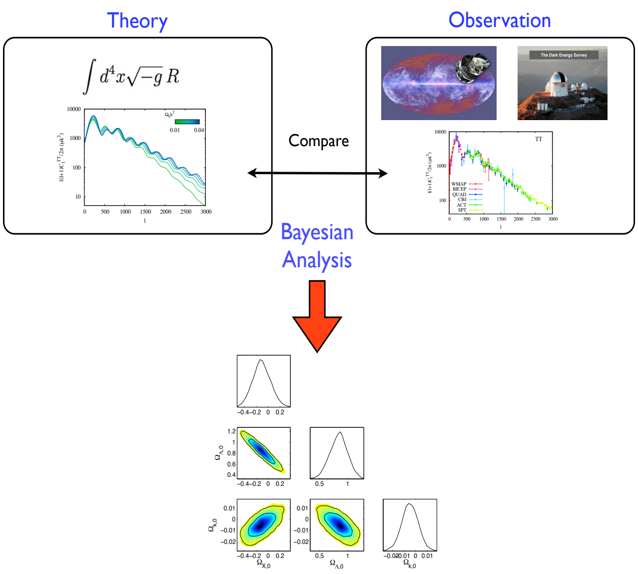
Updated Cosmology

with Python



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0.1 Kinematics

In general, for a particle described with coordinates x^μ, we have the action $S[x^\mu(\lambda)]$ with an associated Lagrangian density, given by

$$S[x^\mu(\lambda)] \equiv L[x^\mu, \dot{x}^\mu]d\lambda, \tag{1}$$

where overdot means derivative respect to an affine parameter λ: $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$.
The variation of the action yields to

Example 0.1.1: The Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0.$$

Pee. Let us consider the motion of a massive particle between points A and B, displayed in Figure 1, the action is given by

$$S = m\int_A^B ds, \tag{2}$$

with boundary conditions defined as

$$\lambda(A) \equiv 0, \qquad \lambda(B) \equiv 1, \tag{3}$$

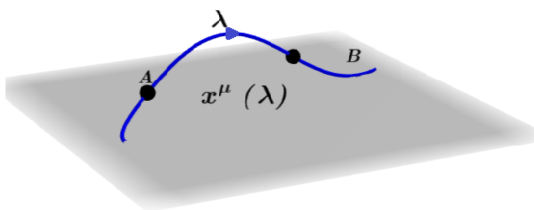


Figure 1: Free particle

where the interval in a generic space is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, and hence

$$S[x^\mu(\lambda)] \quad = \quad m\int_0^1 [g_{\mu\nu}(x)\dot{x^\mu}\dot{x^\nu}]^{1/2}d\lambda. \tag{4}$$

The canonical momenta p_μ are the derivatives of the Lagrangian with respect to the coordinate velocities. Computing the derivatives of the density Lagrangian $L = m(g_{\mu\nu}\dot{x^\mu}\dot{x^\nu})^{1/2}$, and for convenience making $m = 1$[1]:

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{1}{2}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right)^{-1/2} \times g_{\mu\nu}\left[\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha}\dot{x}^\nu + \dot{x}^\mu \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}\right]$$
$$= \frac{1}{2L} g_{\mu\nu}\left[\delta^\mu_\alpha \dot{x}^\nu + \dot{x}^\mu \delta^\nu_\alpha\right] = \frac{1}{2L}\left[g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right] = \frac{1}{L} g_{\mu\alpha}\dot{x}^\mu, \tag{5}$$

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2L}\partial_\alpha g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu. \tag{6}$$

By using the interval ds, we have

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = L^2 \qquad \text{and hence} \qquad \frac{d}{d\lambda} \to L\frac{d}{ds}. \tag{7}$$

Writing the Einstein-Lagrange equations in terms of the interval ds, they yield to

$$\frac{d}{ds}\left(g_{\mu\alpha}\frac{dx^\mu}{ds}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{8}$$

Expanding the first term in the previous expression

$$\left[\partial_\beta g_{\mu\alpha}\frac{dx^\beta}{ds}\right]\frac{dx^\mu}{ds} + g_{\mu\alpha}\frac{d^2x^\mu}{ds^2} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{9}$$

where the first term that contains $\partial_\beta g_{\mu\alpha}$ can be replaced by $\frac{1}{2}(\partial_\beta g_{\mu\alpha} + \partial_\mu g_{\beta\alpha})\frac{dx^\beta}{ds}\frac{dx^\mu}{ds}$. Reordering, we get

$$g_{\mu\alpha}\frac{d^2x^\mu}{ds^2} + \frac{1}{2}(\partial_\beta g_{\mu\alpha} + \partial_\mu g_{\beta\alpha})\frac{dx^\beta}{ds}\frac{dx^\mu}{ds} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{10}$$

By contracting with the inverse metric, relabelling indices and using the Christoffel definition we find the

Geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0.$$

Considering the particle has a four-velocity $u^\mu \equiv \frac{dx^\mu}{ds}$, from the geodesic equation we have

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0, \tag{11}$$

using the chain rule

$$\frac{d}{ds}u^\mu(x^\alpha(s)) = \frac{dx^\alpha}{ds}\frac{\partial u^\mu}{\partial x^\alpha} = u^\alpha \frac{\partial u^\mu}{\partial x^\alpha}, \tag{12}$$

[1] where we have used $\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\mu} = \delta^\nu_\mu$.

so, we get

$$u^\alpha \left(\frac{\partial u^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} u^\beta \right) = 0. \tag{13}$$

We notice the quantity within parenthesis defines the covariant derivative

$$\nabla_\alpha u^\mu \equiv \partial_\alpha u^\mu + \Gamma^\mu_{\alpha\beta} u^\beta, \tag{14}$$

and therefore, we have that $u^\alpha \nabla_\alpha u^\mu = 0$ (same result obtain in GR using parallel transport). Putting back the mass, and using the four-momentum of the particle $p^\mu = -mu^\mu$ [Pee], it yields to

$$p^\alpha \frac{\partial p^\mu}{\partial x^\alpha} = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta. \tag{15}$$

Example 0.1.2: The Einstein-Hilbert action.

Let us consider the Einstein-Hilbert action, given by

$$S_{EH} = \int d^n x \sqrt{-g} R = \int d^n x \sqrt{-g} R_{\mu\nu} g^{\mu\nu},$$

where, as usual, the g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci scalar.

In General Relativity the metric $g_{\mu\nu}$ is the dynamical variable, whereas the Ricci scalar is the product of the metric and its derivatives, hence the integral contains all the dynamical variables that conform the Lagrangian [add Palatini formalism]. Therefore, to minimise the action – by using the variational principle –, we perform the variation of the action equal to zero:

$$\delta S_{EH} = \delta \int d^n x \sqrt{-g} R = 0.$$

Then

$$\begin{aligned} \delta S_{EH} &= \int d^n x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \int d^n x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int d^n x R_{\mu\nu} g^{\mu\nu} \delta \sqrt{-g} \\ &= \delta S_1 + \delta S_2 + \delta S_3. \end{aligned}$$

We compute separately the variation for each term S_i with $i = 1, 2, 3$.
For S_1, we first use the definition of the Christoffel symbols

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\lambda\epsilon} \Gamma^{\epsilon}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\nu\epsilon} \Gamma^{\epsilon}{}_{\mu\lambda}.$$

Then, the corresponding variation is

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \partial_\nu \delta\Gamma^{\lambda}_{\mu\lambda} + \delta\Gamma^{\lambda}_{\lambda\epsilon} \Gamma^{\epsilon}_{\nu\mu} + \Gamma^{\lambda}_{\lambda\epsilon} \delta\Gamma^{\epsilon}_{\nu\mu} - \delta\Gamma^{\lambda}_{\nu\epsilon} \Gamma^{\epsilon}_{\mu\lambda} - \Gamma^{\lambda}_{\nu\epsilon} \delta\Gamma^{\epsilon}_{\mu\lambda} \\ &= (\partial_\lambda \delta\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\lambda\epsilon} \delta\Gamma^{\epsilon}_{\nu\mu} - \Gamma^{\epsilon}_{\mu\lambda} \delta\Gamma^{\lambda}_{\nu\epsilon} - \Gamma^{\epsilon}_{\nu\lambda} \delta\Gamma^{\lambda}_{\mu\epsilon}) \\ &\quad -(\partial_\nu \delta\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\nu\epsilon} \delta\Gamma^{\epsilon}_{\mu\lambda} - \Gamma^{\epsilon}_{\nu\mu} \delta\Gamma^{\lambda}_{\lambda\epsilon} - \Gamma^{\epsilon}_{\nu\lambda} \delta\Gamma^{\lambda}_{\mu\epsilon}). \end{aligned}$$

Using the covariant derivative

$$\nabla_c \delta\Gamma^{c}_{ab} = \partial_c \delta\Gamma^{c}_{ab} + \Gamma^{c}_{cd} \delta\Gamma^{d}_{ba} - \Gamma^{d}_{ac} \delta\Gamma^{c}_{bd} - \Gamma^{d}_{bc}, \delta\Gamma^{c}_{ad},$$

in order to write the previous expression as

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda}.$$

Example 0.1.3:

The first part of the action, S_1, results in the following form:

$$\begin{aligned}\delta S_1 &= \int d^4x\sqrt{-g}g^{\mu\nu}(\nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}) \\ &= \int d^4x\sqrt{-g}[\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - \delta\Gamma^\lambda_{\mu\nu}\nabla_\lambda g^{\mu\nu} - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}) + \delta\Gamma^\lambda_{\mu\lambda}\nabla_\nu g^{\mu\nu}].\end{aligned}$$

Because the covariant derivative of the metric vanishes, thus the previous equation becomes:

$$\begin{aligned}\delta S_1 &= \int d^4x\sqrt{-g}[\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})] \\ &= \int d^4x\sqrt{-g}\nabla_\lambda[g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}]. \qquad (16)\end{aligned}$$

Let be $J^\lambda = g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}$, a vectorial field defined over a region M with frontier Σ. Using the Stokes theorem:

$$\int_M d^4x\sqrt{|g|}\nabla_\lambda J^\lambda = \int_\Sigma d^3x\sqrt{|g|}n_\lambda J^\lambda,$$

with n_λ is a unitary normal vector to the hyper-surface Σ. In infinity J^λ becomes zero on the surfaces due to the variations in $g_{\mu\nu}$ that tend to zero far away from the sources, and the variation of the Christoffel symbols are proporcional to the variations of the metric and its derivatives. Therefore, we have $S_1 = 0$, that is, the first term does not contribute to the variation of the Einstein-Hilbert action.

To compute the variations of S_2 y S_3, let us analyse the behaviour of the metric tensor under their own variations. First, consider that $g_{\lambda\mu}g^{\mu\nu} = \delta^\nu_\lambda$ Then, assuming the metric tensor has inverse, hence it exists a tensor $A^{\nu\mu}$ such that:

$$g^{\mu\nu} = \frac{1}{g}(A^{\mu\nu})^T = \frac{1}{g}A^{\nu\mu},$$

where g is the determinant of $g_{\mu\nu}$. From the two previous expressions, we have $g = g_{\mu\nu}A^{\mu\nu}$. From which we may infer that the partial derivative of the determinant is:

$$\frac{\partial g}{\partial g_{\mu\nu}} = A^{\mu\nu}.$$

Therefore

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = A^{\mu\nu}\delta g_{\mu\nu} = gg^{\nu\mu}\delta g_{\mu\nu}.$$

and given that $g^{\mu\nu}$ is symetric, then:

$$\delta g = gg^{\mu\nu}\delta g_{\mu\nu}.$$

Example 0.1.4:

With the previous calculations in mind, we are able to compute the variation of the $\sqrt{-g}$ term:

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}}g^{\mu\nu}\delta g_{\mu\nu}. \end{aligned} \tag{17}$$

We need $\delta g^{\mu\nu}$ instead of $\delta g_{\mu\nu}$; to do that, we consider the following:

$$\begin{aligned} \delta\delta_{\mu}{}^{\epsilon} = \delta(g_{\mu\lambda}g^{\lambda\epsilon}) &= 0 \\ g^{\lambda\epsilon}\delta g_{\mu\lambda}\delta g^{\lambda\epsilon} &= 0 \\ g^{\lambda\epsilon}\delta g_{\mu\lambda} &= -g_{\mu\lambda}\delta g^{\lambda\epsilon}. \end{aligned}$$

Multiplying both terms of the equation by $g_{\nu\epsilon}$, we have:

$$\begin{aligned} g_{\nu\epsilon}g^{\lambda\epsilon}\delta g_{\mu\lambda} &= -g_{\nu\epsilon}g_{\mu\lambda}\delta g^{\lambda\epsilon} \\ \delta^{\lambda}_{\nu}\delta g_{\mu\lambda} &= -g_{\nu\epsilon}g_{\mu\lambda}\delta g^{\lambda\epsilon} \\ \delta g_{\mu\nu} &= -g_{\mu\lambda}g_{\nu\epsilon}\delta g^{\epsilon\lambda}. \end{aligned} \tag{18}$$

Substituting the last results into equation 17:

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\lambda}g_{\nu\epsilon}\delta g^{\epsilon\lambda} \\ &= -\frac{1}{2}\sqrt{-g}\delta^{\nu}_{\lambda}g_{\nu\epsilon}\delta g^{\epsilon\lambda} \\ &= -\frac{1}{2}\sqrt{-g}g_{\lambda\epsilon}\delta g^{\epsilon\lambda}. \end{aligned} \tag{19}$$

Renaming the indices, then:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

Using that $S_1 = 0$ along with equations 18 and 19, finally we've got:

$$\begin{aligned} \delta S_{EH} &= \int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}\int d^4R\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \\ &= \int d^4x\sqrt{-g}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]\delta g^{\mu\nu}. \end{aligned}$$

Notice the terms within brackets correspond to the definition of the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

Modifications to the Einstein-Hilbert action.

The action of the $f(R)$ models is given by

$$S_{MG} = \int d^n x \sqrt{-g} f(R).$$

See [2], where the equations of motion are (2.15)-(2.16) and the dynamical system (4.63)-(4.66) to find the solutions.

Also, see [5] for a Brane-World Gravity review, where the action to take into account is

$$S_{\text{gravity}} = \frac{1}{2\kappa_{4+d}^2} \int d^4 d^d y \sqrt{-^{(4+d)}g} \left[^{(4+d)}R - 2\Lambda_{4+d} \right],$$

where d is the number of extra dimensions and κ_{4+d}^2 is the gravitation coupling constant.

0.1.1 Geodesics in the FRW metric

The FRW metric (**??**) is written in the following way

$$ds^2 = c^2 dt^2 - R^2(t)\gamma_{ij} dx^i dx^j. \tag{20}$$

HW 0.1: Compute the Christoffel symbols to get

$$\Gamma^0_{ij} = R\dot{R}\gamma_{ij}, \qquad \Gamma^i_{0j} = \frac{\dot{R}}{R}\delta^i_j, \qquad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}),$$

otherwise zero [put back c].

The homogeneity of FRW implies that $\partial_i p^\mu = 0$ and hence only survives $\alpha = 0$. From the geodesic equation (15), we have

$$p^0 \frac{dp^\mu}{dt} = -\Gamma^\mu_{\rho\beta} p^\rho p^\beta \tag{21}$$

$$= -(2\Gamma^\mu_{0j} p^0 + \Gamma^\mu_{ij} p^i) p^j. \tag{22}$$

The implications of the expressions above are:

- A massive particle at rest - in the comoving frame - $p^j = 0$, will stay at rest

$$p^j = 0 \qquad \to \qquad \frac{dp^\mu}{dt} = 0. \tag{23}$$

- Considering the case $\mu = 0$, we have that the first Christoffel vanishes ($\Gamma^0_{0j} = 0$), and hence
$$E\frac{dE}{dt} = -\Gamma^0_{ij}p^i p^j = -\frac{\dot{R}}{R}p^2. \qquad (24)$$
where we have written $p^0 = E$ and the physical three-momentum $p^2 = -g_{ij}p^i p^j = R^2\gamma_{ij}p^i p^j$, and the components of the four momentum satisfy the constraint $g_{\mu\nu}p^\mu p^\nu = m^2$ or $E^2 - p^2 = m^2$. Using the fact that $EdE = pdp$, then the equation can be written as
$$\frac{\dot{p}}{p} = -\frac{\dot{R}}{R} \qquad \rightarrow \qquad p \propto \frac{1}{R}, \qquad (25)$$
the three momentum of any particle (either massive or massless) decays with the expansion of the universe.

  - For massless particle- The energy decays with the expansion of the scale factor
  $$p = E \propto 1/R. \qquad (26)$$
  - For massive
  $$p = \frac{mv}{\sqrt{1-v^2}} \propto \frac{1}{R}, \qquad (27)$$
  where $v^i = dx^i/dt$ is the comoving peculiar velocity of the particles and $v^2 \equiv R^2\gamma_{ij}v^i v^j$. The freely-falling particles left on their own will converge onto the Hubble flow.

0.1.2 Redshift

The light emitted can be viewed either quantum mechanically as a free-propagating photons, or classically propagating electromagnetic waves

- Quantum.
The wavelength $\lambda = h/p$ and since
$$p \propto \frac{1}{R(t)} \rightarrow \lambda \propto R(t). \qquad (28)$$
Light emitted at time t_1 with wavelength λ_1 will be observed at t_0 with
$$\lambda_0 = \frac{R(t_0)}{R(t_1)}\lambda_1. \qquad (29)$$
Since $R(t_0) > R(t_1)$ (with $t_0 > t_1$), then the wavelength of the light increases $\lambda_0 > \lambda_1$, that is, is red-shifted otherwise blue-shifted.

- Classical waves.

  [add Figure]. Consider a galaxy at fixed comoving distance d. At a time η_1, the galaxy emits a signal of short conformal duration $\Delta\eta$. According to the geodesics $\Delta\eta = \Delta\chi$ (??) the light arrives at our telescope at time η_0. The conformal duration of the signal measured by the detector is the same as the source, but the physical time intervals are different at the points of emission and detection.

$$\Delta t_1 = R(\eta_1)\Delta\eta \qquad \& \qquad \Delta t_0 = R(\eta_0)\Delta\eta. \tag{30}$$

  If Δt is the period of the light wave, the light is emitted with wavelength $\lambda_1 = \Delta t_1$, but it is observed with wavelength $\lambda_0 = \Delta t_0$, so that

$$\frac{\lambda_0}{\lambda_1} = \frac{R(\eta_0)}{R(\eta_1)}. \tag{31}$$

For convenience, we express the fractional shift in wavelength of a photon emitted by a distant galaxy at time t_1 with wavelength λ_1 and the observer on Earth today (t_0), as:

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1}, \tag{32}$$

and therefore the gravitational redshift in terms of the scale factor is

$$\boxed{1 + z = \frac{R(t_0)}{R(t_1)}.}$$

Example 0.1.5: Cosmological redshifts

In general it is shown (see [3]) that the redshift z can be computed given the conformal Killing vector field, giving

$$1 + z = \sqrt{\frac{g_{\alpha\beta}(y^\gamma)\xi^\alpha(y^\gamma)\xi^\beta(y^\gamma)}{g_{\alpha\beta}(x^\gamma)\xi^\alpha(x^\gamma)\xi^\beta(x^\gamma)}}.$$

The redshift is used to refer to the time at which the scale factor was a fraction $1/(1+z)$ of its present value. It is also used to refer to the distance that light has travelled since that time [4].

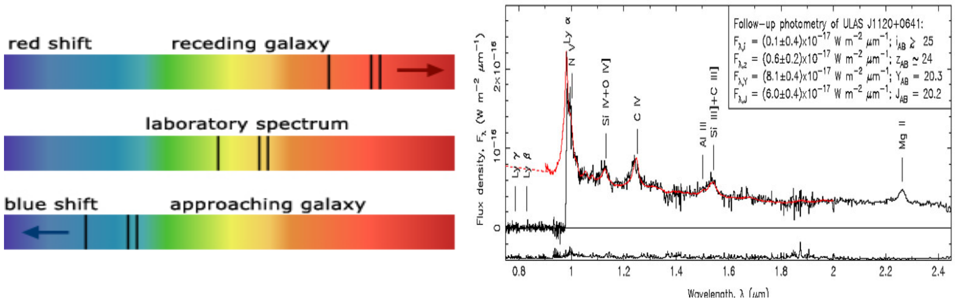


Figure 2: Redshift.

Example 0.1.6: Times in the Universe

Some particular times in the history of the Universe

$$
\begin{aligned}
R &= R_0, \qquad z = 0, \qquad t = 13.8Gys, \\
R &= 0, \qquad z = \infty, \qquad t = 0, \\
R &= R_0/1101, \qquad z = 1100, \qquad t = 380,000ys.
\end{aligned}
$$

0.1.3 Hubble and Deceleration parameter

Let us expand the scale factor as a power series about the present epoch t_0

$$
\begin{aligned}
R(t) &= R[t_0 - (t_0 - t)] = R[t_0 - \Delta] \\
&= R(t_0) - (t_0 - t)\dot{R}|_{t=t_0} + \frac{1}{2}(t_0 - t)^2 \ddot{R}|_{t=t_0} - \cdots \\
&= R(t_0)\left[1 - (t_0 - t)H(t_0) - \frac{1}{2}(t_0 - t)^2 q(t_0) H^2(t_0) - \cdots\right].
\end{aligned} \tag{33}
$$

HW: use simpy (Part II of the course).

The expansion rate of the universe is characterised by the **Hubble parameter** defined as

$$
H(t) \equiv \frac{\dot{R}(t)}{R(t)}, \tag{34}
$$

where the present expansion rate, being $H(t = t_0)$, is called the Hubble constant H_0. Because the Hubble constant is still not known with great accuracy, it is conventional to denote it through the dimensionless parameter h, such that $H_0 = 100\,h$ km s^{-1}Mpc^{-1} $= h/3000$ Mpc^{-1}.

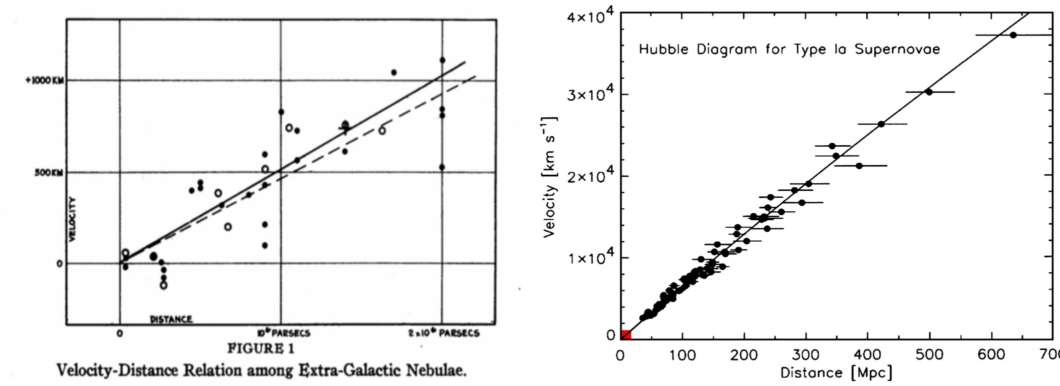


Figure 3: Hubble parameter.

On the other hand, the **deceleration parameter** $q(t)$, is defined by

$$q(t) \equiv -\frac{\ddot{R}(t)R(t)}{\dot{R}^2(t)}. \tag{35}$$

As the name suggests, it describes whether the expansion of the universe is slowing down ($q > 0$) or speeding up ($q < 0$). If the Taylor expansion keeps on going there come out several parameters, for instance the next two ones are

$$\text{jerk}: \qquad j \equiv \frac{R^2\dddot{R}}{\dot{R}^3}, \qquad \text{and} \qquad \text{snap} \qquad s \equiv \frac{R^3\ddddot{R}}{\dot{R}^4}.$$

The coefficients in the power series of the expansion of the scale factor are known as the *cosmography*; see for instance [1].

Now, let us write the redshift parameter in terms of the *look-back* time $t - t_0$. First, we expand the inverse of the scale factor (using expression (33)):

$$\frac{R(t_0)}{R(t)} = \left[1 - (t_0 - t)H_0 - \frac{1}{2}(t_0 - t)^2 q_0 H_0^2 - \cdots\right]^{-1} \approx [1 - \delta x]^{-1} \tag{36}$$

$$\approx 1 + (t_0 - t)H_0 + \frac{1}{2}(t_0 - t)^2 q_0 H_0^2 + (t_0 - t)^2 H_0^2 - \cdots . \tag{37}$$

assuming $|t_0 - t| \ll t_0$ (very close to today). Then, we have

$$z = \frac{R(t_0)}{R(t)} - 1 = (t_0 - t)H_0 + (t_0 - t)^2\left(1 + \frac{1}{2}q_0\right)H_0^2 + \cdots . \tag{38}$$

Since z is an absolute quantity (observable), then the look-back time $t_0 - t$ can be written in terms of z. For $z \ll 1$, from the above equation, we have

$$(t_0 - t)H_0 = z - (t_0 - t)^2 \left(1 + \frac{1}{2}q_0\right) H_0^2 + \cdots . \tag{39}$$

and using the fact, at first order that $(t_0 - t)H_0 \approx z$, therefore

$$t_0 - t = H_0^{-1} z - H_0^{-1} \left(1 + \frac{1}{2}q_0\right) z^2 + \cdots . \tag{40}$$

The approximations depend only on the present-day values of H_0 and q_0, and no knowledge of the complete expansion history $R(t)$ of the universe.

On the other hand, the radial χ coordinate (Eq. **??**) of the emitting galaxy

$$\chi = \int_t^{t_0} \frac{c\ dt}{R(t)} = c\ R_0^{-1} \int_t^{t_0} [1 - (t_0 - t)H_0 + \cdots]^{-1} dt, \tag{41}$$

assuming $|t_0 - t| \ll t_0$, expanding the terms and then integrating, we have

$$\chi = c\ R_0^{-1}[(t_0 - t) + \frac{1}{2}(t_0 - t)^2 H_0 + \cdots]. \tag{42}$$

using the expression (40), and assuming $z \ll 1$,

$$\chi = \frac{c}{R_0 H_0}[z - \frac{1}{2}(1 + q_0)z^2 + \cdots], \tag{43}$$

which only depends on H_0 and q_0 and not on the full expansion $R(t)$.

The proper distance d_p of the emitting galaxy at cosmic time t_0 is $d \equiv R(t_0)\chi$, thus for nearby galaxies $d \approx cz/H_0$. Moreover, using that the cosmological redshift can be written as a Doppler shift due to recession velocity v of the emitting galaxy

$$v \equiv cz = H_0 d.$$

The galaxies appear to recede from us with a recesion speed proportional to their distance: *Hubble's law*. The Hubble constant has the dimensions of the inverse time and $1/H_0$ gives the age of the universe. It is important not to confuse the expansion redshift with a kinematic redshift. Also the redshift, taking into account relativistic velocities, becomes

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{44}$$

Combining the derivative of Eqn. (33) and its inverse to get an expression for the Hubble parameter about the present epoch t_0:

$$H(z) = H_0[1 + (1 + q_0)z - \cdots] \tag{45}$$

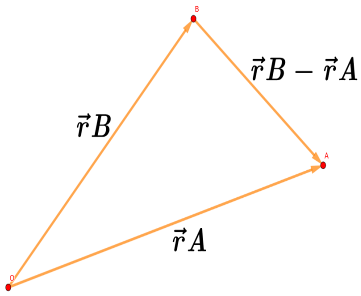
Example 0.1.7: Hubble expansion

The Hubble expansion is a natural property of a homogeneous an isotropic universe. All observers see galaxies with the same Hubble law. For example, consider two observers/-galaxies

$$\vec{v}_A = H_0 \cdot \vec{r}_A, \qquad \vec{v}_B = H_0 \cdot \vec{r}_B, \tag{46}$$

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = H_0\vec{r}_B - H_0\vec{r}_A = H_0(\vec{r}_B - \vec{r}_A). \tag{47}$$

In a homogeneous universe every particle moving with the substratum has a purely radial velocity proporcional to its distance from the observer. Quiz: what would happen if Hubble would have found the velocity behaves differently, i.e. $v = H_0 r^2$?



0.1.4 Integrales

In [1]:
```
import numpy as np
from sympy import *
from gravipy import *
```

$$D = \int_0^R \frac{a}{(a^2 - \rho^2)^{\frac{1}{2}}} d\rho$$

In [11]:
```
init_printing()

a, rho, R = symbols ('a, \\rho, R', positive=True) #Asignamos nuestros simbolos a l
e = Rational(1,2) #Al no poder poner el simbolo 1/2, utilizamo esta forma para pode

D = a / (a**2 - rho**2)**e #La funcion que vamos a integrar
```

```
integrate(D,(rho,0,R))
#integrate(D,rho)
```

Out[11]:

$$a \operatorname{asin}{\left(\frac{R}{a} \right)}$$

$$C = \int_0^{2\pi} R d\phi$$

In [5]:
```
phi = symbols ('\\phi')

C = R

integrate(C,(phi,0,2*pi))
```

Out[5]:

$$2\pi R$$

$$A = \int_0^{2\pi} \int_0^R \frac{a}{(a^2 - \rho^2)^{\frac{1}{2}}} \rho d\rho d\phi$$

In [149]:
```
A = a / (a**2 - rho**2)**e*rho

 simplify(integrate(A,(rho,0,R),(phi,0,2*pi)))
```

Out[149]:

$$2\pi a \left(-\sqrt{-R^2 + a^2} + \sqrt{a^2}\right)$$

$$t = \frac{1}{H_0} \int_0^a [\frac{x}{\Omega_{m,0} + (1 - \Omega_{m,0})x}]^{\frac{1}{2}} dx$$

In [42]:
```
H_0, Omega, x, a = symbols ('H_0, \\Omega_{m0}, x, a')

t =  (1/H_0) * (x / (Omega + (1-Omega)*x))**e

t_1 = t.subs(Omega,1)

integrate(t_1,(x,0,a))
```

Out[42]:

$$\frac{2a^{\frac{3}{2}}}{3H_0}$$

In [151]:
```
#Para Omega > 1

H_0, Omega, x, a = symbols ('H_0, \\Omega_{m0}, x, a')
psi = symbols ('psi')

x1 = Omega / (Omega - 1)*sin(psi/2)**2 # con [0/pi] llamamos nuestra variable x1 q

t_x1 = (1/H_0) * (x / (Omega + (1-Omega)*x))**e

t = factor(t_x1.subs(x,x1))

t

#integrate(t,(psi,0,pi))
```

Out[151]:

$$\frac{\sqrt{\frac{\sin^2\left(\frac{\psi}{2}\right)}{-\Omega_{m0}\sin^2\left(\frac{\psi}{2}\right)+\Omega_{m0}+\sin^2\left(\frac{\psi}{2}\right)-1}}}{H_0}$$

In [154]:
```
# Para Omega < 1
H_0, Omega, x, a = symbols ('H_0, \\Omega_{m0}, x, a')
psi = symbols ('psi')

x2 =  Omega / (1 - Omega) *sinh (psi/2)**2

t_x2 = (1/H_0) * (x / (Omega + (1-Omega)*x))**e

t = factor(t_x2.subs(x,x2))

t

#integrate(t,(psi,0,pi))
```

Out[154]:

$$\frac{\sqrt{\frac{\sinh^2\left(\frac{\psi}{2}\right)}{-\Omega_{m0}\sinh^2\left(\frac{\psi}{2}\right)-\Omega_{m0}+\sinh^2\left(\frac{\psi}{2}\right)+1}}}{H_0}$$

$$t = \frac{1}{H_0}\int_0^a \frac{x}{\sqrt{\Omega_{r,0} + (1-\Omega_{r,0})x^2}}$$

```
In [126]: H_0, Omega_r0, x, a = symbols ('H_0, \\Omega_{r0}, x, a')

          t = 1/H_0* x/sqrt((Omega_r0 + (1 - Omega_r0)*x**2))

          t_1 = t.subs(Omega_r0,1)

          integrate(t_1,(x,0,a))
```

Out[126]:

$$\frac{a^2}{2H_0}$$

```
In [156]: # Para Omega < 1

          H_0, Omega_r0, x, a = symbols ('H_0, \\Omega_{r0}, x, a')

          t = 1/H_0* x/sqrt((Omega_r0 + (1 - Omega_r0)*x**2))

          integrate(t,(x,0,a))
```

Out[156]:

$$-\frac{\sqrt{\Omega_{r0}}\sqrt{1+\frac{a^2 \operatorname{polar\_lift}(-\Omega_{r0}+1)}{\Omega_{r0}}}}{H_0(\Omega_{r0}-1)} + \frac{\sqrt{\Omega_{r0}}}{H_0(\Omega_{r0}-1)}$$

$$t = \frac{1}{H_0}\int_0^a \frac{x}{\sqrt{\Omega_{m,0}x + \Omega_{r,0}}}dx$$

```
In [144]: H_0, Omega_m, Omega_r, x, a = symbols ('H_0, \\Omega_{m0}, \\Omega_{r0} x, a')

          #haciendo
          y = Omega_m * x + Omega_r

          t = 1/H_0 * (x/(sqrt(y)))

          t_1 = simplify(integrate(t,(x,0,a)))

          factor (t_1)
```

Out[144]:

$$\frac{2\sqrt{\Omega_{r0}}\left(\Omega_{m0}a\sqrt{\frac{\Omega_{m0}a}{\Omega_{r0}}+1}-2\Omega_{r0}\sqrt{\frac{\Omega_{m0}a}{\Omega_{r0}}+1}+2\Omega_{r0}\right)}{3H_0\Omega_{m0}^2}$$

$$t=\frac{1}{H_0}\int_0^a\sqrt{\frac{x}{1-\Omega_{\Lambda,0}+\Omega_{\Lambda,0}x^3}}dx$$

In [191]:
```
H_0, Omega_l, x, a , y = symbols ('H_0, \\Omega_{\\Lambda0},x, a ,y')


t = 1/H_0*sqrt(x/(1-Omega_l + Omega_l * x**3))

#Haciendo

y2 = x**3*abs(Omega_l)/(1-Omega_l)

Ht = 2/(3*(abs(Omega_l)))*1/(sqrt(1 + y**2))

integrate(Ht,(y,0,y2))
```

Out[191]:

$$\frac{2\,\mathrm{asinh}\left(\frac{x^3|\Omega_{\Lambda0}|}{-\Omega_{\Lambda0}+1}\right)}{3\,|\Omega_{\Lambda0}|}$$

In [192]:
```
H_0, Omega_l, x, a , y = symbols ('H_0, \\Omega_{\\Lambda0},x, a ,y')


t = 1/H_0*sqrt(x/(1-Omega_l + Omega_l * x**3))

#Haciendo

y2 = x**3*abs(Omega_l)/(1-Omega_l)

Ht = 2/(3*(abs(Omega_l)))*1/(sqrt(1 - y**2))

integrate(Ht,(y,0,y2))
```

Out[192]:

$$\frac{2\,\mathrm{asin}\left(\frac{x^3|\Omega_{\Lambda0}|}{-\Omega_{\Lambda0}+1}\right)}{3\,|\Omega_{\Lambda0}|}$$

In [ ]:

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