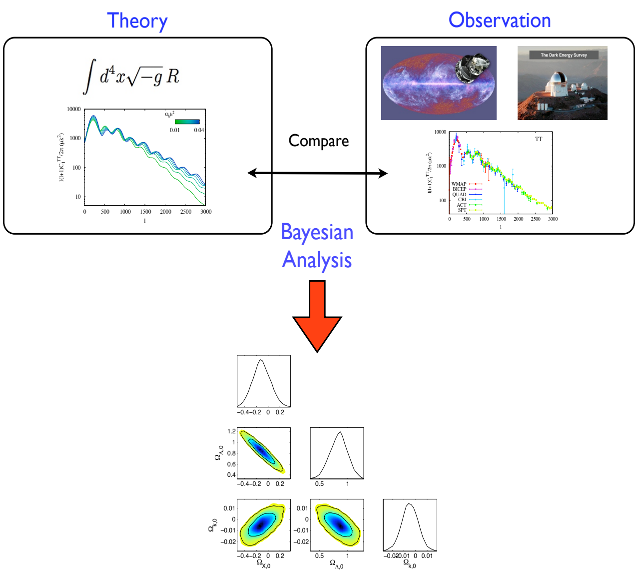
Updated Cosmology

with Python



José-Alberto Vázquez

ICF-UNAM / Kavli-Cambridge

In progress

August 12, 2017

1

Initial conditions from inflation

From quantum to classical

$$\delta\Phi(t,\bar{x}) \equiv \Phi(t,\vec{x}) - \bar{\Phi}(t). \tag{1.1}$$

- Quantum fluctuations induce a non-zero variance in the amplitudes of these oscillations

$$\left\langle |\delta\Phi_k|^2 \right\rangle \equiv \left\langle 0|\delta\Phi_k|^2 0\right\rangle. \tag{1.2}$$

- Then inflationary expansion stretches these fluctuations to super horizon scales.

- At horizon crossing, $k = aH$, switch from inflation fluctuations $\delta\Phi$ to fluctuations in the conserved curvature perturbation $\mathcal{R}$. [In spatially flat gauge $\phi = E = 0$].

$$\mathcal{R} = -\phi - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v) \quad \longrightarrow \mathcal{H}(B+v) \quad \text{zero curvature gauge,}$$

compare to the stress tensor of a scalar field

$$\begin{aligned} \delta T^0_j = g^{0\mu}\partial_\mu\Phi\partial_j\delta\Phi = g^{00}\partial_0\bar{\Phi}\partial_j\delta\Phi = \frac{\bar{\Phi}'}{a^2}\partial_j\delta\Phi \\ \text{recall} \quad \delta T^0_j = -(\bar{\rho}+\bar{p})\partial_j(B+v), \qquad (1.3) \\ \text{hence} \quad B+v = -\frac{\delta\Phi}{\bar{\Phi}'} \\ \text{and} \quad \mathcal{R} = -\frac{\mathcal{H}}{\bar{\Phi}'}\delta\Phi. \qquad (1.4) \end{aligned}$$

For the Energy momentum tensor, using the unperturbed FRW metric

$$S = \int d\tau d^3x \left[\frac{1}{2}a^2\left(\Phi'^2 - (\nabla\Phi)^2\right) - a^4 V(\Phi)\right], \tag{1.5}$$

in the background $\Phi = \bar{\Phi}(\eta)$ is homogeneous

Fluctuations in Φ: $\Phi = \bar{\Phi} + \frac{u}{a}$ where $u = a\delta\Phi$

expanding the fluctuations in u, the term in square brackets becomes

$$\underbrace{a\bar{\Phi}'u'}_{\delta^{(1)}} - \underbrace{a'\bar{\Phi}'u}_{\delta^{(1)}} + \frac{1}{2}u'^2 - u'u\mathcal{H} + \frac{1}{2}u^2\mathcal{H}^2 - \frac{1}{2}(\nabla u)^2 - \underbrace{a^3 u V_{,\Phi}}_{\delta^{(1)}} - \frac{1}{2}a^2u^2V_{,\Phi\Phi}. \tag{1.6}$$

Looking only the terms marked with $\delta^{(1)}$, the first term can be integrated by parts and dropping the boundary term, we have

$$\delta^{(1)} = -\int d\tau d^3x \left[(\bar{\Phi}'a)' + a'\bar{\Phi}' + a^3V_{,\Phi}\right]u, \tag{1.7}$$

expanding

$$\delta^{(1)} = -\int d\tau d^3x a[\bar{\Phi}'' + 2\mathcal{H}\bar{\Phi}' + a^2V_{,\Phi}]u, \tag{1.8}$$

where

$$\bar{\Phi}'' + 2\mathcal{H}\bar{\Phi}' + a^2V_{,\Phi} = 0 \qquad \text{Klein-Gorden for the background field} \tag{1.9}$$

hence, we need to go to the second order in the action

$$\delta^{(2)} = \frac{1}{2}\int d\tau d^3x \left[(u')^2 - (\nabla u)^2 - 2\mathcal{H}uu' + \left(\mathcal{H}^2 - a^2V_{,\Phi\Phi}\right)u^2\right], \tag{1.10}$$

$$\text{using} \quad uu' = \frac{1}{2}(u^2)',$$

and then by parts, we have

$$\delta^{(2)} = \frac{1}{2}\int d\tau d^3x \left[(u')^2 - (\nabla u)^2 + \left(\frac{a''}{a} - a^2V_{,\Phi\Phi}\right)u^2\right]. \tag{1.11}$$

During slow-roll inflation we have

$$\frac{V_{,\Phi\Phi}}{H^2} \approx \frac{3M_p^2V_{,\Phi\Phi}}{V} = 3\eta_V \ll 1, \tag{1.12}$$

since $a' = a^2H$, $H \sim$ constant, deriving

$$\frac{a''}{a} \approx 2a'H = 2a^2H^2 \gg a^2V_{,\Phi\Phi}, \tag{1.13}$$

$$\delta^{(2)} = \int d\tau d^3x \frac{1}{2}\left[(u')^2 + \frac{a''}{a}u^2 - (\nabla u)^2\right]. \tag{1.14}$$

Applying $E - L$

$$u'' - \frac{a''}{a}u - \nabla^2 u = 0 \qquad \text{Mukhanov-Sasaki equation.} \tag{1.15}$$

and for each Fourier mode

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k = 0. \tag{1.16}$$

1.1 Canonical quantization

Follow the quantization of the harmonic oscillator.

Define the momentum conjugate to u

$$\pi_u \equiv \frac{\partial \mathcal{L}}{\partial \dot{u}} = \dot{u}, \tag{1.17}$$

promote π and u to operator-valued, commutation relations

$$[\hat{u}(\tau, \vec{x}), \hat{\pi}(\tau, \vec{x}')] = i\delta(\vec{x} - \vec{x}') \qquad \text{Heinsenberg picture} \tag{1.18}$$

$$\frac{\partial^2 \hat{u}}{\partial \tau^2} - \frac{a''}{a}\hat{u} - \nabla^2 \hat{u} = 0 \tag{1.19}$$

Quantum oscillators $\hat{a}_k^\dagger$, $\hat{a}_k$ creation and annihilation operators

The general solution to the equation

$$\hat{u}(\tau, x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} [\hat{a}(k) u_k(\tau) e^{i\bar{k}\cdot x} + \hat{a}^\dagger(\bar{k}) u_k^*(\tau) e^{-i\bar{k}\cdot\bar{x}}] \tag{1.20}$$

in Fourier

$$\hat{u}(\tau, \vec{k}) = \hat{a}_{\vec{k}} u_k(\tau) + \hat{a}_{\vec{k}}^\dagger u_k^*(\tau). \tag{1.21}$$

1.2 Power spectrum

Power spectrum $P_u(k)$ by computing the two-point correlator of the field u in Fourier space

$$\left\langle 0|\hat{u}(\tau, \vec{k})\hat{u}^\dagger(\tau, \vec{k}')|0\right\rangle = \frac{2\pi^2}{k^3} P_u(k)\delta(\vec{k} - \vec{k}') \tag{1.22}$$

after some algebra

$$\left\langle 0|\hat{u}(\tau, \vec{k})\hat{u}^\dagger(\tau, \vec{k}')|0\right\rangle = |u_k(\tau)|^2 \delta(\vec{k} - \vec{k}') \tag{1.23}$$

the power spectrum is thus $P_u(k) = \frac{k^3}{2\pi^2}|u_k(\eta)|^2$ since $u = a\delta\Phi$

$$P_{\delta\Phi}(k) = \frac{k^3}{2\pi^2}\left|\frac{u_k(\eta)}{a(\eta)}\right|^2 \tag{1.24}$$

we requiere more detailed solutions.

During slow-roll inflation, $H \sim$ constant, or H_k for few e-folds, integrating

$$\frac{a'}{a} = \mathcal{H} = aH_k \quad \Rightarrow \quad a = -\frac{1}{H_k\tau}$$
$$a'' = -\frac{2}{H_k\tau^3} \quad \Rightarrow \quad \frac{a''}{a} = \frac{2}{\tau^2} \tag{1.25}$$

$$u_k{}'' + \left(k^2 - \frac{2}{\tau^2}\right)u_k = 0 \tag{1.26}$$

with solutions

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{1.27}$$

few e-folds after Hubble exit (super horizon)

$$u_k(\tau) \approx -\frac{ie^{-ik\tau}}{\tau\sqrt{2k^3}} \quad \Rightarrow \quad \frac{u_k(\tau)}{a(\tau)} \approx \frac{iH_k e^{-ik\tau}}{\sqrt{2k^3}}. \tag{1.28}$$

Form the modes that have excited the Hubble radius

$$P_{\delta\Phi}(k) = \left(\frac{H_k}{2\pi}\right)^2. \tag{1.29}$$

A light scalar field in quasi the Sitter spacetime acquires an almost-scale-invariant spectrum of fluctuations

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{\mathcal{H}}{\bar{\Phi}'}\right)^2 P_{\delta\Phi} = \left(\frac{H^2}{2\pi\bar{\Phi}'}\right)^2. \tag{1.30}$$

Using slow-roll

$$H^2 \approx \frac{1}{3M_p^2}V(\bar{\Phi}), \qquad 3H\dot{\bar{\Phi}} = -V_{,\Phi} \tag{1.31}$$

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\dot{\bar{\Phi}}}\right)^2 \approx \left(\frac{3H^3}{2\pi V_{,\Phi}}\right)^2 = \frac{\left(V/M_p^2\right)^3}{3(2\pi)^2(V_{,\Phi})^2} = \frac{8}{3}\left(\frac{V^{1/4}}{\sqrt{8\pi}M_p}\right)\frac{1}{\epsilon_V} \tag{1.32}$$

CMB obs. constrain $\mathcal{P}_{\mathcal{R}}(k) \sim 2\times 10^{-9}$.

1.3 Primordial perturbations from inflation

• Slow-roll inflation produces a spectrum of curvature perturbations that is almost scale-invariant (Harrison-Zeldovich)

• We can quantify small departures from scale-invariance, by parameterizing

$$P_{\mathcal{R}} = A_s\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{1.33}$$

$n_s(k)$ spectral index $n_s(k) - 1 \equiv \frac{d \ln P_R(k)}{d \ln k}$ in terms of the potential

$$\frac{d}{d \ln k} = \frac{dt}{d \ln k} \frac{d\Phi}{dt} \frac{d}{d\Phi} \tag{1.34}$$

at Hubble exit

$$k = aH \Rightarrow \frac{d \ln k}{dt} = H \left(1 + \frac{\partial_t H}{H^2} \right) \tag{1.35}$$

$$\begin{aligned}
\frac{\partial_t H}{H^2} &= -\frac{3}{2} \left(\frac{\bar{\rho} + \bar{P}}{\bar{\rho}} \right) \\
&\approx -\frac{3}{2} \frac{(\partial_t \Phi)^2}{V} = -\frac{1}{2} \frac{(3H\partial_t \Phi)^2}{3H^2 V} \\
&\approx -\frac{M_p^2}{2} \left(\frac{V_{,\Phi}}{V} \right)^2 = -\epsilon_V, \qquad \frac{d \ln k}{dt} \approx H(1 - \epsilon_V)
\end{aligned} \tag{1.36}$$

$$\begin{aligned}
\frac{d}{d \ln k} &\approx \frac{1}{H} \frac{d\Phi}{dt} \frac{d}{d\Phi} \\
&\approx \underbrace{-\frac{V_{,\Phi}}{3H^2} \frac{d}{d\Phi}}_{\text{1stFriedmann}} \approx \underbrace{-M_p^2 \frac{V_{,\Phi}}{V} \frac{d}{d\Phi}}_{\text{2ndFriedmann}} \approx -M_p^2 \sqrt{2\epsilon_V} \frac{d}{d\Phi}
\end{aligned} \tag{1.37}$$

differenciate the spectrum

$$\begin{aligned}
n_s - 1 &= -M_p \sqrt{2\epsilon_V} \frac{d}{d\Phi} (\ln V - \ln \epsilon_V) \\
&= -M_p \sqrt{2\epsilon_V} \left(\frac{V_{,\Phi}}{V} - \frac{\epsilon_{V,\Phi}}{\epsilon_V} \right)
\end{aligned} \tag{1.38}$$

second term

$$\begin{aligned}
\frac{d \ln \epsilon_V}{d\Phi} &= 2 \left(\frac{V_{,\Phi\Phi}}{V'} - \frac{V_{,\Phi}}{V} \right) \\
&\approx \frac{\sqrt{2}}{M_p} \left(\frac{\eta_V}{\sqrt{\epsilon_V}} - 2\sqrt{\epsilon_V} \right)
\end{aligned} \tag{1.39}$$

$$n_s - 1 = 2\eta_V(\Phi) - 6\epsilon_V(\Phi). \tag{1.40}$$

→ $\frac{dn_s}{d \ln k}$ second-order in slow roll (running parameter).

- Gravitational waves from inflation

Tensor modes

$$ds^2 = a^2[d\tau^2 - (\delta_{ij} + 2E_{ij}^T) dx^i dx^j] \tag{1.41}$$

remember E^T_{ij} trace-free and $\delta^{ik}\partial_k E^T_{ij} = 0$, there are two degrees of freedom associated $\rightarrow$ behave like massless scalar fields

$$S^{(2)} = \frac{M_p^2}{8} \int d\tau dx^3 a^2 [(E^{T'}_{ij})^2 - (\nabla E^T_{ij})^2] \tag{1.42}$$

define

$$\frac{M_p^2}{2} a E^T_{ij} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f_+ & f_x & 0 \\ f_x & -f_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.43}$$

$$S^{(2)} = \frac{1}{2} \sum_{I=+,x} \int d\tau d^3x [(u_I')^2 - (\nabla u_I)^2 + \frac{a''}{a} u_I^2] \tag{1.44}$$

two copies for the action $u = a\delta\Phi$

$$P_t(k) = \frac{8}{M_p^2} \left(\frac{H_k}{2\pi}\right)^2 \tag{1.45}$$

using the slow-roll inflation

$$P_t(k) \approx \frac{128}{3} \left(\frac{V^{1/4}}{\sqrt{8\pi} M_p}\right)^4 \tag{1.46}$$

defines the tensor-to-scalar ratio

$$r \equiv \frac{P_t(k)}{P_{\mathcal{R}}(k)} \approx 16\epsilon_V$$

$$P_t(k) = A_t \left(\frac{k}{k_0}\right)^{n_t}. \tag{1.47}$$

The tensor spectral index

$$\begin{aligned} n_t = \frac{d\ln P_t(k)}{d\ln k} \approx \frac{d\ln V}{d\ln k} &= -M_p\sqrt{2\epsilon_V}\frac{V_{,\Phi}}{V} \\ &= -M_p\sqrt{2\epsilon_V}\frac{\sqrt{2\epsilon_V}}{M_p} \\ &= -2\epsilon_V \end{aligned} \tag{1.48}$$

Note that $r \approx -8n_t$, the first consistency relation.

1.4 The matter power spectrum

The distribution of matter is a key cosmological observable

MPS $\mathcal{P}_{\Delta m}(\eta; \vec{k})$ is defined by

$$< \Delta_m(\tau, \vec{k}) \Delta^*_m(\tau, \vec{k}') > \equiv \frac{2\pi}{k^3} \mathcal{P}_{\Delta_m}(\tau, k) \delta(\vec{k} - \vec{k}') \tag{1.49}$$

is dimensionless, but frequently

$$P_{\Delta_m}(\tau,k) \equiv \frac{2\pi}{k^3}\mathcal{P}_{\Delta_m}(\tau,k). \tag{1.50}$$

- Real-space measures of matter clustering, σ_R.

$\rightarrow$ the variance of Δ_m averaged in spheres of radius R, equivalent to the variance of Δ_m convolved with

$$\frac{3\Theta(R-|x|)}{4\pi R^3}, \tag{1.51}$$

normalized spherical top hat, which in Fourier space is $W(kR)$

$$W(x) \equiv \frac{3}{x^3}(\sin x - x\cos x) \tag{1.52}$$

$$\sigma_R^2 = \int \frac{d^3\bar{k}}{(2\pi)^3} W^2(kR) P_{\Delta_m}(k). \tag{1.53}$$

Historically $R = 8h^{-1}$ Mpc is chosen

Makes the scale at which perturbation theory breaks down and non-linear effects become important.

Where lineal perturbation holds

$$\Delta_m(\tau,\vec{k}) = T(\tau,k)\mathcal{R}(0,\vec{k}). \tag{1.54}$$

$T(\tau,k)$ transfer function that relates the primordial curvature perturbation to the comoving matter perturbation. [Sometimes you will find it as $\delta(k,\tau) \sim \phi_p D_1(\tau) T(k)$, with the growth function

$$D_1(a) \propto H(a) \int^a \frac{da}{(aH(a))^3}. \tag{1.55}$$

The primordial curvature power spectrum is almost scale-free so it contributes with a factor of k^{-3}.

- First $k < k_{\rm eq}$.

Relates relate Δ_m to ϕ via the poisson equation

$$\Delta_m \sim k^2\phi \qquad \left[\phi(\eta,\bar{k}) = -\frac{3}{5}\mathcal{R}(0,\bar{k}) \qquad \text{matter domination}\right]$$

$$\Rightarrow \quad T(\eta,k) \propto k^2$$

$$P_{\Delta m}(k) \propto \frac{k^4}{k^3} \propto k \qquad \text{on large scales.}$$

• For $k > k_{eq}$

The newtonian-gauge δ_c is constant in time until horizon entry $\delta_c(0,\bar{k}) \propto \phi(0,\bar{k}) \propto \mathcal{R}(0,\bar{k})$.

Then, the Meszaros effect operates inside the horizon and δ_c grows logarithmically with proper time

$$\frac{\delta_c(t_{eq},k)}{\delta_c(0,\bar{k})} \sim 1 + \ln\left(\frac{t_{eq}}{t_k}\right) \tag{1.56}$$

at time of horizon entry $a(t_k)H(t_k) = k$ and $a \propto t^{1/2}$ in radiation domination $\Rightarrow t_k \propto \frac{1}{k^2}$

$$\begin{aligned} &\sim 1 + 2\ln\left(\frac{k}{k_{eq}}\right) \qquad k > k_{eq} \\ &T(\tau,k) \propto \ln\left(\frac{k}{k_{eq}}\right) \\ &P_{\Delta_m} \propto k^{-3}\ln^2\left(\frac{k}{k_{eq}}\right) \qquad k >> k_{eq} \end{aligned} \tag{1.57}$$

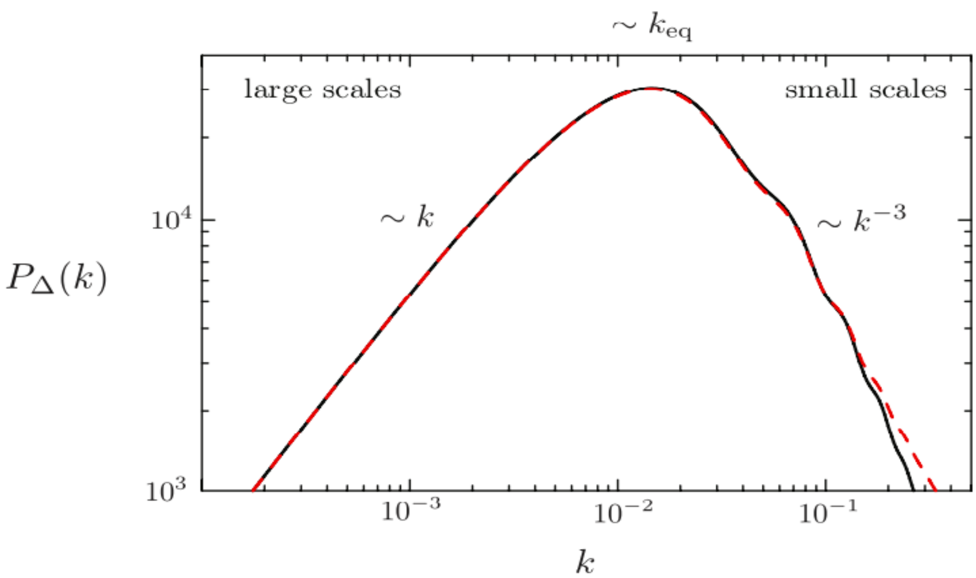


Figure 1.1: Evolution of perturbations.

Bibliography