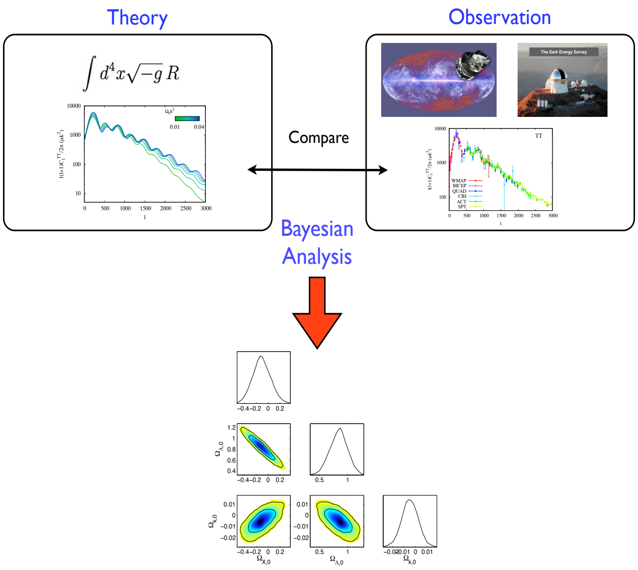
Updated Cosmology

with Python



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0.1 The spacetime metric $g_{\mu\nu}$

The most general expression for the metric $g_{\mu\nu}$ can be represented by a sequence of non-intersecting spacelike hypersurfaces labelled by some parameter t, see Figure 1.

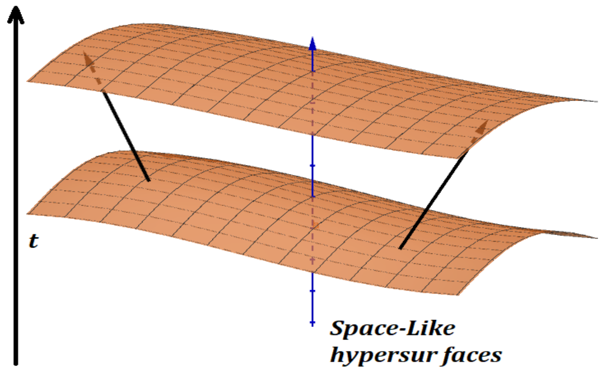

Figure 1: Worldlines.

This parameter may be taken to be the **proper time** along with the worldline of any fundamental observer. The parameter t is then called the *synchronous time coordinate*. In addition, we may also introduce spatial coordinates (x^1, x^2, x^3) that are *constant* along any worldline. Thus, each fundamental observer has fixed (x^1, x^2, x^3) coordinates, and so the latter are called *comoving coordinates*, see for instance Figure 2.

Then the line element takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}c^2dt^2 + 2g_{0i}c\ dtdx^i + g_{ij}dx^i dx^j, \tag{1}$$

where the components of the spatial metric g_{ij} are functions of the coordinates (ct, x^1, x^2, x^3). Because the hypersurfaces $t =$ constant may be naturally constructed in such a way that the 4-velocity of any fundamental observer is orthogonal to the hypersurfaces, then the term g_{0i} must be zero. On the other hand, we may use the proper time of the coordinate system given by the fundamental observers to label the spacelike hypersurfaces (see Figure 1). This choice of coordinate time implies that $g_{00} = 1$, and therefore the space-time interval becomes

$$ds^2 = c^2dt^2 - g_{ij}dx^i dx^j. \tag{2}$$

0.2 The Friedmann-Robertson-Walker metric

Let us now incorporate the postulates of homogeneity and isotropy to the geometry of the Universe. The former demands that all points on a particular spacelike hypersurface are equivalent, whereas the latter demands that all directions on the hypersurface are equivalent for fundamental observers.

Assumption 0.2.1:

Isotropy requires that the distribution of galaxies at two different times must be similar, and **homogeneity** requires that the magnification factor must be independent of the position for the distribution.

It thus follows then, that the time t can enter the g_{ij} only through a common factor, and hence the metric must take the following form

$$ds^2 = c^2 dt^2 - S^2(t) d\sigma^2, \tag{3}$$

where $S(t)$ is a time-dependent scale factor (length-dimensions) and $d\sigma^2 = \gamma_{ij} dx^i dx^j$ contains functions of the coordinates (x^1, x^2, x^3) only. As we will see in the following section, the physical distance (r) is proportional to the comoving distance (x) times the scale factor $S(t)$ and hence it gets larger as time evolves. See Figure 2.

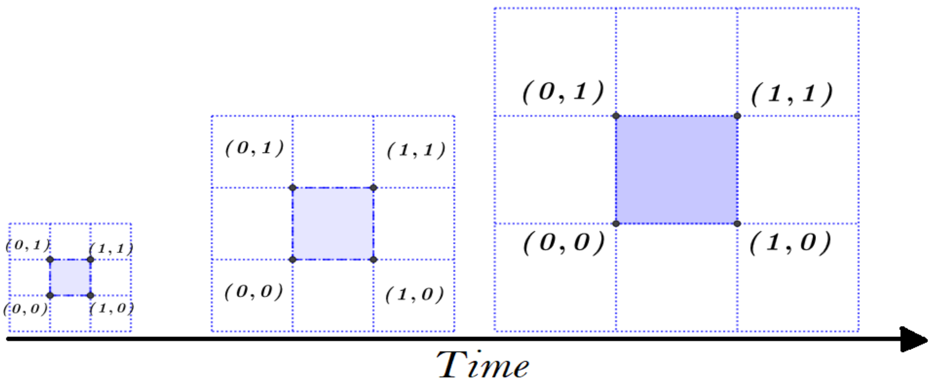

Figure 2: Comoving coordinates.

Over the years, cosmological observations have provided decisive evidence that the universe is currently expanding, therefore the scale factor satisfies $\dot{S}(t) > 0$ as we shall come back later, see Hubble [3], Perlmutter et al. [4], Riess et al. [5].

On the other hand, a *maximally symmetric space* is specified by just one number – the *curvature* K –, which is independent of the coordinates. Such *constant curvature* spaces must be homogeneous and isotropic: the key property we are looking for to describe the Universe at large scales.

Example 0.2.1: Maximally symmetric spaces

A spacetime spatially homogeneous and spatially isotropic is defined as a **maximally symmetric space**. Such space possesses the maximum number of isometries, generated by the Killing vectors, which in an n-dimensional manifold equals $n(n+1)/2$. The following holds for such spaces:

1.- The scalar curvature R is constant, i.e.

$$R = n(n-1)K.$$

2.- The Ricci tensor is proportional to the metric tensor, i.e.

$$R_{\mu\nu} = \frac{1}{n} R g_{\mu\nu}.$$

3.- The Riemman curvature tensor is given by

$$R_{\mu\nu\lambda\rho} = \frac{R}{n(n-1)} (g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho}).$$

Assuming general static isotropy, the line element of an isotropic 3-space in spherical coordinates (r, θ, ϕ) can be written as

$$d\sigma^2 = B(r) dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

and its scalar curvature ${}^3R$ is computed to be

$${}^3R = \frac{2}{r^2} \frac{d}{dr} \left[r \left(1 - \frac{1}{B(r)} \right) \right]. \tag{5}$$

Homogeneity implies that all geometrical properties are independent of r and therefore ${}^3R$ must be constant. That is, equating Eq. (5) to a constant value $6K$ and integrating the result, this yields to the expression

$$r \left(1 - \frac{1}{B} \right) = K r^3 + c, \tag{6}$$

with K and c constants. In order to avoid any singularity at $r = 0$ is compulsory to select $c = 0$ and therefore $B(r) = (1 - K r^2)^{-1}$.

Example 0.2.2: An isotropic 3-space.

For the line element $d\sigma^2 = B(r)dr^2 + r^2 d\Omega^2$ the components of the metric tensor are, along with the inverse components:

$$g_{ab} = \begin{pmatrix} B(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad g^{ab} = \begin{pmatrix} \frac{1}{B^2(r)} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}.$$

By using the symmetric properties of the Christoffel symbols, the identity (**??**), that for three different indices (i.e. $\Gamma^r{}_{\theta\phi}$) the symbols are null, thus the non zero components are:

$$\begin{aligned} \Gamma^r{}_{rr} &= \frac{1}{2B(r)}\frac{dB(r)}{dr}, \qquad \Gamma^r{}_{\theta\theta} = -\frac{r}{B(r)}, \qquad \Gamma^r{}_{\phi\phi} = -\frac{r\sin^2\theta}{B(r)}. \\ \Gamma^\theta{}_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi{}_{\phi\theta} = \cot\theta. \end{aligned}$$

The Riemann tensor components

$$\begin{aligned}
R^r{}_{\theta r\theta} &= \Gamma^r{}_{\theta\theta,r} - \Gamma^r{}_{r\theta,\theta} + \Gamma^r{}_{r\lambda}\Gamma^\lambda{}_{\theta\theta} - \Gamma^r{}_{\theta\lambda}\Gamma^\lambda{}_{r\theta} = \frac{r}{2B^2(r)}\frac{dB(r)}{dr}. \\
R^r{}_{\phi r\phi} &= \Gamma^r{}_{\phi\phi,r} - \Gamma^r{}_{r\phi,\phi} + \Gamma^r{}_{r\lambda}\Gamma^\lambda{}_{\phi\phi} - \Gamma^r{}_{\phi\lambda}\Gamma^\lambda{}_{r\phi} = \frac{r\sin^2\theta}{2B^2(r)}\frac{dB(r)}{dr}. \\
R^\theta{}_{r\theta r} &= \Gamma^\theta{}_{rr,\theta} - \Gamma^\theta{}_{\theta r,r} + \Gamma^\theta{}_{\theta\lambda}\Gamma^\lambda{}_{rr} - \Gamma^\theta{}_{r\lambda}\Gamma^\lambda{}_{\theta r} = \frac{1}{2rB(r)}\frac{dB(r)}{dr}. \\
R^\theta{}_{\phi\theta\phi} &= \Gamma^\theta{}_{\phi\phi,\theta} - \Gamma^\theta{}_{\theta\phi,\phi} + \Gamma^\theta{}_{\theta\lambda}\Gamma^\lambda{}_{\phi\phi} - \Gamma^\theta{}_{\phi\lambda}\Gamma^\lambda{}_{\theta\phi} = \sin^2\theta\left(1 - \frac{1}{B(r)}\right). \\
R^\phi{}_{r\phi r} &= \Gamma^\phi{}_{rr,\phi} - \Gamma^\phi{}_{\phi r,r} + \Gamma^\phi{}_{\phi\lambda}\Gamma^\lambda{}_{rr} - \Gamma^\phi{}_{r\lambda}\Gamma^\lambda{}_{\phi r} = \frac{1}{2rB(r)}\frac{dB(r)}{dr}. \\
R^\phi{}_{\theta\phi\theta} &= \Gamma^\phi{}_{\theta\theta,\phi} - \Gamma^\phi{}_{\phi\theta,\theta} + \Gamma^\phi{}_{\phi\lambda}\Gamma^\lambda{}_{\theta\theta} - \Gamma^\phi{}_{\theta\lambda}\Gamma^\lambda{}_{\phi\theta} = 1 - \frac{1}{B(r)}.
\end{aligned}$$

The non-null Ricci tensor components:

$$R_{rr} = \frac{1}{rB}\frac{dB}{dr}, \qquad R_{\theta\theta} = 1 - \frac{1}{B} + \frac{r}{2B^2}\frac{dB}{dr}, \qquad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta.$$

Finally, the curvature scalar:

$$R = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{2}{r^2}\frac{d}{dr}\left[r\left(1 - \frac{1}{B(r)}\right)\right].$$

Considering $R_{jk} = 2Kg_{jk}$

$$\frac{1}{rB}\frac{dB}{dr} = 2KB(r), \qquad \& \qquad 1 - \frac{1}{B} + \frac{r}{2B^2}\frac{dB}{dr} = 2Kr^2.$$

Integrating the first equation

$$B(r) = \frac{1}{A - Kr^2}.$$

Substituting into the second expression, then it gives $1 - A + Kr^2 = Kr^2$, from which we see that $A = 1$. Thus, we have constructed the line element for the maximally symmetric 3-space.

Finally, with the previous results we have that the spatial part of the metric is written as

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2, \tag{7}$$

where r is the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the 2-sphere. Notice it has a similar form as the metric for a 3-sphere embedded in four-dimensional Euclidean space. The metric contains a 'hidden symmetry', since the origin of the radial coordinate is completely arbitrary. We can choose any point in this space as our origin since all points are equivalent. There is no centre in this space.

HW 0.2: Take metric (4) and compute Christoffel and Riemann to get (7). Make sure you do $R_{ij} = 2Kg_{ij}$.

Plugging everything together into (3), we get the Friedmann-Robertson-Walker metric

$$ds^2 = c^2 dt^2 - S^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]. \tag{8}$$

Let us assume that $K \neq 0$, then we can define the variable $\tilde{k} = K/|K|$ such that $\tilde{k} = \pm 1$ depending on the sign of K. Moreover we introduce the rescale coordinate

$$\tilde{r} = |K|^{1/2} r, \tag{9}$$

so Eqn. (8) becomes

$$ds^2 = c^2 dt^2 - \frac{S^2(t)}{|K|} \left[\frac{d\tilde{r}^2}{1 - \tilde{k}\tilde{r}^2} + \tilde{r}^2 d\Omega^2 \right]. \tag{10}$$

and then, introducing the rescaled function $R(t)$ by (we keep $R(t)$ as the factor, as it does contain the units and the coordinates are still comoving [dimensionless])

$$R(t) = \begin{cases} \frac{S(t)}{|K|^{1/2}} & \text{if } K \neq 0, \\ S(t) & \text{if } K = 0. \end{cases} \tag{11}$$

Eqn. (10) yields to

$$ds^2 = c^2 dt^2 - R^2(t) \left[\frac{d\tilde{r}^2}{1 - \tilde{k}\tilde{r}^2} + \tilde{r}^2 d\Omega^2 \right]. \tag{12}$$

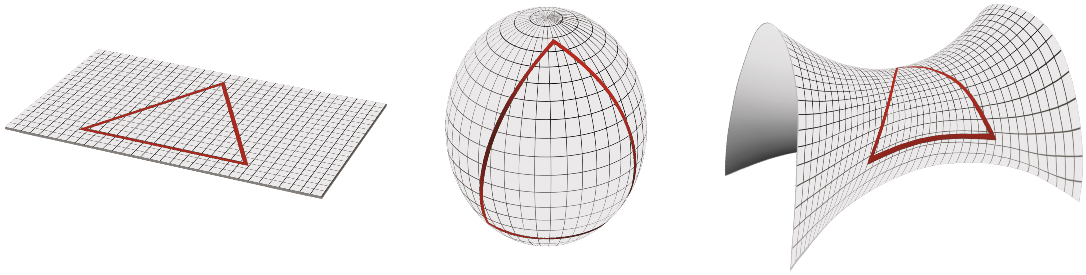

Figure 3: Three curvatures allowed for an Isotropic and Homogeneous space-time.

The constant $\tilde{k}$ classifies the curvature of the spatial sections, with closed (S^3), flat (R^3) and open (H^3) universes corresponding to $\tilde{k} = +1, 0, -1$, respectively (see Figure 3).

Example 0.2.3: Curvature metrics.

For the flat case $\tilde{k} = 0$ the spatial metric is

$$\begin{aligned} d\sigma^2 &= dr^2 + r^2 d\Omega^2 \\ &= dx^2 + dy^2 + dz^2, \end{aligned}$$

which is simply a flat Euclidean space.
For the closed case $\tilde{k} = +1$ we can define $r = \sin\chi$ to write the metric as

$$d\sigma^2 = d\chi^2 + \sin^2\chi d\Omega^2,$$

which is the metric of a three-sphere.
In the open $\tilde{k} = -1$ case we can set $r = \sinh\chi$ to obtain

$$d\sigma^2 = d\psi^2 + \sinh^2\psi d\Omega^2.$$

This is the metric for a three-dimensional space of constant negative curvature.

Notice the line element (12) has a rescaling symmetry, that leaves the metric invariant

$$R \to \lambda R, \qquad \tilde{r} \to r/\lambda, \qquad \tilde{k} \to \lambda^2 k. \tag{13}$$

A convenient form to express the FRW metric is by choosing the rescaling factor as $\lambda = 1/R_0$. That is, using coordinates normalised to the present time, labelled with subscript '0', to defined *the normalised scale factor*

$$a(t) \equiv \frac{R(t)}{R_0}. \tag{14}$$

Curvature	Geometry	Angles of triangle	circumference of circle	Type of Universe
$k > 0$	Spherical	$> 180°$	$c < 2\pi r$	Closed
$k = 0$	Flat	$180°$	$c = 2\pi r$	Flat
$k > 0$	Hyperbolic	$< 180°$	$c > 2\pi r$	Open

Table 1: A Summary of possible geometries

Therefore the scale factor is set to unity today $a_0 \equiv a(t_0) \equiv 1$, $\tilde{r} \to R_0 r$ has units of [length] and the curvature parameter $\tilde{k} \to k/R_0^2$ has dimensions of $[\text{length}]^{-2}$. Note that in this case, k can take any value and not just be restricted to $\{+1, 0, -1\}$. The general properties of these three spaces can be summarised in Table 1. The *general FRW metric* written in terms of the normalised scale factor $a(t)$ is thus given by

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right]. \tag{15}$$

In general and throughout this book, we will use the metric (12) but dropping the tilde for convenience.

0.2.1 Geometric properties of the FRW metric

The physical meaning of the curvature term becomes more apparent by redefining the **radial coordinate** $d\chi \equiv dr/\sqrt{1 - kr^2}$ in the metric (12), that leads to

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S_k^2(\chi) d\Omega^2\right], \tag{16}$$

where the function $S_k(\chi)$ is specified by the curvature term:

$$S_k(\chi) = \begin{cases} \sin\chi, & \text{for} \quad k = 1 \quad \text{(closed universe)} \\ \chi, & \text{for} \quad k = 0 \quad \text{(flat universe)} \\ \sinh\chi, & \text{for} \quad k = -1 \quad \text{(open universe)} \end{cases} \tag{17}$$

where the comoving coordinates remained. When using the symmetry shown above they get units by $\chi \to \chi/\lambda$ and $S_k^2 \to S_k^2/\lambda$.
The comoving radial χ-coordinate, on a null geodesic ($ds^2 = 0$), is computed from

$$\chi = \int \frac{c\ dt}{R(t)}. \tag{18}$$

The form of the metric (16) is particularly convenient to study the propagation of light. For this purpose, it is useful to introduce the **conformal time**:

$$d\eta = \frac{c\ dt}{R(t)}. \tag{19}$$

so that (16) becomes

$$ds^2 = \underbrace{R^2(\eta)}_{\text{Conformal}}\ \underbrace{\left[d\eta^2 - (d\chi^2 + S_k^2(\chi)d\Omega^2)\right]}_{\text{Minkowski}}. \tag{20}$$

We notice the presence of the static Minkowski space multiplied by a conformal factor $R^2(\eta)$. Because light moves along null geodesics, $ds^2 = 0$, the propagation of light in a FRW is the same as in Minkowski space firstly transformed to conformal time, and along the path we have

$$d\eta = d\chi. \tag{21}$$

Therefore the dynamics of the space-time, in a homogeneous and isotropic universe, reduces to determining the scale factor $R(t)$, which is computed from Einstein's equations once the matter content is specified, as we shall see below.

Introduction

In the Λ-CDM model a basic assumption is given by the cosmological principle, which establishes that the Universe where we live is **homogeneous** and **isotropic** at large scales. However, in this paper some models that do not meet those requirements are discussed.

Non-isotropic cosmological models

These type of models are characterised for being homogeneous but not necessarily isotropic in its spatial part, therefore they can be seen as generalisation of the FLRW Universe. Among the most famous are those known as Bianchi models, they are described by the metric (in natural units)[1]:

$$ds^2 = -dt^2 + a_x(t)dx^2 + a_y(t)dy^2 + a_z(t)dz^2.$$

Supposing a comoving test particle in this solution, it will follow the geodesic where (x, y, z) keep constant, however, since the scale factor is different in each direction, its volume and shape could be modified in general.

In order to test this models according to the experimental data, several calculation of nucleosynthesis and CMB anisotropies have been realised, nevertheless, the results have shown that these models are inconsistent with some cosmological parameters, ergo, they are usually only considered as toy models that are tractable exact solutions of Einstein's field equations.

Non-homogeneous cosmological models

These inhomogeneous models are those exact solutions of Einsteins equations that in analogy with the non-isotropic ones, can reproduce the FLRW solution as a limit. There are several proposals in these directions, where the best known are [2]:

(a) The Szekeres - Szafron family: These models are characterised by the metric:

$$ds^2 = dt^2 - e^{2\alpha(t,x,y,r)}dr^2 - e^{2\beta(t,x,y,r)}(dx^2 + dy^2),$$

and they meet the following properties

- They obey the Einstein equations with a perfect fluid source.
- The flow-lines of the perfect fluid are geodesic and nonrotating.

– The hypersurfaces orthogonal to the flow-lines are conformally flat.

– The Ricci tensor of those hypersurfaces has two of its eigenvalues equal. 5. The shear tensor has two of its eigenvalues equal.

(b) The Lemaitre model: This describes a spherically symmetric inhomogeneous fluid with anisotropic pressure. In comoving coordinates it has the following form.

$$ds^2 = e^{A(t,r)}dt^2 - e^{B(t,r)}dr^2 - R^2(t,r)(d\upsilon^2 + \sin^2 \upsilon d\phi^2).$$

(c) In the special case of dust with the cosmological constant, the above model reproduces the Lemaitre-Tolman (LT) model which is described by the metric

$$ds^2 = dt^2 - \frac{R,_r^2}{1+2E}dr^2 - R^2(t,r)(d\upsilon^2 + \sin^2 \upsilon d\phi^2).$$

Some non-standard cosmological models have been discussed. It is worth mentioning that these models can be seen as a generalisation of the FLRW solution which provides a good phenomenological landscape, and since the existence of gravitational lenses, we know that we do not live in a FLRW Universe, thus, considering more general and tractable exact solutions to Einstein's equations is very important.

Bibliography

[1] Scientific American. Could silicon be the basis for alien life forms, just as carbon is on earth? https://www.scientificamerican.com/article/could-silicon-be-the-basi/, 1998. 9

[2] Krzysztof Bolejko, Marie-Noëlle Célérier, and Andrzej Krasiński. Inhomogeneous cosmological models: exact solutions and their applications. Classical and Quantum Gravity, 28(16): 164002, aug 2011. doi: 10.1088/0264-9381/28/16/164002. URL https://dx.doi.org/10.1088/0264-9381/28/16/164002. 9

[3] Edwin Hubble. A relation between distance and radial velocity among extra-galactic nebulae. Proceedings of the National Academy of Sciences, 15(3):168–173, 1929. doi: 10.1073/pnas.15.3.168. URL http://www.pnas.org/content/15/3/168.short. 3

[4] S. Perlmutter, G. Aldering, G. Goldhaber, and et al. Measurements of Ω and Λ from 42 High-Redshift Supernovae. The Astrophysical Journal, 517(2):565, 1999. URL http://stacks.iop.org/0004-637X/517/i=2/a=565. 3

[5] Adam G. Riess, Alexei V. Filippenko, Peter Challis, and et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. The Astronomical Journal, 116(3):1009, 1998. URL http://stacks.iop.org/1538-3881/116/i=3/a=1009. 3