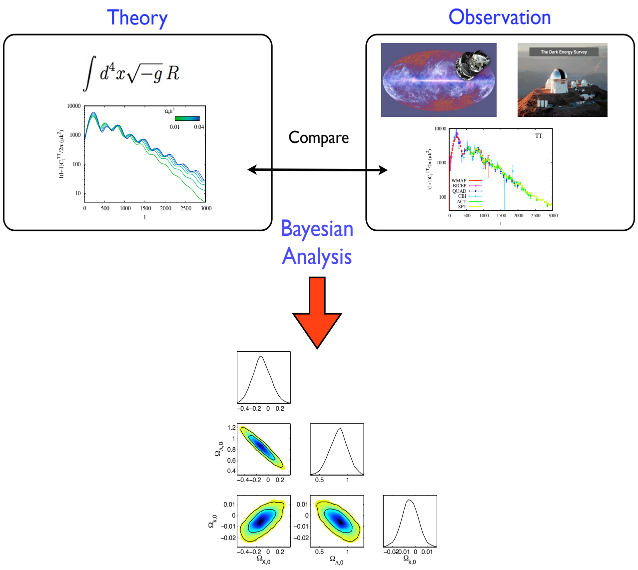
Updated Cosmology

with Python



José-Alberto Vázquez

ICF-UNAM / Kavli-Cambridge

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Relativistic Perturbation Theory

Scales larger than the Hubble radius and for relativistic fluids (Newton is inadequate), so now we use the basic idea:

- Perturb the metric.
- Perturb the stress-energy tensor.
- In Einstein equations, for linear perturbations, drop products of small quantities.

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \Lambda \delta g_{\mu\nu}.$$

3.1 Perturbed Spacetime

Let us start by perturbing the metric of the space-time.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu},$$

where the background metric corresponds to the spatially-flat FLRW

$$ds^2 = a^2(\tau)(d\tau^2 - \delta_{ij} dx^i dx^j) = a^2 \eta_{\mu\nu} dx^\mu dx^\nu, \tag{3.1}$$

with Friedmann equations, in conformal time,

$$\mathcal{H}^2 = \frac{1}{3} a^2 (8\pi G\rho + \Lambda), \tag{3.2}$$

$$\mathcal{H}' = \frac{1}{6} a^2 [2\Lambda - 8\pi G(\rho + 3P)], \tag{3.3}$$

with $\mathcal{H}' \equiv \partial_\tau a/a = aH$. The most general perturbation to the background metric is:

$$ds^2 = a^2(\tau)\left\{(1+2\psi)d\tau^2 - 2B_i dx^i d\tau - [(1-2\phi)\delta_{ij} + 2E_{ij}]dx^i dx^j\right\}. \tag{3.4}$$

- ϕ, ψ are scalar functions of τ and x^i.
- B_i transforms like a three-vector

$$x^i \to x'^i, \quad B_i \to \frac{\partial x^j}{\partial x'^i} B_j.$$

- E_{ij} is symmetric ($E_{ij} = E_{ji}$) and trace-free ($\delta^{ij}E_{ij} = 0$) three-tensor

$$x^i \to x'^j, \quad E_{ij} \to \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} E_{kl},$$

latin indices on spatial vectors and tensors are raised and lowered with $\delta_{ij} \to B^i = \delta^{ij}B_j$, $E^i_j = \delta^{ik}E_{kj}$.

3.2 Scalar, vector and tensor decomposition

A formal proof can be found in J.M. Stewart, Class. Quantum Grav. 7 (1990) 1169.

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{B_i^T}_{\text{vector}}.$$

The vector part is transverse (divergence free) $\delta^{ij}\partial_j B_i^T = 0$, whereas the tensor component

$$E_{ij} = \underbrace{\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{\partial_{(i}E_{j)}^T}_{\text{vector}} + \underbrace{E_{ij}^T}_{\text{tensor}}.$$

the scalar part is trace-free and satisfies

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E,$$

the vector part E_i^T is transverse: $\delta^{ij}\partial_j E_i^T = 0$, with

$$\partial_{(i}E_{j)}^T \equiv \frac{1}{2}\left(\partial_i E_j^T + \partial_j E_i^T\right),$$

and the tensor E_{ij}^T is symmetric, trace-free and transverse $\delta^{ik}\partial_k E_{ij}^T = 0$.

Number of degrees of freedom from each contribution:

- Scalar - one degree of freedom $E(\tau, \vec{x})$, and they are ψ, ϕ, E, B: 4 degrees of freedom.

- Vector - two degrees of freedom, E_i^T - three components but one constraint (divergence-vanishes). They are E_i, B_i, then 4 degrees of freedom.

- Tensors - two degrees fo freedom, E_{ij}^T - five components (6: symmetric - 1 for being trace-free), but $\delta^{ij}\partial_k, E_{ij}^T$ are three constraints.
  Total 4 + 4 + 2 = 10 degrees of freedom (number of equations we need - minus the coordinates).

3.3 Orthogonal frame vectors

Remember the stress-energy tensor for a perfect fluid is

$$\bar{T}^{\mu}{}_{\nu} = (\bar{\rho} + \bar{P})\bar{u}^{\mu}\bar{u}_{\nu} - \bar{P}\delta^{\mu}_{\nu}. \tag{3.5}$$

where $\bar{u}_\mu = a\delta^0_\mu$, $\bar{u}^\mu = a^{-1}\delta^\mu_0$ for a comoving observer, with $u^\mu = \frac{dx^\mu}{dt}$.

$$g_{\mu\nu}u^\mu u^\nu = 1, \quad \text{and} \quad \bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = 1. \tag{3.6}$$

$$\begin{aligned} 1 &= g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \\ &= g_{\mu\nu}\left(\frac{d\tau}{dt}\right)^2\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ &= \left(\frac{d\tau}{dt}\right)^2\left(g_{00} + 2\underbrace{g_{0i}\frac{dx^i}{d\tau}}_{\mathcal{O}(2)} + g_{ij}\underbrace{\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}_{\mathcal{O}(2)}\right), \end{aligned}$$

where $v^i \equiv \frac{dx^i}{d\tau}$ is the coordinate velocity - small perturbation.

$$\Rightarrow \quad 1 = \left(\frac{d\tau}{dt}\right)^2 g_{00} = a^2(1+2\psi)\left(\frac{d\tau}{dt}\right)^2,$$

and, at linear order

$$\frac{d\tau}{dt} = \frac{1}{a}(1-\psi). \tag{3.7}$$

Therefore, the fluid 4-velocity is $u^\mu = a^{-1}[1-\psi, v^i]$, and

$$u_0 = g_{00}u^0 + g_{0i}u^i = a^2(1+2\psi)a^{-1}(1-\psi) + \mathcal{O}(2) = a(1+\psi), \tag{3.8}$$

$$u_i = g_{i0}u^0 + g_{ij}u^j = -a^2B_i a^{-1}(1-\psi) - a^2[\delta_{ij} + \mathcal{O}(1)]a^{-1}v^j = -a(B_i + v_i). \tag{3.9}$$

We need to construct explicitly an orthonormal frame of 4-vectors $(E_0)^\mu$ and $(E_i)^\mu$ in the perturbed metric. Since u^μ is the perturbed 4-velocity of an observer at rest, relative to the coordinate system, in the perturbed metric

• Timelike $(E_0)^\mu$ - Then, perturbing (3.6):

$$\delta g_{\mu\nu}\bar{u}^\mu\bar{u}^\nu + 2\bar{u}_\mu\delta u^\mu = 0.$$

Using $\bar{u}^\mu = a^{-1}\delta^\mu_0$ and $\delta g_{00} = 2a^2\psi$, then $\rightarrow$ $\delta u^0 = -\psi a^{-1}$. Therefore

$$(E_0)^\mu = a^{-1}(1-\psi)\delta^\mu_0. \tag{3.10}$$

Check orthogonality: $g_{\mu\nu}(E_0)^\mu(E_0)^\nu = g_{00}a^{-2}(1-2\psi) + 0 = a^2(1+2\psi)a^{-2}(1-2\psi) = 1.$

• Spacelike $(E_i)^\mu$ -

$$(E_i)^\mu = a^{-1}[B_i\delta^\mu_0 + (1+\phi)\delta^\mu_i - E^j_i\delta^\mu_j]. \tag{3.11}$$

Check orthogonality: $g_{\mu\nu}(E_0)^\mu(E_i)^\nu = g_{0\nu}a^{-1}(1-\psi)(E_i)^\nu = a^{-2}(1-\psi)[g_{00}B_i + g_{0i}(1+\phi) - g_{0j}E^j_i] = B_i - B_i = 0.$

HW: check

$$g_{\mu\nu}(E_i)^\mu(E_j)^\nu = -\delta_{ij}. \tag{3.12}$$

3.4 Matter perturbations

We can derive the perturbed expressions by using (3.5) with

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu,$$

$$\delta T^\nu_\mu = (\delta\rho + \delta P)\bar{u}^\mu\bar{u}_\nu + (\bar{\rho} + \bar{P})(\delta u^\mu\bar{u}_\nu + \bar{u}^\mu\delta u_\nu) - \delta P\delta^\mu_\nu - \Pi^\mu_\nu.$$

In an orthonormal frame

$$\begin{aligned} T^{\hat{0}\hat{0}} &= \bar{\rho}(\tau) + \delta\rho, && \text{energy density.} \\ T^{\hat{0}\hat{i}} &= q^i, && \text{momentum density.} \\ T^{\hat{i}\hat{j}} &= [\bar{P}(\tau) + \delta P]\delta^{ij} - \Pi^{ij}, && \text{momentum flux,} \end{aligned}$$

where Π^{ij} is the trace-free anisotropic stress. If there're several contributions $T_{\mu\nu} = \sum_I T^I_{\mu\nu}$

$$\delta\rho = \sum \delta\rho_I, \quad \delta P = \sum \delta P_I, \quad q^i = \sum \delta q^i_I, \quad \Pi^{ij} = \sum_I \Pi^{ij}_I,$$

and apply the SVT decomposition for q_i, Π_{ij}.

• Construct the coordinate components of the energy-momentum tensor in terms of the 4-vectors (3.10) and (3.11), with $T^{\mu\nu} = (E_\alpha)^\mu (E_\beta)^\nu T^{\hat\alpha\hat\beta}$,

$$\begin{aligned} T^{00} &= (E_0)^0(E_0)^0 T^{\hat 0\hat 0} + 2(E_0)^0(E_i)^0 T^{\hat 0\hat\imath} + (E_i)^0(E_j)^0 T^{\hat\imath\hat\jmath} \\ &= a^{-2}(1-2\psi)\bar\rho(1+\delta) + \mathcal{O}(2) + \mathcal{O}(2) \\ &= a^{-2}\bar\rho(1-2\psi+\delta). \end{aligned} \tag{3.13}$$

$$\begin{aligned} T^{0i} &= (E_0)^0(E_0)^i T^{\hat 0\hat 0} + (E_0)^0(E_j)^i T^{\hat 0\hat\jmath} + (E_j)^0(E_0)^i T^{\hat 0\hat\jmath} + (E_j)^0(E_k)^i T^{\hat\jmath\hat k} \\ &= 0 + a^{-2}\delta^i_j q^j + 0 + a^{-2}B_j\delta^i_k \bar P\delta^{jk} \\ &= a^{-2}(q^i + \bar P B^i). \end{aligned} \tag{3.14}$$

$$\begin{aligned} T^{ij} &= (E_0)^i(E_0)^j T^{\hat 0\hat 0} + (E_0)^i(E_k)^j T^{\hat 0\hat k} + (E_k)^i(E_0)^j T^{\hat k\hat 0} + (E_k)^i(E_l)^j T^{\hat k\hat l} \\ &= 0 + 0 + 0 + a^{-2}[(1+\phi)\delta^i_k - E^i_k][(1+\phi)\delta^j_l - E^j_l][(\bar P + \delta P)\delta^{kl} - \Pi^{kl}] \\ &= a^{-2}[\bar P\delta^{ij} + (2\bar P\phi + \delta P)\delta^{ij} - 2\bar P E^{ij} - \Pi^{ij}]. \end{aligned} \tag{3.15}$$

Things look neater in mixed coordinate components

$$\begin{aligned} T^0_0 &= g_{\mu 0}T^{0\mu} = g_{00}T^{00} + g_{0i}T^{0i} \\ &= a^2(1+2\psi)a^{-2}\bar\rho(1-2\psi+\delta) + \mathcal{O}(2) \\ &= \bar\rho(1+\delta). \end{aligned} \tag{3.16}$$

Peculiar velocity of the matter, using eqn. (3.5) with the components u^μ and u_μ

$$\begin{aligned} q^i = T^i_0 &= (\rho + P)u^i u_0 - P\delta^i_0 \\ &= (\bar\rho + \bar P)a^{-1}v^i a(1+\psi) = (\bar\rho + \bar P)v^i. \end{aligned} \tag{3.17}$$

HW:

$$T^i_0 = q^i.$$

$$T^i_j = -(\bar P + \delta P)\delta^i_j + \Pi^i_j.$$

3.5 The gauge problem

The metric perturbations aren't uniquely defined, but depend on our choice of coordinates or the *gauge choice.*

• We implicitly chose a *specific time slicing* of the spacetime and *specific spatial coordinate* on these slices.

⇒ Making a different choice of coordinates can change the values of the perturbation variables and may even introduce fictitious perturbations.

i.e. **FRW spacetime**, change the spatial coordinates

$$\begin{aligned} x^i &\rightarrow \tilde{x}^i + \xi^i(\tau, \vec{x}), \\ \Rightarrow dx^i &= d\tilde{x}^i - \partial_\tau \xi^i d\tau - \partial_k \xi^i d\tilde{x}^k. \end{aligned}$$

sometimes we'll use $\frac{\partial}{\partial \tau} ='$ such that $\xi_i' \equiv \partial_\tau \xi_i$. Then

$$ds^2 = a^2(\tau)[d\tau^2 - 2\xi_i' d\tilde{x}^i d\tau - (\delta_{ij} + 2\partial_{(i}\xi_{j)})d\tilde{x}^i d\tilde{x}^j].$$

We *apparently have introduced* metric perturbations, but these are just *fictitious gauge modes* that can be removed by going back to the old coordinates.

Similarly, we can change our time slicing $\tau \rightarrow \tau + \xi^0(\tau, \vec{x})$:

$$\rho(\tau) \rightarrow \rho(\tau + \xi^0(\eta, \vec{x})) = \bar{\rho}(\tau) + \bar{\rho}'\xi^0.$$

Even in an unperturbed universe a change of coordinates can introduce a fictitious density perturbation. Therefore, we need a more physical way to identify true perturbations → define them in such a way the don't change under change of coordinates.

3.6 Gauge transformations

Consider small changes in coordinates $x^\mu \rightarrow \tilde{x}^\mu \equiv x^\mu + \xi^\mu(\tau, \vec{x})$

$$\tilde{\tau} = \tau + T(\tau, x^i) \qquad \tilde{x}^i = x^i + L^i(\tau, x^j).$$

where T and L^i are arbitrary functions still to be defined. For a Lorentz scalar field Φ (i.e. inflation), the new perturbation at the same event is (remember $\bar{\Phi}$ is homogeneous)

$$\begin{aligned} \delta\tilde{\Phi} &= \tilde{\Phi} - \bar{\Phi}(\tilde{\eta}) \\ &= \Phi - \bar{\Phi}(\eta + T) \\ &= \underbrace{\Phi - \bar{\Phi}(\eta)}_{\delta\Phi} - T\bar{\Phi}', \end{aligned}$$

to first order

$$\delta\tilde{\Phi} = \delta\Phi - T\bar{\Phi}'. \tag{3.18}$$

For metric perturbations

Remember $\bar{g}_{\mu\nu}$ is the metric in the background

$$\begin{aligned}
\delta\tilde{g}_{\mu\nu} &= \tilde{g}_{\mu\nu} - \bar{g}_{\mu\nu}(\tilde{\tau},\tilde{x}^i) \\
&= \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} g_{\alpha\beta} - \bar{g}_{\mu\nu}(\tilde{\tau},\tilde{x}^i) \\
&= \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}[\delta g_{\alpha\beta} + \bar{g}_{\alpha\beta}(\tau,x^i)] - \bar{g}_{\mu\nu}(\tilde{\tau},\tilde{x}^i) \\
&= \delta g_{\mu\nu} + \left(\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} - \delta^\alpha_\mu\delta^\beta_\nu\right)\bar{g}_{\alpha\beta}(\tau,x^i) - T\bar{g}'_{\mu\nu}(\tau,x^i) - L^i\partial_i\bar{g}_{\mu\nu}(\tau,x^i),
\end{aligned}$$

where for the first term we have that $\delta g_{\alpha\beta}$ is a perturbation, hence taking only the 0-order of $\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}$. We still need to know the inverse of $\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}$, to linear order.

The matrix of derivates

$$\frac{\partial\tilde{x}^\alpha}{\partial x^\mu} = \begin{pmatrix} \partial\tilde{\tau}/\partial\tau & \partial\tilde{\tau}/\partial x^i \\ \partial\tilde{x}^i/\partial\tau & \partial\tilde{x}^i/\partial x^j \end{pmatrix} = \begin{pmatrix} 1+T' & \partial_i T \\ \partial_\tau L^i & \delta^i_j + \partial_j L^i \end{pmatrix}.$$

The inverse of matrix with form $1+\Delta$ is $1-\Delta$, to first order in Δ, then

$$\frac{\partial x^\alpha}{\partial\tilde{x}^\mu} = \begin{pmatrix} \partial\tau/\partial\tilde{\tau} & \partial\tau/\partial\tilde{x}^i \\ \partial x^i/\partial\tilde{\tau} & \partial x^i/\partial\tilde{x}^j \end{pmatrix} = \begin{pmatrix} 1-T' & -\partial_i T \\ -\partial_\tau L^i & \delta^i_j - \partial_j L^i \end{pmatrix}.$$

Substituting

$$\delta\tilde{g}_{00} = \delta g_{00} + \left(\frac{\partial x^\alpha}{\partial\tilde{\tau}}\frac{\partial x^\beta}{\partial\tilde{\tau}} - \delta^\alpha_0\delta^\beta_0\right)\bar{g}_{\alpha\beta} - T\bar{g}'_{00} - L^i\partial_i\bar{g}_{00}.$$

Therefore

$$\begin{aligned}
2a^2\tilde{\psi} &= 2a^2\psi + \left(\frac{\partial\tau}{\partial\tilde{\tau}}\frac{\partial\tau}{\partial\tilde{\tau}} - 1\right)\bar{g}_{00} + 2\frac{\partial\tau}{\partial\tilde{\tau}}\frac{\partial x^i}{\partial\tilde{\tau}}\bar{g}_{0i} + \frac{\partial x^i}{\partial\tilde{\tau}}\frac{\partial x^j}{\partial\tilde{\tau}}\bar{g}_{ij} - T\partial_\tau a^2 \\
&= 2a^2\psi - 2T'a^2 + \mathcal{O}(2) + \mathcal{O}(2) - 2T\mathcal{H}a^2.
\end{aligned}$$

$$\therefore\ \tilde{\psi} = \psi - T' - \mathcal{H}T. \tag{3.19}$$

HW:

$$\tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\partial_i L^i. \tag{3.20}$$

$$\tilde{B}_i = B_i + \partial_i T - L'_i. \tag{3.21}$$

$$\tilde{E}_{ij} = E_{ij} - \partial_{<i}L_{j>}. \tag{3.22}$$

3.7 Gauge invariant perturbations

SVT - decomposition and considering only the scalar modes

$$\tilde{\tau} = \tau + T, \qquad \tilde{x}^i = x^i + \delta^{ij}\partial_j L.$$

We have

$$\tilde{\psi} = \psi - T' - \mathcal{H}T. \tag{3.23}$$

$$\tilde{\phi} = \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L. \tag{3.24}$$

$$\tilde{B} = B + T - L'. \tag{3.25}$$

$$\tilde{E} = E - L. \tag{3.26}$$

4 functional degrees of freedom and 2 gauge functions (T and L) ⇒ construct two gauge invariant quantities (do not change under gauge transformation).

Bardeen variables

$$\Psi \equiv \psi + \mathcal{H}(B - E') + B' - E''. \tag{3.27}$$

$$\Phi \equiv \phi - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E. \tag{3.28}$$

$$\Phi_i \equiv {E_i}' - B_i. \tag{3.29}$$

HW: show that Φ, Ψ, Φ_i don't change under a coordinate transformation.

3.7.1 Gauge fixing

An alternative solution to the gauge problem is to **fix the gauge** and keep track of all perturbations i.e. use the freedom of T and L to set two scalar metric perturbations to zero.

Newtonian gauge (conformal)

Choose T and L such that

$$E = B = 0.$$

$$ds^2 = a^2(\tau)[(1 + 2\psi)d\tau^2 - (1 - 2\phi)\delta_{ij}dx^i dx^j].$$

$$\Psi = \psi, \qquad \Phi = \phi \qquad \text{Bardeen potentials.}$$

The physics appears simple since:

- Hypersurfaces of constant time are orthogonal to the observers at rest ($B = 0$).
- Induced geometry of the constant-time hypersurfaces is isotropic ($E = 0$).
- In the absence of anisotropic stress $\Psi = \Phi$.
- Similarity of the metric to the usual weak-field limit of GR about Minkowski.
- Ψ plays the role of the gravitational potential (will see).

Therefore, the preferred gauge for studying the formation of large-scale structures and CMB anisotropies.

Spatially flat gauge

$$\phi = E = 0.$$

Convenient for computing inflationary perturbations. Fluctuations in the inflaton field.

Synchronous gauge

$$\psi = B = 0.$$

See: Ma and Betschinger.

3.8 Perturbations of the Stress-Energy tensor

Repeat the analysis (more convenient with mixed components):

$$\begin{aligned}
\delta\tilde{T}^{\mu}_{\nu} &= \frac{\partial\tilde{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}}T^{\alpha}_{\beta} - \bar{T}^{\mu}_{\nu}(\tau + T, x^i + L^i) \\
&= \frac{\partial\tilde{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}}[\delta T^{\alpha}_{\beta} + \bar{T}^{\alpha}_{\beta}] - \bar{T}^{\mu}_{\nu}(\tau + T, x^i + L^i) \\
&= \delta T^{\mu}_{\nu} + \left(\frac{\partial\tilde{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}} - \delta^{\mu}_{\alpha}\delta^{\beta}_{\nu}\right)\bar{T}^{\alpha}_{\beta} - T\partial_{\tau}\bar{T}^{\mu}_{\nu} - L^i\partial_i\bar{T}^{\mu}_{\nu}.
\end{aligned}$$

For $T^0_0 = \bar{\rho} + \delta\rho$

$$\begin{aligned}
\delta\tilde{\rho} &= \delta\rho + \left(\frac{\partial\tilde{\tau}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial\tilde{\tau}} - \delta^{0}_{\alpha}\delta^{\beta}_{0}\right)\bar{T}^{\alpha}_{\beta} - T\partial_{\tau}\bar{\rho} - L^i\partial_i\bar{\rho} \\
&= \delta\rho + \left(\frac{\partial\tilde{\tau}}{\partial\tau}\frac{\partial\tau}{\partial\tilde{\tau}} - 1\right)\bar{\rho} - \frac{\partial\tilde{\tau}}{\partial x^i}\frac{\partial x^j}{\partial\tilde{\tau}}\bar{\rho}\delta^i_j - T\bar{\rho}' \\
&= \delta\rho + [(1 + \bar{T}')(1 - \bar{T}') - 1]\bar{\rho} - \mathcal{O}(2) - T\bar{\rho}'.
\end{aligned}$$

$$\therefore \ \delta\tilde{\rho} = \delta\rho - T\bar{\rho}'. \tag{3.30}$$

HW:

$$\begin{aligned} \delta\tilde{P} &= \delta P - T\dot{\bar{P}}. & (3.31)\\ \tilde{q}^i &= q^i + (\bar{\rho}+\bar{P})\dot{L}^i. & (3.32)\\ \tilde{\Pi}^i_j &= \Pi^i_j. & (3.33) \end{aligned}$$

3.9 Gauge invariant perturbations

One useful combination

$$\begin{aligned} \bar{\rho}\Delta &\equiv \delta\rho + \bar{\rho}'(v+B) \\ &= \delta\rho - 3\mathcal{H}(\bar{\rho}+\bar{p})(B+v), \end{aligned} \tag{3.34}$$

where $q_i = (\bar{\rho}+\bar{p})\partial_i v$, Δ comoving-gauge density perturbation

HW: Δ is gauge-invariant.

3.10 Gauge fixing

Define the gauge in the matter sector.

Uniform density gauge

Use the freedom in the time-slicing to set the total density perturbation to zero

$$\delta\rho = 0.$$

Comoving gauge

Scalar momentum density to vanish

$$q = 0.$$

naturally connected to the inflationary initial conditions.

3.11 Adiabatic fluctuations

• Simple inflation models predict initial fluctuations that are adiabatic.

⇒ Energy densities of all species are constant on hypersurfaces and all species have the same peculiar velocities.

The quantity

$$\frac{\delta\rho_I}{\bar\rho_I + \bar P_I} - \frac{\delta\rho_J}{\bar\rho_J + \bar P_J}, \tag{3.35}$$

is gauge invariant since $\delta\tilde\rho_I \to \delta\rho_I - T\bar\rho_I'$, I and J labelled the species, and $\bar\rho_J' = -3\mathcal{H}(\bar\rho + \bar P)[= -(1+w_i)\rho_i]$.

Moreover, it vanished for adiabatic fluctuations since then all $\delta\rho_J = 0$, in a gauge for which the constant-time hypersurfaces coincide with those of uniform total density

$$\delta = \frac{\delta\rho_I}{\bar\rho_I'} = \frac{\delta\rho_J}{\bar\rho_J'} \qquad \Rightarrow \qquad \frac{\delta_I}{1+w_I} = \frac{\delta_J}{1+w_J} \qquad \text{for all species I and J.} \tag{3.36}$$

For adiabatic perturbations, matter ($w_{\rm m} \approx 0$), radiation ($w_{\rm r} = \frac{1}{3}$) obey $\delta_{\rm r} = \frac{4}{3}\delta_{\rm m}$ and the total density perturbation

$$\delta\rho_{\rm tot} = \bar\rho_{\rm tot}\delta_{\rm tot} = \sum_I \bar\rho_I \delta_I, \tag{3.37}$$

is dominated by the especies that is dominated in the background since all the δ_I are comparable.

3.12 Isocurvature fluctuations

Adiabatic perturbations correspond to a change in the total energy density.

Isocurvature perturbations only correspond to perturbation betwen the different components

$$\delta_{IJ} \equiv \frac{\delta_I}{1+w_I} - \frac{\delta_J}{1+w_J}, \tag{3.38}$$

single-field inflation predicts that the primordial perturbations are purely adiabatic $\delta_{IJ} = 0$

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