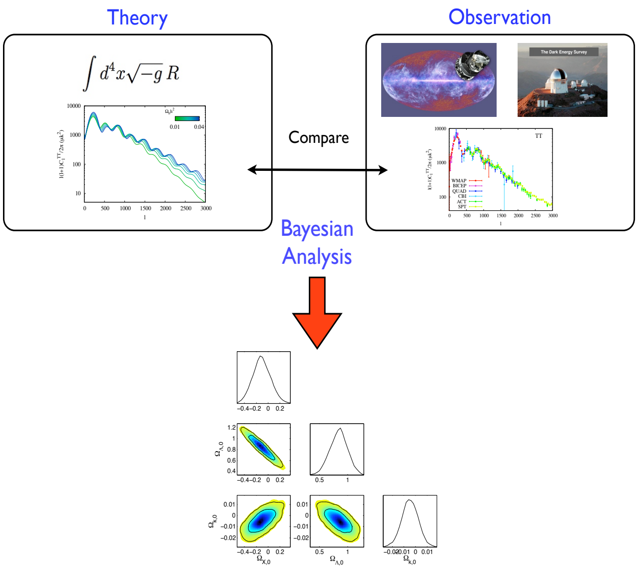
Updated Cosmology

with Python



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In progress

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Curvature Perturbation

1.0.1 Conserved curvature perturbation

An important quantity conserved on super Hubble scales for adiabatic scalar fluctuation irrespective of $w(z)$: the comoving curvature perturbation.

$\Rightarrow$ Perturbation to the intrinsic curvature scalar of comoving hypersurfaces ($q^i = 0$) allows to match the perturbations from inflation to those in the radiation dominated universe on large scales without needing to know the details of the reheating phase at the end inflation.

• Work out the intrinsic curvature of surfaces of constant time

- The induced metric γ_{ij} (just the spatial part)

$$\gamma_{ij} \equiv a^2[(1-2\phi)\delta_{ij} + 2E_{ij}],$$

with inverse $\gamma^{ij} = a^{-2}[(1+2\phi)\delta^{ij} - 2E^{ij}]$, and

$${}^{(3)}\Gamma^i{}_{jk} = \frac{1}{2}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}),$$

we need only γ to zero order, cause $\partial\gamma_{ij}$ are first order perturbations

$$\begin{aligned} {}^{(3)}\Gamma^i{}_{jk} &= \delta^{il}\partial_j(-\phi\delta_{kl} + E_{kl}) + \delta^{il}\partial_k(-\phi\delta_{jl} + E_{jl}) - \delta^{il}\partial_l(-\phi\delta_{jk} + E_{jk}) \\ &= -(2\delta^i_{(j}\partial_{k)}\phi - \delta^{il}\delta_{jk}\partial_l\phi) + (2\partial_{(j}E^i_{k)} - \delta^{il}\partial_l E_{jk}). \end{aligned} \tag{1.1}$$

The intrinsic curvature is the associated Ricci scalar

$${}^{(3)}R = \gamma^{ik}\partial_l^{(3)}\Gamma^l_{ik} - \gamma^{ik}\partial_k^{(3)}\Gamma^l_{il} + \underbrace{\gamma^{ik(3)}\Gamma^{l(3)}_{ik}\Gamma^m_{lm}}_{=0} \underbrace{-\gamma^{ik(3)}\Gamma^{m(3)}_{il}\Gamma^l_{km}}_{=0}, \tag{1.2}$$

to first-order

$$a^{2\,(3)}R = \delta^{ik}\partial_l{}^{(3)}\Gamma^l_{ik} - \delta^{ik}\partial_k{}^{(3)}\Gamma^l_{il}, \tag{1.3}$$

$$\begin{aligned}
\text{first term} &= -\delta^{ik}(2\delta^l_{(i}\partial_{k)}\phi - \delta^{il}\delta_{ik}\partial_j\phi) + \delta^{ik}(2\partial_{(i}E^l_{k)} - \delta^{jl}\partial_j E_{ik}) \\
&= -2\delta^{kl}\partial_k\phi + 3\delta^{jl}\partial_j\phi + 2\partial_i E^{il} - \delta^{jl}\partial_j(\underbrace{\delta^{ik}E_{ik}}_{=0}) \\
&= \delta^{kl}\partial_k\phi + 2\partial_k E^{kl},
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
\text{second term} &= -\delta^l_l\partial_i\phi - \delta^l_i\partial_l\phi + \partial_i\phi + \partial_l E^l_i + \partial_i E^l_l - \partial_l E^l_i \\
&= -3\partial_i\phi.
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
a^{2(3)}R &= \partial_l(\delta^{kl}\partial_k\phi + 2\partial_k E^{kl}) + 3\delta^{ik}\partial_k\partial_i\phi \\
&= \nabla^2\phi + 2\partial_i\partial_j E^{ij} + 3\nabla^2\phi \\
&= 4\nabla^2\phi + 2\partial_i\partial_j E^{ij}.
\end{aligned}$$

Vector and tensor perturbations are zero. Scalar perturbations, $E_{ij} = \partial_{\langle i}\partial_{j\rangle}E$

$$\begin{aligned}
\partial_i\partial_j E^{ij} &= \delta^{il}\delta^{jm}\partial_i\partial_j(\partial_l\partial_m E - \frac{1}{3}\delta_{lm}\nabla^2 E) \\
&= \nabla^2\nabla^2 E - \frac{1}{3}\nabla^2\nabla^2 E = \frac{2}{3}\nabla^4 E,
\end{aligned}$$

$$a^{2(3)}R = 4\nabla^2(\phi + \frac{1}{3}\nabla^2 E). \tag{1.6}$$

Define $-(\phi + \nabla^2 E/3) \equiv$ curvature perturbation, and the comoving curvature perturbation $\mathcal{R}$ evaluated in the comoving gauge ($B_i = 0 = q^i$), to form a gauge-invariant combination that equals $\mathcal{R}$ in the comoving gauge.

HW: $\mathcal{R} = -\phi - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v)$ is a gauge-invariant expression.

1.0.2 A conservation law

$\mathcal{R}$ is conserved on large scales for adiabatic perturbations. In the conformal newtonian gauge

$$\mathcal{R} = -\phi + \mathcal{H}v. \tag{1.7}$$

Using Einstein $0i$, we have

$$\mathcal{R} = -\phi - \frac{\mathcal{H}(\phi' + \mathcal{H}\psi)}{4\pi G a^2(\bar{\rho} + \bar{P})}. \tag{1.8}$$

Then,

$$\begin{aligned} -4\pi G a^2(\bar{\rho} + \bar{P})\mathcal{R}' &= 4\pi G a^2(\bar{\rho} + \bar{P})\phi' + \mathcal{H}'(\phi' + \mathcal{H}\psi) + \mathcal{H}(\phi'' + \mathcal{H}'\psi + \mathcal{H}\psi') \\ &\quad - \frac{\mathcal{H}(\phi' + \mathcal{H}\psi)}{(\bar{\rho} + \bar{P})}(2\mathcal{H}(\bar{\rho} + \bar{P}) + \bar{\rho}' + \bar{P}'), \end{aligned}$$

bearing in mind that $\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{P})$, we have

$$\begin{aligned} &= 4\pi G a^2(\bar{\rho} + \bar{P})\phi' + \mathcal{H}'(\phi' + \mathcal{H}\psi) + \mathcal{H}(\phi'' + \mathcal{H}'\psi + \mathcal{H}\psi') \\ &- \mathcal{H}(\phi' + \mathcal{H}\psi)\left[2\mathcal{H} - 3\mathcal{H}\left(1 + \frac{\bar{P}'}{\bar{\rho}'}\right)\right], \end{aligned} \tag{1.9}$$

in $\bar{P}'$ use Poisson, and $\mathcal{H}^2 - \mathcal{H}' = 4\pi G a^2(\bar{\rho} + \bar{P})$, to get

$$\begin{aligned} -4\pi G a^2(\bar{\rho} + \bar{P})\mathcal{R}' &= (\mathcal{H}^2 - \mathcal{H}')\phi' + \mathcal{H}'(\phi' + \mathcal{H}\psi) + \mathcal{H}(\phi'' + \mathcal{H}'\psi + \mathcal{H}\psi') \\ &\quad + \mathcal{H}^2(\phi' + \mathcal{H}\psi) + \frac{\mathcal{H}\bar{P}'}{\bar{\rho}'}(\nabla^2\phi - 4\pi G a^2\bar{\rho}\delta), \end{aligned} \tag{1.10}$$

adding and subtracting $4\pi G a^2 \mathcal{H}\delta P$

$$\begin{aligned} -4\pi G a^2(\bar{\rho} + \bar{P})\mathcal{R}' &= \mathcal{H}[\phi'' + \mathcal{H}\psi' + 2\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\psi - 4\pi G a^2\delta P] \\ &\quad + 4\pi G a^2\mathcal{H}\underbrace{\left(\delta P - \frac{\bar{P}'}{\bar{\rho}'}\bar{\rho}\delta\right)}_{\delta P_{\rm nad}} + \frac{\mathcal{H}\bar{P}'}{\bar{\rho}'}\nabla^2\phi, \end{aligned} \tag{1.11}$$

with $\delta P_{\rm nad}$, the non-adiabatic pressure perturbation. It is gauge invariant since $\delta P \quad \rightarrow \quad \delta P - T\bar{P}', \qquad \delta\rho \quad \rightarrow \quad \delta\rho - T\bar{\rho}'$.

It vanishes for a barotropic equation of state $\bar{P} = \bar{P}(\bar{\rho})$, and more generally it vanishes for adiabatic fluctuations. Using the trace-part of the ij Einstein equation

$$-4\pi G a^2(\bar{\rho} + \bar{P})\mathcal{R}' = \frac{1}{3}\mathcal{H}\nabla^2(\phi - \psi) + 4\pi G a^2\mathcal{H}\delta P_{\rm nad} + \frac{\mathcal{H}\bar{P}'}{\bar{\rho}'}\nabla^2\phi. \tag{1.12}$$

If $\delta P_{\rm nad}$ vanishes, the $RHS \sim \mathcal{H}k^2\phi \sim \mathcal{H}k^2\mathcal{R}$

$$\mathcal{H}^2\mathcal{R}' \sim \mathcal{H}k^2\mathcal{R} \quad \Rightarrow \quad \frac{d\ln\mathcal{R}}{d\ln a} \sim \left(\frac{k}{\mathcal{H}}\right)^2. \tag{1.13}$$

$\mathcal{R}$ doesn't evolve on super-Hubble scales $k \ll \mathcal{H}$.

$\mathcal{R}$ at horizon crossing during inflation survives unaltered until later times.

1. CURVATURE PERTURBATION

2

Evolution of fluctuations

Inflation sets the initial conditions up [for those superhorizon modes].

Spectrum of inflation for the curvature perturbation $\mathcal{R}$ $\leftrightarrow$ The gravitational potential ϕ, in the Newtonian gauge $\phi = \psi$

$$\mathcal{R} = -\phi - \frac{2}{3(1+w)} \left(\frac{\phi'}{\mathcal{H}} + \phi \right), \tag{2.1}$$

w is the equation of state for the background.

Adiabatic perturbations and constant equation of state $w \approx c_s^2$ for a single component

$$\phi'' + 3(1+w)\mathcal{H}\phi' + wk^2\phi = 0. \tag{2.2}$$

Homework: Derive the above relation

2.0.1 Super horizon limit

Consider scales $k \ll \mathcal{H}$, which means drop the last term in order to get

$$\phi'' + 3(1+w)\mathcal{H}\phi' = 0, \tag{2.3}$$

where the growing-mode solution is ϕ = constant, independent of w (for w = constant).

Meaning: The gravitational potential is frozen outside the horizon during both radiation and matter eras.

From the Poisson equation

$$\delta = -\frac{2}{3}\frac{k^2}{\mathcal{H}^2}\phi - \frac{2}{\mathcal{H}}\phi' - 2\phi \qquad \left[\frac{3}{2}\mathcal{H}^2 = 4\pi G a^2 \bar{\rho}\right], \tag{2.4}$$

on super horizon scales $k \ll \mathcal{H}$ and $\phi \approx$ constant,

$$\delta \approx -2\phi = \text{constant}, \tag{2.5}$$

primordial gravitational potential $\Rightarrow$ δ is also frozen on super-horizon scales.

For adiabatic initial conditions

$$\delta_{\rm m} = \frac{3}{4}\delta_{\rm r} \approx -\frac{3}{2}\phi_{\rm rad}, \tag{2.6}$$

($\delta_{\rm r} \approx \delta$ during radiation era). On super horizon scales, the density perturbations are therefore simply proportional to the curvature perturbations set up by inflation.

Radiation to matter transition

Exploit the conservation or $\mathcal{R}$ on super-Hubble scales $k \ll \mathcal{H}$

$$\mathcal{R} = -\frac{5+3w}{3+3w}\phi, \tag{2.7}$$

(super horizon) relating fluctuations the CMB to primordial fluctuation.
Considering $w = \frac{1}{3}$, $w = 0$

$$\mathcal{R} = -\frac{3}{2}\phi_{\rm rad} = -\frac{5}{3}\phi_{\rm mat} \quad \Rightarrow \quad \phi_{\rm mat} = \frac{9}{10}\phi_{\rm rad}, \tag{2.8}$$

ϕ decreases by a factor of 9/10 in the transition from radiation dominated to matter dominated.

2.1 Evolution fo fluctuations

• Gravitational potential (??)

- Radiation Era

$$\delta P = \frac{\delta\rho}{3}, \quad \mathcal{H}^2 \propto a^{-2}, \quad a \propto \tau, \quad \mathcal{H} = \frac{1}{\tau}, \tag{2.9}$$

$$\phi'' + \frac{3}{\tau}\phi' - \frac{1}{\tau^2}\phi = \frac{4\pi G a^2}{3}\bar{\rho}\delta_{\rm r}. \tag{2.10}$$

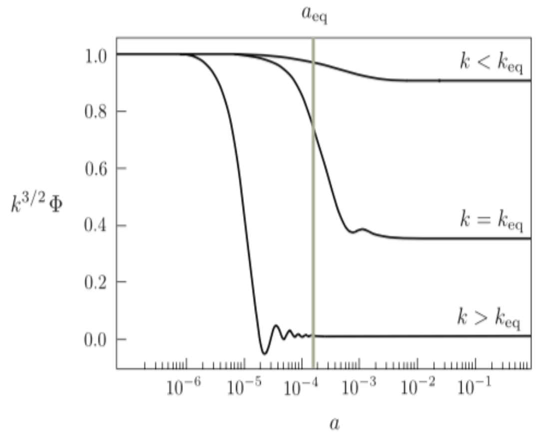

Figure 2.1: Evolution of perturbations.

using Einstein 00

$$\phi'' + \frac{3}{\tau}\phi' - \frac{1}{\tau^2}\phi = \frac{1}{3}\nabla^2\phi - \frac{1}{\tau}\left(\phi' + \frac{1}{\tau}\phi\right)$$
$$\Rightarrow \phi'' + \frac{4}{\tau}\phi' - \frac{1}{3}\nabla^2\phi = 0$$
$$\phi'' + \frac{4}{\tau}\phi' + \frac{k^2}{3}\phi = 0, \tag{2.11}$$

damped wave equation with propagation speed $\frac{1}{\sqrt{3}}$.
Now, using $\phi = u(x)/x$ and $x = k\tau/\sqrt{3}$, we have

$$u'' + \frac{2}{x}u' + \left(1 - \frac{2}{x^2}\right)u = 0, \tag{2.12}$$

with solutions

$$\phi(x) = A_k\frac{J_1(x)}{x} + B_k\frac{n_1(x)}{x}. \tag{2.13}$$

Where the Bessel and Neumann are described by

$$J_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} = \frac{x}{3} + \mathcal{O}(x^3), \tag{2.14}$$
$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} = -\frac{1}{x^2} + \mathcal{O}(x^3), \tag{2.15}$$

however, the Neumann blows up for small x (early times): hence reject that solution.

The proportionality constant A_k can be found by matching the solutions to the primordial value of the potential

$$\phi_k(0) = -\frac{2}{3}\mathcal{R}_k(0), \tag{2.16}$$

and therefore

$$\phi_k(\tau) = -2\mathcal{R}_k(0)\left(\frac{\sin x - x\cos x}{x^3}\right) \qquad \text{all scales} \tag{2.17}$$

- Outside the sound horizon $x = \frac{1}{\sqrt{3}}k\tau \ll 1 \quad \Rightarrow \quad \phi = \text{constant}.$

- On subhorizon scales $x \gg 1$ [use that $J_\ell(x) \sim \frac{1}{x}\sin\left(x - \frac{\ell\pi}{3}\right)$] to get

$$\phi_k(\tau) = 6\mathcal{R}_k(0)\frac{\cos\left(\frac{1}{\sqrt{3}}k\tau\right)}{(k\tau)^2}. \tag{2.18}$$

During the radiation era, subhorizon modes of ϕ oscillate with $f = \frac{1}{\sqrt{3}}k$ and amplitude that decays as $\tau^{-2} \propto a^{-2}$.

2.2 Matter era

$w = 0$

$$\phi'' + \frac{6}{\tau}\phi = 0, \tag{2.19}$$

$$\phi = \begin{cases} \text{constant} \\ \tau^{-5} \propto a^-5/2 \end{cases} \tag{2.20}$$

$\phi =$ cons is frozen on all scales during matter domination.

2.3 Radiation

2.3.1 Evolution of perturbations

Radiation era (radiation perturbations)

$$\delta_r = -\frac{2}{3}(k\tau)^2\phi - 2\tau\phi' - 2\phi \tag{2.21}$$

$$\Delta_r = -\frac{2}{3}(k\tau)^2\phi. \tag{2.22}$$

(using $\Delta = \nabla^2\phi/4\pi Ga^2\rho$, $3/2\mathcal{H}^2 = 4\pi Ga^2\rho$, $\mathcal{H} \propto 1/\tau \propto 1/a$).
Outside the horizon ($k\tau \ll 1$) $\delta_\mathrm{r} = -2\phi$ and is constant, Δ_r grows as $\tau^2 \propto a^2$.

Inside the horizon ($k\tau \gg 1$)

$$\delta_\mathrm{r} \approx \Delta_\mathrm{r} = -\frac{2}{3}(k\tau)^2\phi = 4\mathcal{R}(0)\cos\left(\frac{1}{\sqrt{3}}k\tau\right), \tag{2.23}$$

oscillates w/constant amplitude and $4\mathcal{R}(0)$ is the solution $\delta_{\rm r}'' - \frac{1}{3}\nabla^2\delta_{\rm r} = 0$.

Matter era

Radiation perturbations are subdominant

On sub-horizon

$$\left.\begin{aligned} \delta_{\rm r}' &= -\tfrac{4}{3}\nabla\cdot\vec{v}_{\rm r} \\ v_{\rm r}' &= -\tfrac{1}{4}\nabla\delta_{\rm r} - \nabla\phi \end{aligned}\right\} \delta_{\rm r}'' - \frac{1}{3}\nabla^2\delta_{\rm r} = \frac{4}{3}\nabla^2\phi = \text{cons}, \tag{2.24}$$

is an harmonic oscillator w/constant driving force, the acoustic oscillator $\rightarrow$ give rise to the peaks in the CMB spectrum.

Dark matter

Early times $(r+m)$ (no baryons)

$$\mathcal{H}^2 = \frac{\mathcal{H}_0^2\Omega_{\rm m}^2}{\Omega_{\rm r}}\left(\frac{1}{y} + \frac{1}{y^2}\right) \qquad y \equiv \frac{a}{a_{\rm eq}}. \tag{2.25}$$

- Subhorizon scales

$$\left.\begin{aligned} \delta_{\rm m}' &= -\nabla\cdot\vec{v}_{\rm m} \\ \vec{v}_{\rm m}{}' &= -\mathcal{H}\vec{v}_{\rm m} - \nabla\phi \end{aligned}\right\} \delta_{\rm m}'' + \mathcal{H}\delta_{\rm m}' = \nabla^2\phi. \tag{2.26}$$

(only sourced by matter fluctuations)

$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2\bar{\rho}_m\delta_m \approx 0. \tag{2.27}$$

can be written as the Meszaros equation

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0 \tag{2.28}$$

with solutions.

$$\delta_m \propto \begin{cases} 2+3y \\ (2+3y)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 6\sqrt{1+y} \end{cases} \tag{2.29}$$

- For $y \ll 1$ (RD) $\quad\Rightarrow\quad \delta_{\rm m} \propto \ln y \propto \ln a$.
- For $y \gg 1$ (MD) $\quad\Rightarrow\quad \delta_{\rm m} \propto y \propto a$.

		$\mathcal{R}D$		mD	
		ϕ	$\delta_m(\Delta_m)$	ϕ	$\delta_m(\Delta_m)$
$k >> k_{eq}$	super	constant	constant (a^2)	-	-
	sub	a^{-2}	$\ln a$	constant	a
$k << k_{eq}$	super	constant	constant (a^2)	constant	constant (a)
	sub	-	-	constant	a

Intermediate times

From the gravitational potential

$$\Delta_{\mathrm{m}} = \frac{\nabla^2 \phi}{4\pi G a^2 \bar{\rho}} \propto \begin{cases} a \\ a^{-\frac{3}{2}} \end{cases} \tag{2.30}$$

Late times $(m + \Lambda)$

$$\nabla^2 \phi = 4\pi G a^2 \bar{\rho}_m \Delta,$$

pressure fluctuations are negligable

$$\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 0$$

since $a^2 \bar{\rho}_m \propto a^{-1}$, $\phi \propto \Delta_m / a$

$$\begin{aligned} &\partial_\eta^2 \left(\frac{\Delta_m}{a}\right) + 3\mathcal{H}\partial_\eta \left(\frac{\Delta_m}{a}\right) + (2\mathcal{H}' + \mathcal{H}^2)\left(\frac{\Delta_m}{a}\right) = 0 \\ &\Delta_m'' + \mathcal{H}\Delta_m' + (\mathcal{H}' - \mathcal{H}^2)\Delta_m = 0 \qquad \text{w/Friedmann} \\ &\Delta_m'' + \mathcal{H}\Delta_m' - 4\pi G a^2 \bar{\rho}_m \Delta_m = 0 \qquad \text{valid on all scales} \end{aligned} \tag{2.31}$$

Baryons

• Before decoupling.

$z > z_{\mathrm{dec}} \approx 1100$, photons and baryons are coupled strongly to each other via Compton scattering

$$v_\gamma = v_{\mathrm{b}}, \qquad \delta_\gamma = \frac{4}{3}\delta_{\mathrm{b}}.$$

The pressure of the photons supports oscillations on small scales

• After decoupling

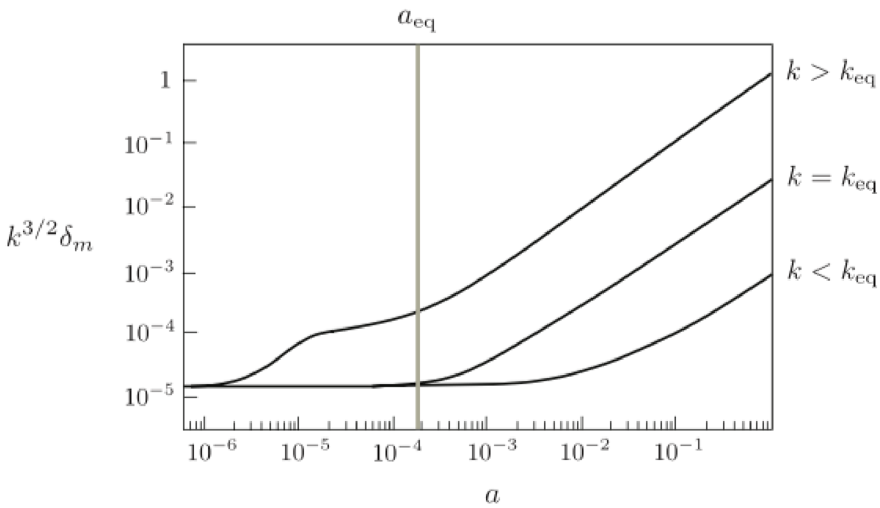

Figure 2.2: Evolution of perturbations.

$$
\begin{aligned}
&\delta_b' + \mathcal{H}\delta_b' = 4\pi G a^2(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c) \\
&\delta_c' + \mathcal{H}\delta_c' = 4\pi G a^2(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c) \\
&\bar{\rho}_m\delta_m = \bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c.
\end{aligned}
$$

Define $D \equiv \delta_b - \delta_c$, subtracting

$$
D'' + \frac{2}{\tau}D' = 0 \qquad D \propto \begin{cases} \text{constant} \\ \tau^{-1} \end{cases}
$$

while δ_m

$$
\delta_m'' + \frac{2}{\tau}\delta_m' - \frac{6}{\tau^2}\delta_m = 0 \qquad \delta_m \propto \begin{cases} \tau^2 \\ \tau^{-3} \end{cases}
$$

since

$$
\frac{\delta_b}{\delta_c} = \frac{\bar{\rho}_m\delta_m + \bar{\rho}_c D}{\bar{m}\delta_m - \bar{\rho}_b D} \quad \rightarrow \quad \frac{\delta_m}{\delta_m} = 1 \tag{2.32}
$$

δ_b approaches δ_c.

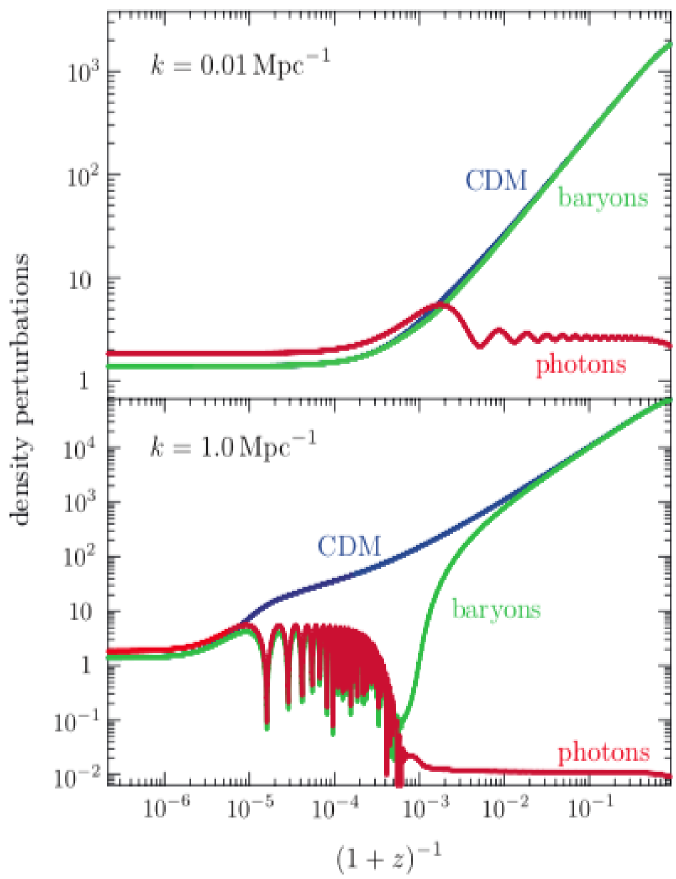

Figure 2.3: Evolution of perturbations.

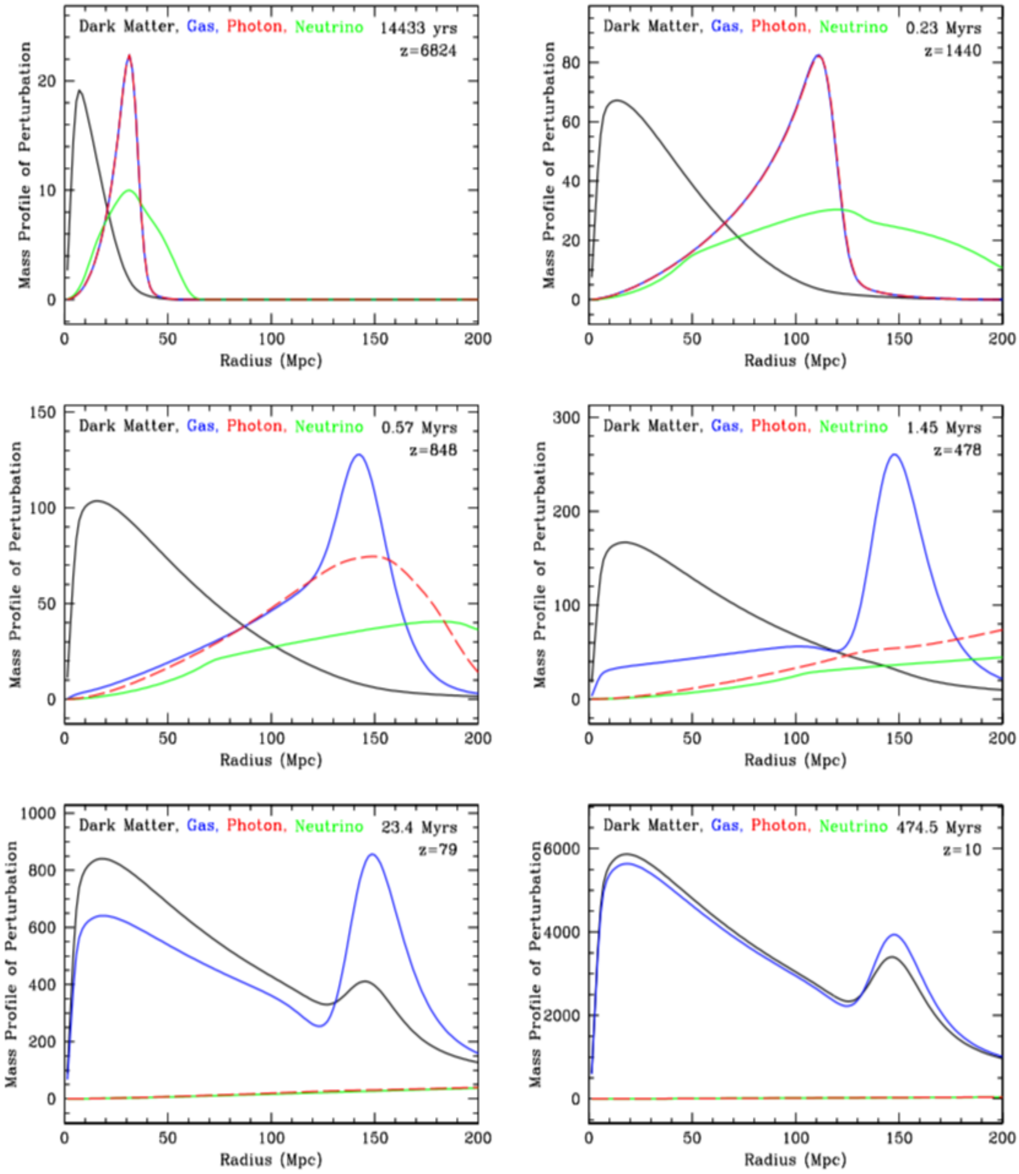

Figure 2.4: Evolution of perturbations.

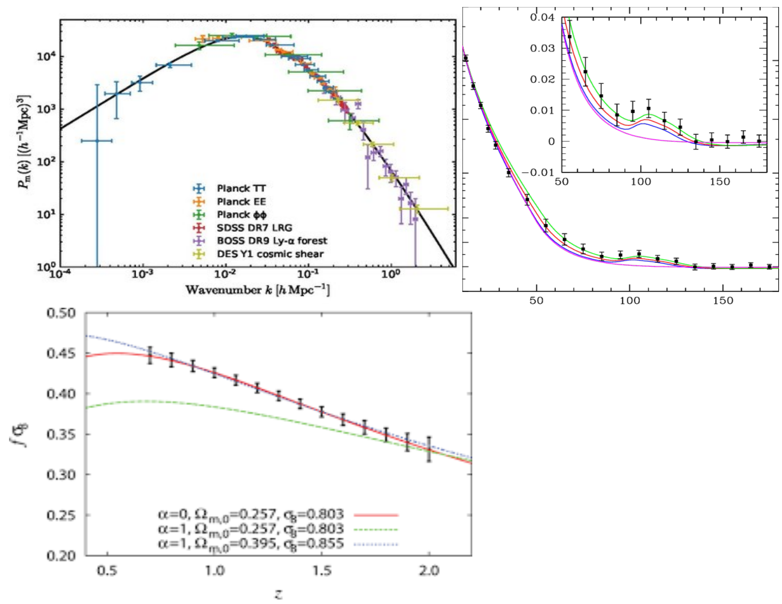

Figure 2.5: Evolution of perturbations.

Bibliography