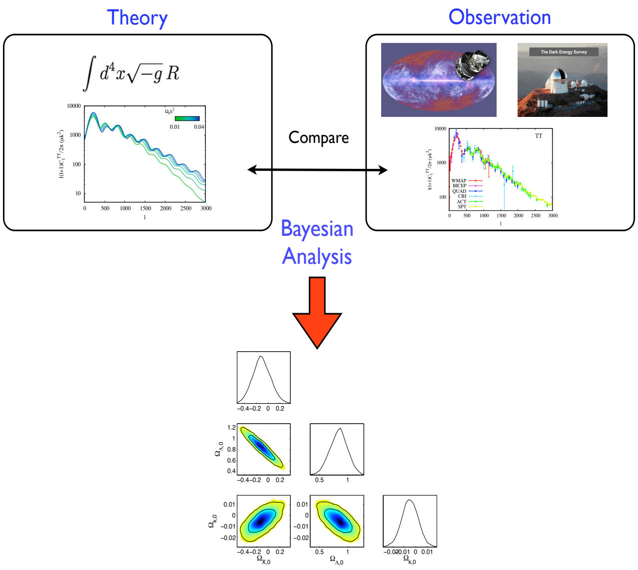
Updated Cosmology

with Python



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In progress

August 12, 2017

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CMB

1. Cosmic Microwave Background – Observational cosmology, history.
   Recombination, Decoupling, Last Scattering – Pettini
   Black body radiation

2. Stats
   Perturbations
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The cosmic microwave background (CMB) is the thermal radiation left over from the "Big Bang", also known as "relic radiation". It is fundamental to observational cosmology because it is the oldest light in the universe, dating to the epoch of recombination. With a traditional optical telescope, the space between stars and galaxies (the background) is completely dark. However, a sufficiently sensitive radio telescope shows a faint background glow, almost exactly the same in all directions, that is not associated with any star, galaxy, or other object. This glow is strongest in the microwave region of the radio spectrum.

The CMB is a snapshot of the oldest light in our Universe, imprinted on the sky when the Universe was just 380,000 years old. It shows tiny temperature fluctuations that correspond to regions of slightly different densities, representing the seeds of all future structure: the stars and galaxies of today.

1.1 Isotropic CMB

The Cosmic Microwave Background radiation was discovered in 1965 by two American radio astronomers, Arno Penzias and Robert Wilson, while trying to identify sources of noise in microwave satellite communications at Bell Laboratories in New Jersey. Their discovery was announced alongside the interpretation of the CMB as relic thermal radiation from the Big Bang by Robert Dicke and collaborators working at the nearby Princeton University. Interestingly, the possibility of a cosmic thermal background were first entertained by Gamow, Alpher and Herman in 1948 as a consequence of Big Bang nucleosynthesis, but the idea was so beyond the experimental

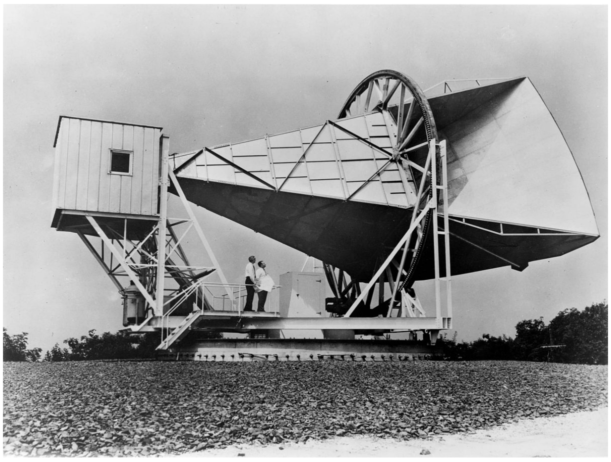

Figure 1.1: Discovery of the CMB.

The original detection by Penzias and Wilson was at a wavelength of 73.5 mm, this being the wavelength of the telecommunication signals they were working with; this wavelength is two orders of magnitude longer than $\lambda_{\text{peak}} = 1.1\text{mm}$ of a $T = 2.7255\text{K}$ blackbody.

At any angular position (θ, ϕ) on the sky, the spectrum of the CMB is a near-perfect blackbody (see Figure 1.2). The CMB is in fact the closest approximation we have to an ideal black-

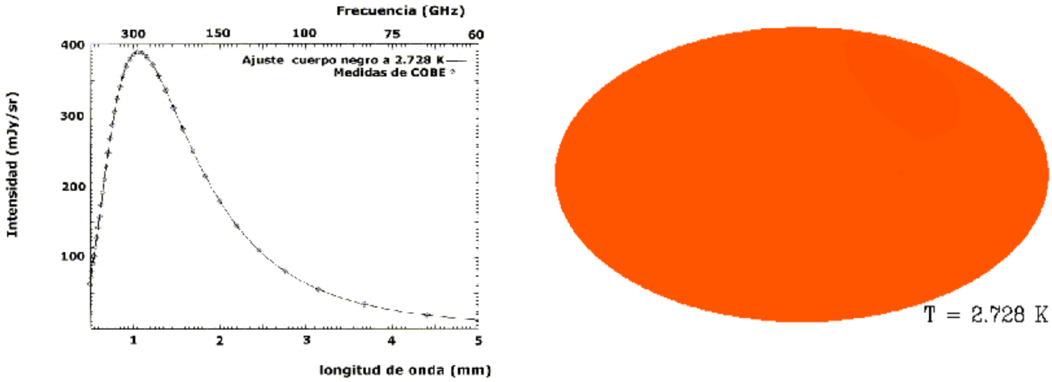

Figure 1.2: Blackbody radiation.

body. With $T(\theta, \phi)$ denoting the temperature at a given point on the sky, the mean temperature averaged over the whole sky is

$$\langle T \rangle = \frac{1}{4\pi} \int T(\theta, \phi) \sin\theta d\theta d\phi = 2.7255 \pm 0.0006K \tag{1.1}$$

The deviations from this mean temperature from point to point on the sky are tiny. Defining the dimensionless T fluctuations:

$$\frac{\delta T}{T}(\theta, \phi) = \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle} \tag{1.2}$$

is found that

$$\left\langle \left(\frac{\delta T}{T} \right) \right\rangle^{1/2} = 1.1 \times 10^{-5} \tag{1.3}$$

Such deviations were first reported in 1992 by the COBE team. Subsequent CMB missions (WMAP and Planck) have significantly improved the angular resolution and precision in the mapping of the CMB sky, as illustrated in Figure 1.3.

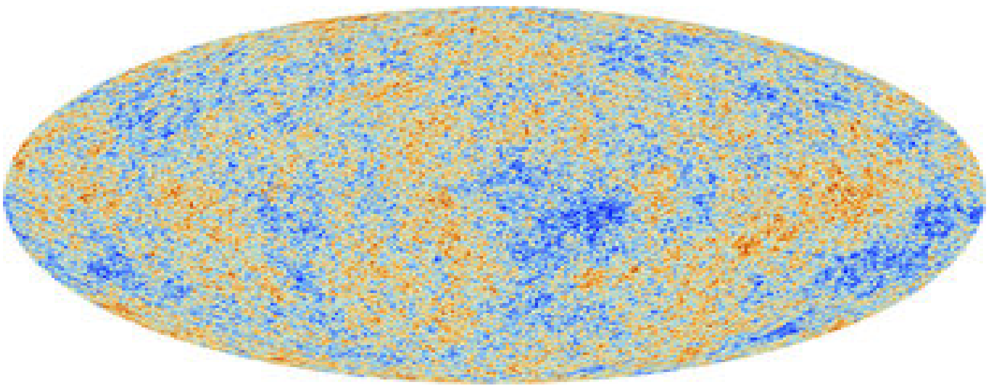

Figure 1.3: CMB seen by Planck.

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The Boltzmann equation

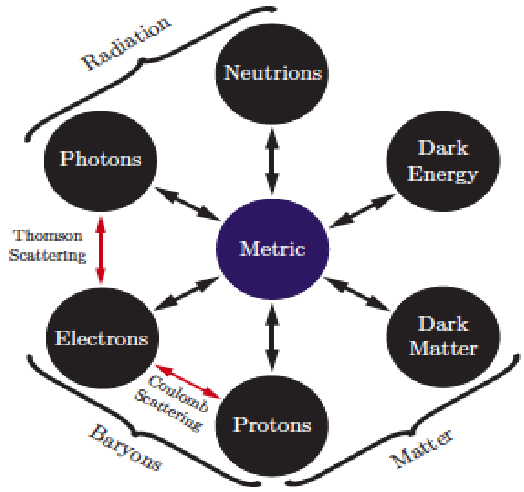

Figure 2.1: Interactions between the different forms of matter in the universe.

The Boltzmann equation

$$\frac{df}{d\eta} = C[f] \tag{2.1}$$

The expansion of the Universe is slow compared to the microwave frequency of the CMB. It is hence adiabatic, as far as the photons are concerned. The distribution function of the cosmic microwave background with temperature $\bar{T}$ is

$$\bar{f} = \left[\exp\left(\frac{E}{\bar{T}}\right) - 1\right]^{-1}. \tag{2.2}$$

We see that $\bar{f}$ depends just upon the energy E of a photon. Writing $T = T_0 a^{-1}$, we see that $\bar{f}$ is a function of aE only:

$$\bar{f}(aE) = \left[\exp\left(\frac{aE}{\bar{T}_0}\right) - 1\right]^{-1}. \tag{2.3}$$

To make the connection, we note that in General Relativity, the energy of a photon is given by

$$E = -u_\mu p^\mu, \tag{2.4}$$

and for observers in the unperturbed background at rest, i.e. $u_\mu = a(1,0,0,0)$, we haven then $E = -a\,p$, where $p \equiv |\mathbf{p}| = \sqrt{p^i p^j \delta_{ij}}$, in the background $p^0 = p$.

Them $\bar{f}$ depends solely of $P = a^2 p$, and therefore use P as an argument for f.

The full distribution function is naively a function of (x^μ, p^μ). Yet, the physics governing the evolution of f respects the mass shell condition $p_\mu p^\mu = m^2$. Therefore, f is a function $f(x^\mu, p^i)$ only. Use p^0 as a function of p^i right from the start. In order to do this, let us split the spatial momentum

$$p^i \equiv p n^i \tag{2.5}$$

into its magnitude p and the unit vector of photon momentum $\mathbf{n}$, so $\delta_{ij} n^i n^j = 1$. Hence, we arrive at our final set of variables for f

$$f = f(\eta, \mathbf{x}, P, \mathbf{n}) \tag{2.6}$$

The complete distribution function for each species can be split into background plus a perturbation part:

$$f(\eta, \mathbf{x}, P, \mathbf{n}) = \bar{f}(P) + F(\eta, \mathbf{x}, P, \mathbf{n}), \tag{2.7}$$

Useful relations, got them from $g_{\mu\nu} p^\mu p^\nu = 0$

$$p^0 = p(1 - \Psi - \Phi) \tag{2.8}$$

$$p_0 = -a^2 p(1 + [\Psi - \Phi]) \tag{2.9}$$

$$p_i = a^2 p n_i (1 - 2\Phi) \tag{2.10}$$

$$\sqrt{-g} = a^4(1 + \Psi - 3\Phi) \tag{2.11}$$

and $\mu = \frac{\mathbf{k}\cdot\mathbf{n}}{k}$, and a useful formula

$$\frac{\partial p}{\partial p^i} = \frac{\partial\sqrt{\delta_{mn} p^m p^n}}{\partial p^i} = \frac{1}{2}\frac{2p_i}{p} = n_i \tag{2.12}$$

2.1 Collisionless Part

The evolution of perturbations in the universe is quantified by the Boltzmann equation:

$$\left(\frac{\partial f}{\partial \eta}\right)_P + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \eta} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial \eta} + \frac{\partial f}{\partial n^i}\frac{\partial n^i}{\partial \eta} = C[f,G], \tag{2.13}$$

which relates the effects of gravity on the photon distribution function f to the rate of interactions with other species, given by the *collision term* $C[f,G]$. The previous distribution applies to polarization as well by simply replacing $F \to G$ (we use G to denote the linear polarization distribution function) and $\bar{f} = \bar{f}' \to 0$

On the Boltzmann equation the last term vanishes, because it is of second order in perturbation theory: $\bar{f}$ does not depend on n^i and hence $\partial f/\partial n^i$ is a perturbation. In addition $\partial n^i/\partial \eta$, is a change in photon direction that can only come from a spatially inhomogeneous scattering process. So all in all the last term is of second order and we can safely discard it.

The most difficult term is the third one.

$$\begin{aligned} \frac{\partial P}{\partial \eta} &= \frac{\partial}{\partial \eta} a^2 p && (2.14) \\ &= 2\frac{\dot{a}}{a}a^2 p + a^2 \frac{\partial p}{\partial \eta} && (2.15) \end{aligned}$$

and using the equation

$$\frac{\partial p}{\partial \eta} = \frac{\partial p}{\partial p^i}\frac{\partial p^i}{\partial \eta} = n^i \frac{\partial p^i}{\partial \eta} \tag{2.16}$$

The third term can be computed from the geodesic equation

$$p^0 \frac{\partial p^i}{\partial \eta} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta = 0 \tag{2.17}$$

then

$$n_i \frac{\partial p^i}{\partial \eta} = -(p^0)^{-1} n_i \Gamma^i_{\alpha\beta} p^\alpha p^\beta \tag{2.18}$$

Collecting all the terms, we have

$$n_i \Gamma^i_{\beta\gamma} p^\beta p^\gamma = 2\frac{a'}{a}p^0 p + p^2 \left[i\mu k(\Psi - \Phi) - 2\Phi'\right] \tag{2.19}$$

$$\frac{\partial f}{\partial P}\frac{\partial P}{\partial \eta} = -P\bar{f}_{,P}\{i\mu k[\Phi + \Psi] + 2\Phi'\}, \tag{2.20}$$

and the spatial part

$$\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \eta} = i\mu k F(\eta, \mathbf{x}, P, \mathbf{n}). \tag{2.21}$$

Collecting the terms involving the background only

$$\left(\frac{\partial f}{\partial \eta}\right)_P = 0 \tag{2.22}$$

The change in a distribution function of massless particles which depends solely on P is zero: the preservation of the background black body spectrum.

As far as the perturbed distribution is concerned, it is much more exciting:

$$\left(\frac{\partial F}{\partial \eta}\right)_P + i\mu k F - P\bar{f}_{,P}\{i\mu k[\Phi + \Psi] + 2\Phi'\} = C[f, G] \tag{2.23}$$

Finally, making the substitution $F \to G$, $\bar{f}' \to 0$, we get the simple evolution equation for the linear polarization G

$$\left(\frac{\partial G}{\partial \eta}\right)_P + i\mu k G = C_G[f, G] \tag{2.24}$$

where $C_G[f, G]$ is the collision term for G.

2.1.1 Perturbed temperature

Writing the temperature function T in terms of the photon *brightness temperature perturbation* $\Delta \equiv \Delta T/\bar{T}$, we have

$$T(\eta, \mathbf{x}, \mathbf{n}) = \bar{T}(\eta)[1 + \Delta(\eta, \mathbf{x}, \mathbf{n})], \tag{2.25}$$

$$\begin{aligned} f = \bar{f}\left(\frac{P}{1+\Delta}\right) &= \bar{f} + \frac{\partial \bar{f}}{\partial P}\left[\frac{P}{1+\Delta} - P\right] \\ &= \bar{f} + \frac{\partial \bar{f}}{\partial P}P\left(\frac{1}{1+\Delta} - 1\right) \\ &= \bar{f} + \frac{\partial \bar{f}}{\partial P}P(1 - \Delta - 1) \\ &= \bar{f} - \frac{\partial \bar{f}}{\partial P}P\Delta \end{aligned} \tag{2.26}$$

and therefore F and Δ are connected via

$$F(\eta, \mathbf{x}, P, \mathbf{n}) = -P\frac{\partial \bar{f}}{\partial P}\Delta(\eta, \mathbf{x}, \mathbf{n}). \tag{2.27}$$

So,

$$G(\eta, \mathbf{x}, P, \mathbf{n}) = -P\frac{\partial \bar{f}}{\partial P}\mathcal{Q}(\eta, \mathbf{x}, \mathbf{n}). \tag{2.28}$$

Then, the simplify Boltzmann equation becomes

$$\Delta' + ik\mu\Delta = -i\mu k[\Phi + \Psi] - 2\Phi' + \hat{C}[f, G] \tag{2.29}$$

where $\hat{C}[f, G] \equiv C[f, G]/(P\bar{f}_{,P})$

2.2 The Collision Term from Compton Scattering

The dominant term for the coupling of photons to the baryons is via inverse Compton scattering

$$e^-(\mathbf{q}) + \gamma(\mathbf{p}) \rightleftharpoons \mathbf{e}^-(\mathbf{q}') + \gamma(\mathbf{p}') \tag{2.30}$$

where we are interested how the photon distribution as a function of momentum p changes [Thomson scattering is the low-energy limit of Compton scattering]. The amplitude can be calculated from the Feynman rules.

$$C[f, G] = an_e\sigma_T\bar{f}_{,P}P\left\{i\mu v_b + \Delta(\eta, \mathbf{x}, \mathbf{n}) - \frac{1}{4}\int_{-1}^{1}\Delta(\eta, \mathbf{x}, \mathbf{n}')[P_2(\lambda)P_2(\mu) + 2]d\lambda\right. \tag{2.31}$$

$$\left. -\frac{1}{4}\int_{-1}^{1}\mathcal{Q}(\eta, \mathbf{x}, \mathbf{n}')P_2(\mu)[-2\sqrt{6\pi 5}{}_2Y_2^0(\lambda)]d\lambda\right\} \tag{2.32}$$

The expansion of the temperature perturbation (Δ) and polarisations (Q and U), in terms of the spherical harmonics $Y_l^m(\mathbf{n})$, are

$$\Delta(\eta, \mathbf{x}, \mathbf{n}) = \sum_l (-i)^l \Delta_l(k, \eta) P_l(\hat{\mathbf{k}} \cdot \mathbf{n}), \tag{2.33}$$

$$(Q \pm iU)(\eta, \mathbf{x}, \mathbf{n}) = \sum_{l=2} (-i)^l (E_l^0 \pm iB_l^0)\sqrt{\frac{4\pi}{2l+1}}\,{}_{\mp 2}Y_l^0(\mathbf{n}), \tag{2.34}$$

where E and B are the electric and magnetic modes and the P_l's represent the Legendre polynomials. So

$$C[f, G] = an_e\sigma_T\bar{f}_{,P}P\left\{i\mu v_b + \Delta(\eta, \mathbf{k}, \mathbf{n}) + \frac{1}{10}\Delta_2 P_2(\mu) - \Delta_0 - \frac{\sqrt{6}}{10}[E_2 - \Delta_2]\right\} \tag{2.35}$$

The Boltzmann equation thus yields to the evolution equation of temperature perturbations [5]:

$$\Delta' + ik\mu\Delta + \kappa'\Delta = -i\mu k[\Phi + \Psi] - 2\Phi' + \kappa' \left\{ \frac{1}{4}\delta_\gamma - \Phi - i\mu v_b + \frac{1}{10}P_2(\mu)[\sqrt{6}E_2 - \Delta_2] \right\}. \quad (2.36)$$

$$Q' + ik\mu Q + \kappa' Q = \frac{\kappa'}{10}\{P_2(\mu) - 1\}\left[\sqrt{6}E_2 - \Delta_2\right]. \quad (2.37)$$

Note that the temperature perturbation $\Delta(\mathbf{n})$ is a function of either $\Delta(\eta, \mathbf{x}, \mathbf{n})$ or, in Fourier space, $\Delta(\eta, \mathbf{k}, \mathbf{n})$; $\kappa' \equiv a n_e \sigma_T$ is the differential optical depth and $\mu = k^{-1}\mathbf{k} \cdot \mathbf{n}$ the direction cosine.

We have use the expressions for the first few moments of the distribution function

$$T^\mu_\nu = \int \sqrt{-g}\frac{p^\mu p_\nu}{|p_0|} f(p, x) d^3p \quad (2.38)$$

$$\delta = 4\Phi + \frac{1}{\pi}\int \Delta(\mathbf{n})\mathbf{d\Omega} \quad (2.39)$$

We notice that (2.36) is not manifestly gauge-invariant, however by defining the gauge invariant temperature perturbation $\mathcal{M} = \Delta + 2\Phi$, and its multipole decomposition

$$\mathcal{M}(\eta, \mathbf{x}, \mathbf{n}) = \sum_l (-i)^l \mathcal{M}_l(\eta, \mathbf{k}) P_l(\mathbf{n}), \quad (2.40)$$

the evolution equation (2.36), in gauge-invariant components, becomes:

$$\mathcal{M}' + ik\mu\mathcal{M} + \kappa'\mathcal{M} = i\mu k[\Phi - \Psi] + \kappa' \left\{ \frac{1}{4}D^\gamma_g - i\mu v_b + \frac{1}{10}P_2(\mu)\left[\sqrt{6}E_2 - \mathcal{M}_2\right] \right\}. \quad (2.41)$$

The procedure is as follows: For each Legendre polynomials P_l

- replace $\mathcal{M}(\eta, \mu)$ by its multipole expansion
- multiply by $P_l(\mu)$
- integrate both l.h.s. and r.h.s. of the new equation over μ : $\int_{-1}^{1} d\mu$
- use the orthogonality relation $\int_{-1}^{1} d\mu P_l(\mu) P_n(\mu) = 2\delta_{ln}/(2l + 1)$

After integrating (2.41) for each l and applying orthogonality relations of the Legendre polynomials, the hierarchy for $\mathcal{M}$ is hence given by [6]:

$$\begin{aligned}
\mathcal{M}_0' &= -\frac{k}{3}V_\gamma, && (2.42)\\
\mathcal{M}_1' &= \kappa'(V_b - V_\gamma) + k(\Psi - \Phi) + k\left(\mathcal{M}_0 - \frac{2}{5}\mathcal{M}_2\right), && (2.43)\\
\mathcal{M}_2' &= -\kappa'(\mathcal{M}_2 - \mathcal{C}) + k\left(\frac{2}{3}V_\gamma - \frac{3}{7}\mathcal{M}_3\right), && (2.44)\\
\mathcal{M}_l' &= -\kappa'\mathcal{M}_l + k\left(\frac{l}{2l-1}\mathcal{M}_{l-1} - \frac{l+1}{2l+3}\mathcal{M}_{l+1}\right), \quad l > 2, && (2.45)
\end{aligned}$$

and similarly for the polarisation

$$\begin{aligned}
E_2' &= -\frac{k\sqrt{5}}{7}E_3 - \kappa'(E_2 + \sqrt{6}\mathcal{C}), && (2.46)\\
E_l' &= k\left(\frac{{}_2\kappa_l}{2l-1}E_{l-1} - \frac{{}_2\kappa_{l+1}}{2l+3}E_{l+1}\right) - \kappa' E_l, \quad l > 2. && (2.47)
\end{aligned}$$

Here $\mathcal{C} = \mathcal{M}_2 - \sqrt{6}E_2/10$ and ${}_2\kappa_l = \sqrt{l^2 - 4}$ are combinatorial factors.

Massless neutrinos follow the same multipole hierarchy as $\mathcal{M}$, however without polarisation and Thompson scattering. Hence, the perturbed neutrino distribution $\mathcal{N}$ satisfies [5]:

$$\begin{aligned}
\mathcal{N}_0' &= -\frac{k}{3}V_\nu, && (2.48)\\
\mathcal{N}_0' &= k(\Psi - \Phi) + k\left(\mathcal{N}_0 - \frac{2}{5}\mathcal{N}_2\right), && (2.49)\\
\mathcal{N}_l' &= k\left(\frac{l}{2l-1}\mathcal{N}_{l-1} - \frac{l+1}{2l+3}\mathcal{N}_{l+1}\right), \quad l > 1. && (2.50)
\end{aligned}$$

For completeness, we quote the hierarchy for the tensor multipoles, temperature $\tilde{\Delta}_l^T$, polarisation $\tilde{\Delta}_l^P$ and cross-correlation $\tilde{\Delta}_l^{T,P}$ [5, 12]:

$$\begin{aligned}
\tilde{\Delta}_0^T &= -k\tilde{\Delta}_1^T - \kappa'[\tilde{\Delta}_0^T - \psi] - h', && (2.51)\\
\tilde{\Delta}_0^P &= -k\tilde{\Delta}_2^T - \kappa'[\tilde{\Delta}_1^T + \psi], && (2.52)\\
\tilde{\Delta}_l^{T,P} &= \frac{k}{2l+1}\left[l\tilde{\Delta}_{l-1}^{T,P} - (l+1)\tilde{\Delta}_{l+1}^{T,P}\right] - \kappa'\tilde{\Delta}_l^{T,P}; \quad l \geq 1, && (2.53)
\end{aligned}$$

where h is the longitudinal-scalar part of tensor decomposition in (**??**), and ψ is given by

$$\psi = \frac{1}{10}\tilde{\Delta}_0^T + \frac{1}{7}\tilde{\Delta}_2^T + \frac{3}{70}\tilde{\Delta}_4^T - \frac{3}{5}\tilde{\Delta}_0^P + \frac{6}{7}\tilde{\Delta}_2^P - \frac{3}{70}\tilde{\Delta}_4^P. \quad (2.54)$$

The Boltzmann hierarchy is nowadays solved numerically with software packages such as CMBFAST [11] to produce the CMB spectrum. Also, a widely used implementation is the

CAMB code [8], often embedded in the analysis package CosmoMC. Different codes have also been implemented to compute the CMB spectrum, i.e. CMBEASY is fully object oriented C++ [4], CLASS is written in C [7], and CMBquick is written in Mathematica, but is unavoidably slow [9].

2.3 The Line of Sight Strategy

So usually, we are interested in $\mathcal{M}(\eta_0, \mu)$. It turns out that there is a clever way to obtain this that even highlights the different contributions towards the final anisotropy. Let us develop this Line of Sight strategy. Inspecting, one notices that the l.h.s can be written as

$$e^{-i\mu k\eta} e^{-\kappa(\eta)} \dot{L} \tag{2.55}$$

where

$$L \equiv e^{i\mu k\eta} e^{\kappa(\eta)} \mathcal{M} \tag{2.56}$$

Hence, the Boltzmann equation translates into

$$\dot{L} = e^{i\mu k\eta} e^{\kappa(\eta)} \left[i\mu k(\Phi - \Psi) + \kappa' \left(\frac{1}{4} D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C} \right) \right] \tag{2.57}$$

and integrated over conformal time

$$L(\eta_0) = \int_0^{\eta_0} d\eta e^{i\mu k\eta} e^{\kappa(\eta)} \left[i\mu k(\Phi - \Psi) + \kappa' \left(\frac{1}{4} D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C} \right) \right] \tag{2.58}$$

The photon perturbation today is given by

$$\mathcal{M}(\mu, \eta_0) = \int_0^{\eta_0} d\eta e^{i\mu k(\eta - \eta_0)} e^{\kappa(\eta) - \kappa(\eta_0)} \times \left[i\mu k(\Phi - \Psi) + \kappa' \left(\frac{1}{4} D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C} \right) \right] \tag{2.59}$$

The product $g \equiv \kappa' \exp(\kappa(\eta) - \kappa(\eta_0))$ plays an important role and is called the visibility function. Its peak defines the epoch of recombination.

Each term in the above Equation containing factors of μ, can be integrated by parts, in order to get rid of μ. Applying this procedure to all terms involving μ yields

$$\mathcal{M}(\mu, \eta_0) = \int_0^{\eta_0} e^{\imath\mu k(\eta - \eta_0)} S_T(k, \eta) d\eta \tag{2.60}$$

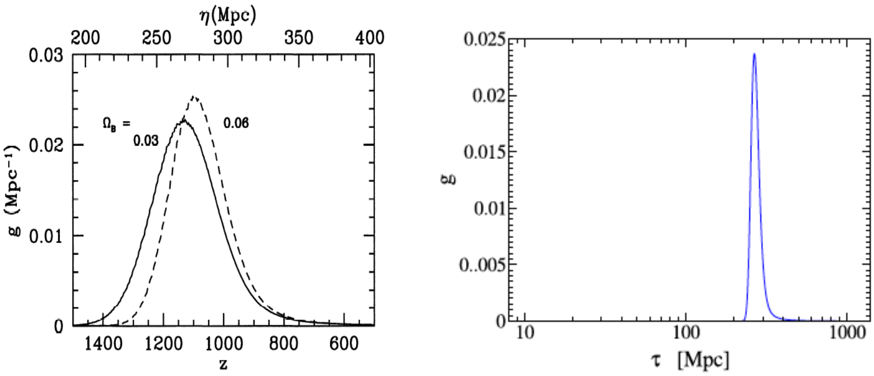

Figure 2.2: Visibility function. Its peak at about $\eta \approx 300$Mpc defines the epoch of last scattering.

where the source is

$$\begin{aligned} S_T &= -e^{\kappa(\eta)-\kappa(\eta_0)}[\Phi' - \Psi'] + g'\left[\frac{V_b}{k} + \frac{3}{k^2}\mathcal{C}'\right] + g''\frac{3}{2k^2}\mathcal{C} \\ &+ g\left[\frac{1}{4}D_g^\gamma + \frac{V_b'}{k} - (\Phi - \Psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\mathcal{C}''\right], \end{aligned} \tag{2.61}$$

Let us examine in more detail the temperature perturbations. The density contrast D_g^γ is the main contribution, driving the spectrum towards the oscillatory behaviour. It can be seen as an intrinsic temperature variation over the background last-scattering surface: $\delta T/T \propto D_g^\gamma/4$. The Doppler shift, V_b-term, describes the blueshift caused by last scattering electrons moving towards the observer. The term involving time derivatives of the potentials, $(\Phi' - \Psi')$, is known as the *integrated Sachs-Wolfe* effect (ISW) [10]. It describes the change of the CMB photon energy due to the evolution of the potentials along the line of sight. The terms involving $\mathcal{C}$ and its derivatives describe polarisation effects and are far less important than the D_g^γ term. Finally, the $(\Phi - \Psi)$ term arises from the gravitational redshift when climbing out of the potential well at last scattering. The combination $D_g^\gamma/4 - (\Phi - \Psi)$ is known as the *ordinary Sachs-Wolfe effect* (SW). This gives the main contribution on scales that at decoupling were well outside the horizon [2, 5].

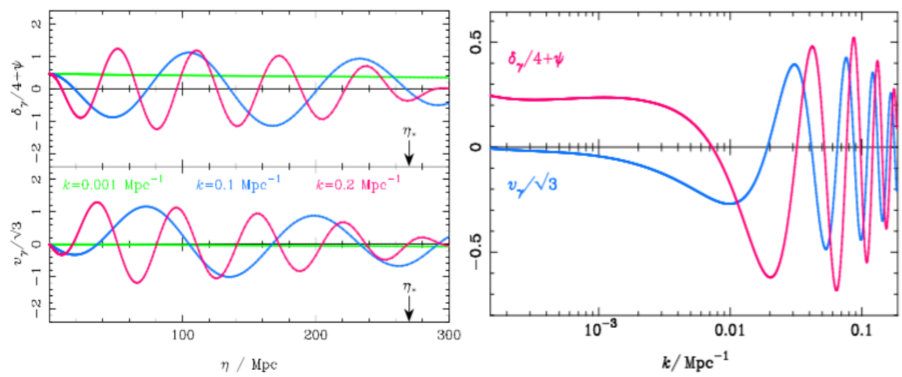
$\delta_\gamma/4-\psi$
$v_\gamma/\sqrt{3}$
k=0.001 Mpc^{-1}
k=0.1 Mpc^{-1}
k=0.2 Mpc^{-1}
η_*
η / Mpc
$\delta_\gamma/4+\psi$
$v_\gamma/\sqrt{3}$
k/ Mpc^{-1}

Figure 2.3

3

Statistics of Random Fields

The density contrast δ, introduced in the previous section, can be considered statistically as a random field with zero mean, $\langle \delta(\mathbf{x}) \rangle = 0$. The measure of the clustering degree in the spatial direction $\mathbf{r}$ is determined by the *correlation function* ξ, which is defined as the product of the density contrast at two separate points, $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$:

$$\xi(r) \equiv \langle \delta(\mathbf{x}) \delta(\mathbf{x}+\mathbf{r}) \rangle. \tag{3.1}$$

Because of statistical homogeneity and isotropy of a random field, the two-point correlator depends only on the distance $r = |\mathbf{r}|$ between the two points. On the other hand, the amplitude of fluctuations on different lengths are described by the *power spectrum* $\mathcal{P}(k)$, which is simply the inverse Fourier transform of the correlation function ξ:

$$\langle \hat{\delta}(\mathbf{k}) \hat{\delta}(\mathbf{k}') \rangle = \frac{2\pi^2}{k^3} \mathcal{P}(k) \delta_D(\mathbf{k} - \mathbf{k}'), \tag{3.2}$$

where $\hat{\delta}$ is the Fourier transform of the density contrast δ. The Dirac's delta distribution δ_D guarantees that modes relative to different wave-numbers are uncorrelated in order to preserve homogeneity; $\mathcal{P}(k)$ has dependency only on the magnitude of the momenta no on $\mathbf{k}$ direction because of isotropy. The normalisation factor $2\pi^2/k^3$ in the definition of the power spectrum is conventional and has the virtue of making $\mathcal{P}(k)$ dimensionless if $\delta(\mathbf{x})$ is.

3.0.1 CMB power spectrum

The primary anisotropies carried out by physical effects before the recombination epoch, encoded in the fractional temperature perturbation, are expanded in terms of the spherical harmonics on the surface of last scattering by

$$\frac{\Delta T}{\bar{T}}(\eta_0, \mathbf{x_0}, \mathbf{n}) = \sum_{l,m} a_{lm} Y_{lm}(\mathbf{n}), \tag{3.3}$$

where the a_{lm}'s define the multipoles of the CMB anisotropy; $\mathbf{x_0}$ is our position and η_0 the present conformal time. Assuming the $a_{l,m}$'s are Gaussian random fields, the two-point correlator gives

$$\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}. \tag{3.4}$$

The angular *CMB power spectrum* C_l^{TT} is computed through the two-point correlation function (3.1) by

$$C(\theta) \equiv \left\langle \frac{\Delta T(\mathbf{n})}{\bar{T}} \frac{\Delta T(\mathbf{n}')}{\bar{T}} \right\rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\mathbf{n} \cdot \mathbf{n}'). \tag{3.5}$$

where $\mathbf{n} \cdot \mathbf{n'} = \cos\theta$, and we have used the addition theorem for spherical harmonics to express the sum of products of Y_{lm}'s in terms of the Legendre polynomials. We consider initial conditions in terms of the conformal Newtonian gauge potential $\Phi_{\rm ini} = \mathcal{R}$. Because the evolution equations for Δ are independent of the direction $\mathbf{k}$, we may write

$$\Delta_l(\eta_0, \mathbf{k}, \mathbf{n}) = \Phi_{\rm ini}(\mathbf{k}) \Delta_l(\eta_0, k, \mathbf{n}). \tag{3.6}$$

Therefore the C_l's are found to be

$$C_l^{XY} = \frac{4\pi}{(2l+1)^2} \int \frac{d^3k}{(2\pi)^3} \, \mathcal{P}_{\mathcal{R}}(k) \, \Delta_l^X(k) \Delta_l^Y(k), \tag{3.7}$$

where X and Y represent the *temperature* (T) and *polarisations* (E or B); $\mathcal{P}_{\mathcal{R}}(k)$ is the power spectrum of the initial curvature perturbations

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_0}\right)^{n_s - 1} \tag{3.8}$$

and A_s the initial scalar amplitude, quoted at a reference scale k_0 (one chooses $k_0 = 0.05$Mpc) and the spectrum is a featureless power law with scalar spectral index n_s.

The moments obtained from the line of sight integration method [11], in terms of the spherical Bessel functions j_l, are given by

$$\Delta_l^T = (2l+1) \int d\eta j_l(k[\eta - \eta_0]) S_T(k, \eta), \tag{3.9}$$

$$\Delta_l^E = (2l+1) \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^{\eta_0} d\eta S_E(k, \eta) j_l(x), \tag{3.10}$$

with the sources

$$S_T = -e^{\kappa(\eta)-\kappa(\eta_0)}[\Phi' - \Psi'] + g'\left[\frac{V_b}{k} + \frac{3}{k^2}\mathcal{C}'\right] + g''\frac{3}{2k^2}\mathcal{C}$$
$$+ \; g\left[\frac{1}{4}D_g^\gamma + \frac{V_b'}{k} - (\Phi - \Psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\mathcal{C}''\right], \quad (3.11)$$
$$S_E = \frac{3g\mathcal{C}}{4x^2}, \quad (3.12)$$

where we have used $x \equiv k(\eta_0 - \eta)$ and the visibility function $g \equiv \kappa' \exp(\kappa(\eta) - \kappa(\eta_0))$.

For completeness, we quote the hierarchy for the tensor multipoles, temperature $\tilde{\Delta}_l^T$, polarisation $\tilde{\Delta}_l^P$ and cross-correlation $\tilde{\Delta}_l^{T,P}$ [5, 12]:

$$C_{XY;l}^{\rm tens} = \frac{4\pi}{(2l+1)^2}\int \frac{d^3k}{(2\pi)^3}\,\mathcal{P}_{\mathcal{T}}(k)\,\Delta_{X;l}^{\rm tens}(k)\Delta_{Y;l}^{\rm tens}(k), \quad (3.13)$$

where $\mathcal{P}_{\mathcal{T}}(k)$ is the initial tensor power spectrum, and the moments:

$$\Delta_{T;l}^{\rm tens} = \sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_0} d\eta S_T^{\rm tens}(k,\eta)\frac{j_l(x)}{x^2}, \quad (3.14)$$
$$\Delta_{E,B;l}^{\rm tens} = \int_0^{\eta_0} d\eta S_{E,B}^{\rm tens}(k,\eta) j_l(x), \quad (3.15)$$

with the sources (using (3.22)):

$$S_T^{\rm tens}(k,\eta) = h'\exp(-\kappa) + g\psi, \quad (3.16)$$
$$S_E^{\rm tens}(k,\eta) = g\left\{\psi - \frac{\psi''}{k^2} + \frac{2\psi}{x^2} - \frac{\psi'}{kx}\right\}$$
$$-g'\left\{\frac{2\psi'}{k^2} + \frac{4\psi}{kx}\right\} - 2g''\frac{\psi}{k^2}, \quad (3.17)$$
$$S_B^{\rm tens}(k,\eta) = g\left\{\frac{4\psi}{x} + \frac{2\psi'}{k}\right\} + 2g'\frac{\psi}{k}. \quad (3.18)$$

$$\tilde{\Delta}_0^T = -k\tilde{\Delta}_1^T - \kappa'[\tilde{\Delta}_0^T - \psi] - h', \quad (3.19)$$
$$\tilde{\Delta}_0^P = -k\tilde{\Delta}_2^T - \kappa'[\tilde{\Delta}_1^T + \psi], \quad (3.20)$$
$$\tilde{\Delta}_l^{T,P} = \frac{k}{2l+1}\left[l\tilde{\Delta}_{l-1}^{T,P} - (l+1)\tilde{\Delta}_{l+1}^{T,P}\right] - \kappa'\tilde{\Delta}_l^{T,P}; \quad l \geq 1, \quad (3.21)$$

where h is the longitudinal-scalar part of tensor decomposition in (**??**), and ψ is given by

$$\psi = \frac{1}{10}\tilde{\Delta}_0^T + \frac{1}{7}\tilde{\Delta}_2^T + \frac{3}{70}\tilde{\Delta}_4^T - \frac{3}{5}\tilde{\Delta}_0^P + \frac{6}{7}\tilde{\Delta}_2^P - \frac{3}{70}\tilde{\Delta}_4^P. \quad (3.22)$$

The slow way would be to get the C_l's directly from the (vast) multipole hierarchy of the photon distribution and the multipole hierarchy up to $l \equiv 3000$. In contrast, the line of sight

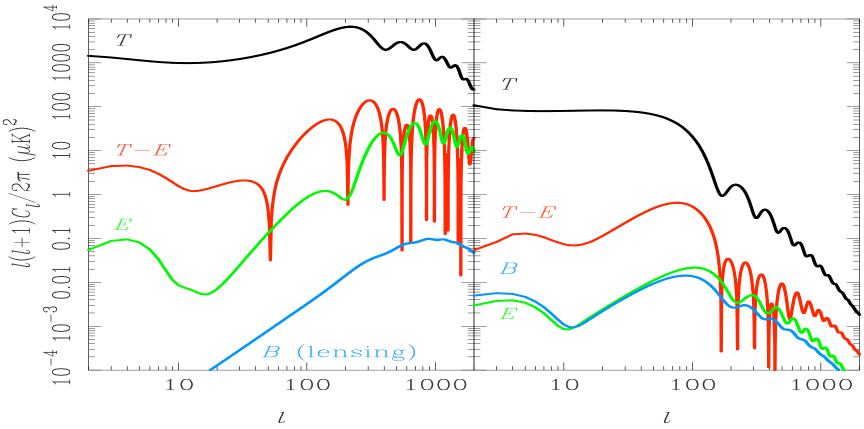

Figure 3.1: CMB spectra for all the contributions: Temperature, E-mode, B-mode and T-E cross-correlation. The left-hand-side displays the scalar perturbations whereas the right-hand-side tensor perturbations (gravitational waves). Figure reprinted from Challinor [3]

integration gets the Δ_l's by folding the source term S with the spherical Bessel functions j_l. While the Bessel functions oscillate rapidly in this convolution, the source term is most of the time rather slowly changing. It thus suffices to calculate the sources at few (cleverly chosen) points and interpolate between.

Figure 3.1 shows the adiabatic CMB spectra for all the contributions: Temperature, E-mode, B-mode and T-E cross-correlation. The left-hand-side of the panel displays the CMB spectra for scalar perturbations, whereas the right-hand-side tensor perturbations (gravitational waves). All of them in units of $l(l+1)/2\pi[\mu K]^2$.

3.1 Codes

The Boltzmann hierarchy is nowadays solved numerically with software packages such as CMB-FAST [11] to produce the CMB spectrum. Also, a widely used implementation is the CAMB code [8], often embedded in the analysis package CosmoMC. Different codes have also been implemented to compute the CMB spectrum, i.e. CMBEASY is fully object oriented C++ [4], CLASS is written in C [7], and CMBquick is written in Mathematica, but is unavoidably slow [9].

1995/ Fortran 77 — COSMICS: Cosmological initial conditions and microwave anisotropy
* Chung-Pei Ma, Edmund Bertschinger. arXiv:astro-ph/9506072
http://ascl.net/cosmics.html

1996/ Fortran 77 — CMBFAST: A microwave anisotropy code
* Seljak Uros, Zaldarriaga Matias. arXiv:astro-ph/9603033, arXiv:astro-ph/9704265
http://ascl.net/cmbfast.html

2000/ Fortran 90 — CAMB: Code for Anisotropies in the Microwave Background.
* Antony Lewis, Anthony Challinor and Anthony Lasenby. arXiv:astro-ph/9911177
http://camb.info/

2003/ C++ — CMBEASY: an Object Oriented Code for the Cosmic Microwave Background
* Michael Doran arXiv:astro-ph/0302138v2
http://www.thphys.uni-heidelberg.de/ robbers/cmbeasy/index.html

2001 — Davis Anisotropy Shortcut (DASh)
DASh incorporates many analytic and semianalytic approximations that have been presented elsewhere in the literature and also some new ones. The Astrophysical Journal, 578:665-674, 2002

TABLE I. Comparison between CMB Codes [a]

	CAMB	CLASS	CMBEASY	CMBquick	CosmoLib [b]
Language	F90	C	C++	Mathematica	F90[c]
gauge [d]	syn.	syn./Newt. [e]	syn./gauge-inv.	Newt.	Newt.
open/close universe	Yes	No	No	No	No
massive neutrinos	Yes	Yes	Yes	Yes	No
tensor perturb.	Yes	Yes	Yes	Yes	Yes
CDM isocurvature mode	Yes	Yes	Yes	Yes	Yes
dark energy perturb.	Yes	Yes	Yes	No	Yes
nonzero $c_{s,b}^2$	Yes	Yes	Yes	No	Yes
dark energy EOS.	constant	$w_0 + w_a(1-a)$	arbitrary	-1	arbitrary
non-smooth primordial power	No	No	No	No	Yes
MCMC driver	Yes	No	Yes	No	Yes
periodic proposal density	No	No	No	No	Yes
data simulation	No	No	No	No	Yes
second-order perturb. [f]	No	No	No	Yes	No [g]

[a] Here we do not include CMBFast, which is no longer supported by its authors or available for download.

http://lambda.gsfc.nasa.gov/toolbox/

Figure 3.2: a

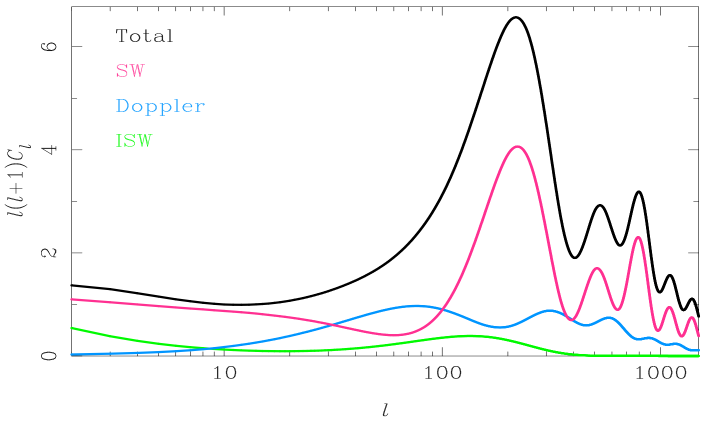

Figure 3.3: Total CMB temperature-spectrum and its different contributions: Sachs-Wolfe (SW) $D_g^\gamma/4 - (\Phi - \Psi)$; Doppler effect V_b^γ; and the integrated Sachs-Wolfe effect (ISW) coming from evolution of the potential along the line of sight. Figure from Challinor [3]

3.2 Description of fluctuations

- The $l = 0$ term of the correlation function (the monopole) vanishes if the mean temperature has been defined correctly.

- The $l = 1$ (the dipole) reflects the motion of the Earth through space. What we are seeing is the effect of the Earth's motion relative to the local comoving frame of reference. The Earth is moving with a velocity $v = 369kms^{-1}$ towards a point on the boundary of the constellations of Crater and Leo.

- The Sachs-Wolfe effect ($l < 100$) - The gravitational effects are the dominant contributions at large angular scales. $C_l \propto \int d\ln k \mathcal{P}_{\mathcal{R}}(k) j_l^2(k[\eta - \eta_0])$, and if we make use of the integral

$$\int_0^\infty j_l^2(x)dx = \frac{1}{2l(l+1)} \tag{3.23}$$

and assume a nearly scale-invariant scalar spectrum $n_s \approx 1$, then

$$\frac{l(l+1)C_l}{2\pi} = \frac{1}{25}A_s \tag{3.24}$$

is approximately constant, shown as a flat plateau at low multipoles. More generally, a primordial spectrum that varies as a power-law in k gives an angular power spectrum going like

$$C_l \sim \frac{\Gamma(l + n_s/2 - 1/2)}{\Gamma(l - n_s/2 + 5/2)} \tag{3.25}$$

- Intermediate scales ($100 < l < 1000$) - Perturbations inside the horizon have evolved causally and produced the anisotropy at the last scattering epoch ($l_{\rm hor} \approx 200$). The balance between the gravitational force and radiation pressure is presented as series of characteristic peaks called *acoustic oscillations*.

- Small scales ($l > 1000$) - The thickness of the last scattering surface leads to a damping of $C_l^T \sim l^{-4}$ at the highest multipoles, commonly called the *Silk effect*. The total mean-squared distance that a photon will have moved by such a random walk by the time η_* is therefore

$$\int_0^{\eta_*} \frac{d\eta'}{a n_e \sigma_T} \sim \frac{1}{k_D^2} \tag{3.26}$$

which defines a damping scale k_D^{-1}.

At these scales, important contributions are also provided by secondary anisotropies: gravitational lensing, Rees-Sciama effect (RS), Sunyaev-Zel'dovich effect (SZ), kinetic Sunyaev-Zel'dovich effect, Ostriker-Vishniac effect (OV), foregrounds from discrete sources [1].

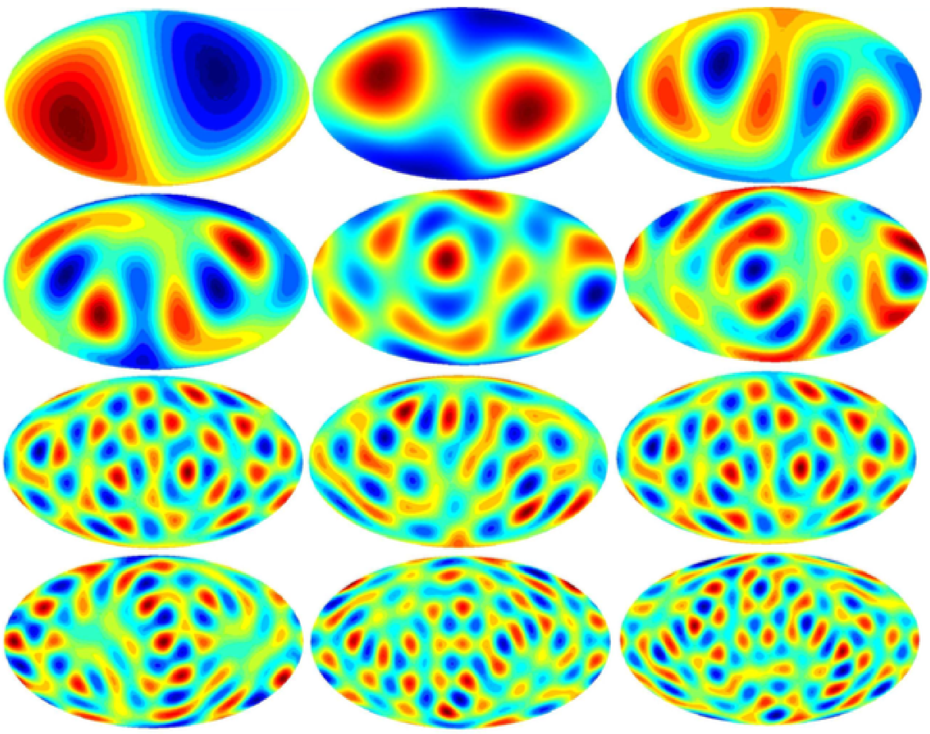

Figure 3.4

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