walls [1, 14]. We explain it in more detail about this new term in Chapter **??**.

A summary of the main components of the universe, along with their behaviour, is shown in Table 1. Before solving the cosmological equations for the whole mixture of perfect-fluid components, we include some essential notation:

Density parameter: We also introduce the ratio of the energy-density relative to the critical density $\rho_{\rm c} \equiv 3H^2/8\pi G$, as the dimensionless density parameter:

$$\Omega_i(t) \equiv \frac{\rho_i(t)}{\rho_{\rm c}(t)}, \tag{34}$$

where the index 'i' labels a single type of component, such as matter, radiation, etcetera. The *critical density* is the energy density for which the universe is spatially flat, that is, from Eqn. (31) we have $\rho_c(t) = 3H^2/8\pi G$; and its present value is $\rho_{c,0} = 1.88h^2 \times 10^{-22}$ gcm^{-3}.

0.2.2 The cosmological field equations

We have computed the evolution of the scale factor for a universe made up by single-independent fluids: radiation, matter, vacuum, spatial curvature, vacuum energy and missing energy. To make the basic Friedmann models more realistic, we need to take into account the whole mixture of these components. Suppose that within the mixture, the distinct fluids do not interact with each other but only through their mutual gravitation (perfect fluids). The total energy-momentum tensor of a multiple-component fluid is thus given by

$$T^{\mu\nu} = \sum_i (T^{\mu\nu})_i, \tag{35}$$

where i labels the sum over various constituents, each of them individually modelled as a single perfect-fluid with $p_i = w_i \rho_i$. Using the definitions introduced above, the Friedmann equations (12) and (13) for a multi-fluid universe are now written in the following way

$$H^2 = H_0^2 \left[\sum_i \Omega_{i,0} \left(\frac{R}{R_0} \right)^{-3(1+w_i)} + \Omega_{k,0} \left(\frac{R}{R_0} \right)^{-2} \right], \tag{36}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3} \sum_i \rho_i (1 + 3w_i) = \frac{R\ddot{R}}{\dot{R}^2}. \tag{37}$$

The density parameters at any given time are

$$\Omega_i = \Omega_{i,0} \left(\frac{H_0}{H} \right)^2 \left(\frac{R}{R_0} \right)^{-3(1+w_i)}. \tag{38}$$

Therefore, these equations, at any cosmic time, can be written in the elegant forms:

$$\Omega_T \equiv \sum_i \Omega_i + \Omega_\Lambda = 1 - \Omega_k, \tag{39}$$

$$q = \frac{1}{2} \sum_i \Omega_i (1 + 3w_i). \tag{40}$$

In particular, the curvature density-parameter $\Omega_k = -kc^2/H^2R^2$, determines the normalisation of the scale factor (**??**), or *curvature radius*:

$$R_0 = H_0^{-1} \sqrt{-kc^2/\Omega_{k,0}} = \frac{cH_0^{-1}}{\sqrt{|\Omega_{k,0}|}}. \tag{41}$$

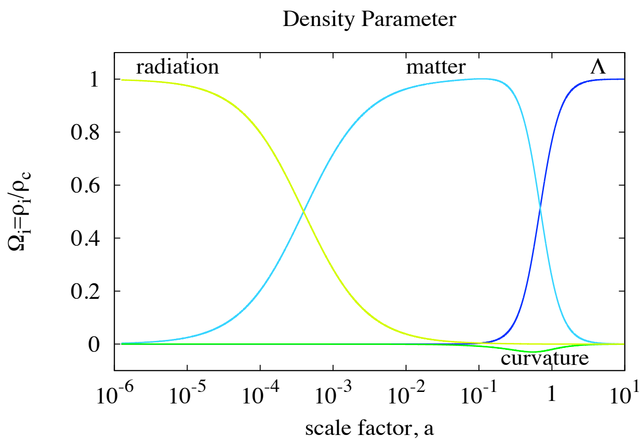


Figure 3: The evolution of density parameters $\Omega_i(a)$ are seen as a succession of several epochs, each of them dominated by different components: radiation, matter, curvature and cosmological constant.

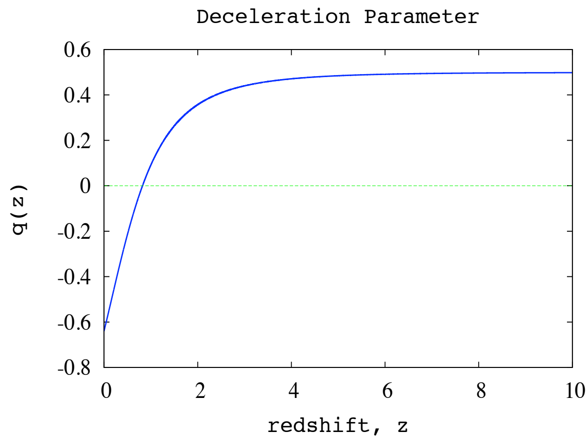


Figure 4: Deceleration parameter $q(z)$ as a function of redshift z for a multi-fluid universe. Notice that the universe is currently accelerating ($q|_{z=0} < 0$).

In a universe with positive curvature, R_0 is just the radius of the 3-sphere. Notice that the matter distribution (39) clearly determines the spatial geometry of the universe: $\Omega_T < 1$ (open), $\Omega_T = 1$ (flat) and $\Omega_T > 1$ (closed).

On the other hand, we can go back and for on the scale factors $a \leftrightarrow R$ from Eqn. (**??**), thus from now on, and for simplicity, we shall use the normalised scale factor a.

The Friedmann equations have exact solutions in just a few simple cases, for instance in a universe modelled by perfect-fluids. For this particular case, the density parameters and their dependence on the scale factor are plotted in Figure 3. In this Figure, the cosmic evolution of the different constituents are seen as a succession of several epochs, each of them corresponding to a different perfect-fluid. At the earliest stages, radiation dominates because of its behaviour $\rho_{\rm r} \propto a^{-4}$. Then, at $a_{\rm eq} \approx 4.2 \times 10^{-5} h^{-2}$, the radiation contribution equals that of matter, which starts dominating afterwards. It is noticeable that the curvature term is almost negligible due to the initial conditions taken (see Section **??**). Finally, the cosmological constant term dominates over the late-time evolution of the universe, and remains so for all time due to its constant energy-density behaviour.

From expression (40), we observe that the sign of $(1+3w_i)$ determines whether the universe is accelerating $(q < 0)$ or decelerating $(q > 0)$. If the major contribution comes from a fluid(s) with $w_i > -1/3$ the expansion of the universe will gradually slow-down due to the action of gravity. On the other hand, if $w_i < -1/3$ the pressure component acts as a 'repulsive' term leading to an accelerated expansion. For instance the cosmological constant term, which dominates over the dynamics of the universe at low-redshift, is considered the principal responsible for the present accelerated expansion of the universe, seen in Figure 4.

0.3 Scalar Fields

Another important component of the Universe to bear in mind are the scalar fields, as they may play the role as the dark matter or dark energy component, and we'll see more about them in the section of Inflation. The action for a particle with 0-spin, or *scalar field* ϕ with potential $V(\phi)$, is given by:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \tag{42}$$

with an associated energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \, [\frac{1}{2} \partial_\sigma \phi \partial^\sigma \phi - V(\phi)]. \tag{43}$$

The corresponding dynamics for ϕ is obtained from the Euler-Lagrange equations, which describes a relativistic wave equation, called as the *Klein-Gordon* equation:

$$\Box^2 \phi + \frac{dV}{d\phi} = 0. \tag{44}$$

Comparing the energy-momentum tensor of the field with the analogous for the perfect fluids, we may associate an effective energy density ρ_ϕ and pressure p_ϕ to the field:

$$T_{00} = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}(\vec{\nabla}\phi)^2, \tag{45}$$

$$T_{ii} = p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}(\vec{\nabla}\phi)^2, \tag{46}$$

Therefore, the Friedmann and the acceleration equations for a Universe only filled up with a homogeneous single-scalar field become

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{47}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\dot{\phi}^2 - V(\phi)\right], \tag{48}$$

with a corresponging equation of state

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \tag{49}$$

Therefore, the condition to be satisfied for the field to accelerate the universe ($p_\phi \approx -\rho_\phi$) becomes $\dot{\phi}^2 \ll V(\phi)$, which is easily fulfilled with a suitable flat potential. When the condition $\langle\dot{\phi}^2\rangle = V(\phi)$ is satisfied, then the field beahaves as a dust component, ie. $w_\phi \approx 0$, thus a good candidate to be the dark matter component.

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