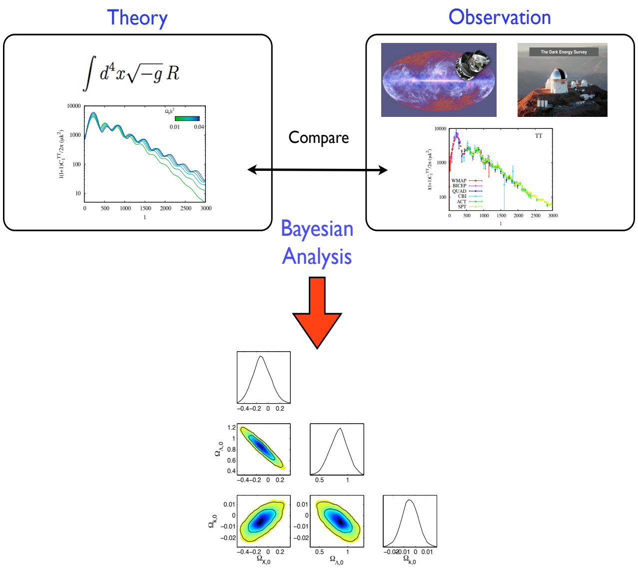
Updated Cosmology

with Python



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In progress

August 12, 2017

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The Perturbed Universe

The homogeneous and isotropic model provides an accurate description of the physical properties of the universe on large scales: the expansion history and the evolution of its energy content. Nevertheless, at small scales homogeneity and isotropy are no longer valid approximations, and therefore we have to make use of a more complex theory to describe, for instance, the temperature anisotropies observed in the CMB and the matter distribution. We have seen that the temperature of the photons in the CMB presents small anisotropies compared to the background temperature $\Delta T/\bar{T} \sim 10^{-5}$. This tiny value supports the use of linear perturbation theory to predict accurately its statistical distribution. The idea of perturbation theory is straightforward: perturb the metric and the stress-energy tensor in the Einstein's equations about the background and, to first order, drop products of small quantities. Then solve the coupled system of equations

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{1.1}$$

This section is aimed to present an outline of linear perturbation theory, but for an extended review, we refer to Bardeen [1], Dodelson [3], Hu and Dodelson [6], Kodama and Sasaki [8], Liddle and Lyth [10], Ma and Bertschinger [11], Mukhanov [12], Mukhanov et al. [13]; special attention is paid to the papers written by Challinor [2], Doran [4], Durrer [5], Knobel [7], Kurki-Suonio [9].

An outline of the theoretical concepts revised in this chapter is displayed in Figure 1.1 . The quantities shown in the bottom panel will allow us to establish the connection with current and future cosmological observations, as we shall see in the next chapter.

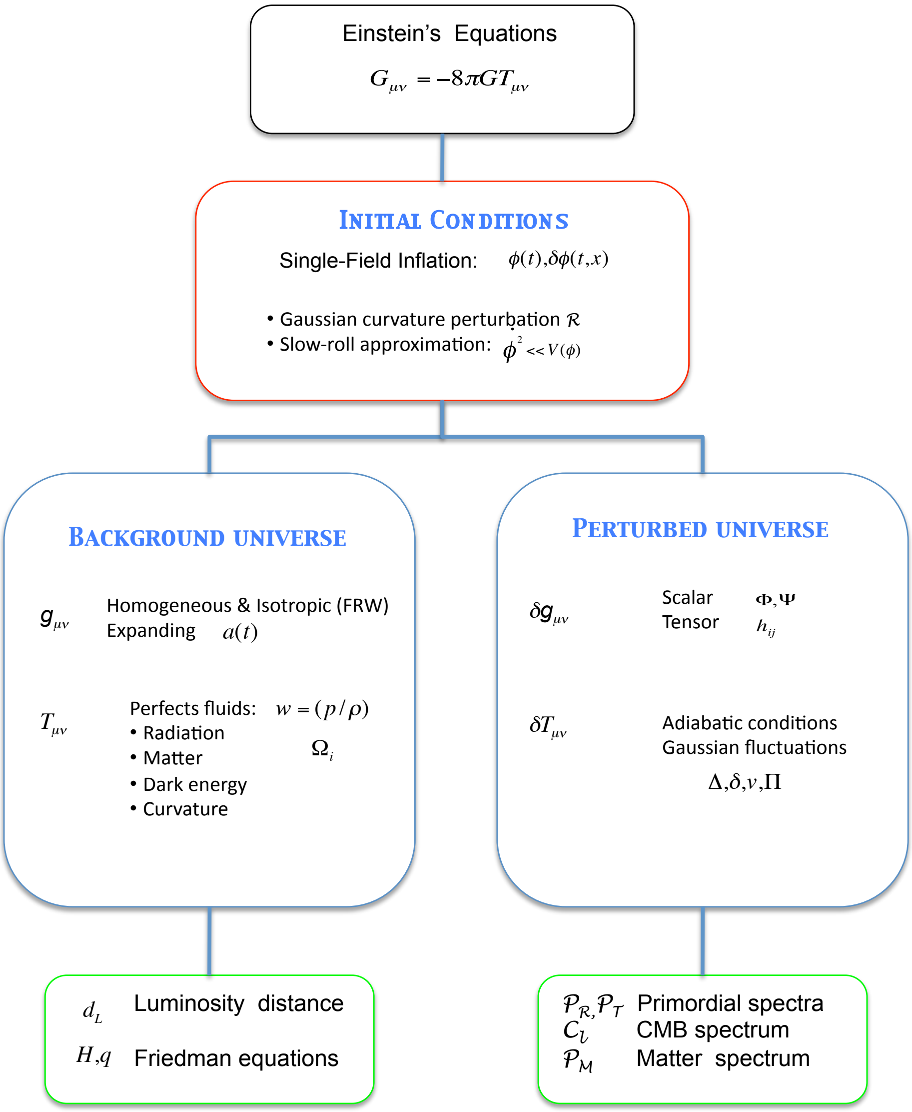

Figure 1.1: Outline of the theoretical concepts reviewed throughout the following chapters.

1.1 Newtonian Perturbation Theory

On scales well inside the Hubble radius and when describing non-relativistic matter, Newtonian gravity is an adequate approximation of GR in cosmology. Consider an ideal, self-gravitating non-relativistic fluid described by ρ, $P \ll \rho$, $\vec{u}$, $\vec{r}$ (vector position), the equations that describe this type of fluid are

$$\text{Continuity} \quad : \quad \partial_t \rho + \nabla_{\vec{r}} \cdot (\rho \vec{u}) = 0, \qquad \text{(energy density conservation)}, \tag{1.2}$$

$$\text{Euler} \quad : \quad (\partial_t + \vec{u} \cdot \nabla_{\vec{r}})\vec{u} = -\frac{1}{\rho}\nabla_{\vec{r}} P - \nabla_{\vec{r}} \Phi, \qquad \text{(momentum conservation)}, \tag{1.3}$$

where $g = -\nabla_{\vec{r}}\Phi$ represents the gravitational acceleration and term within the parenthesis, $\partial_t + \vec{u} \cdot \nabla_{\vec{r}}$, the convective derivate (follows the fluid element as it moves).

$$\text{Poisson :} \quad \nabla^2_{\vec{r}}\Phi = 4\pi G\rho - \Lambda, \tag{1.4}$$

Φ being the gravitational potential, and we hace incorporated the cosmological constant.

1.1.1 Perturbation analysis

Then, the key idea is to perturb the background quantities

$$\rho \quad \rightarrow \quad \bar{\rho}(t) + \delta\rho \equiv \bar{\rho}(1 + \delta). \tag{1.5}$$

$$P \quad \rightarrow \quad \bar{P}(t) + \delta P. \tag{1.6}$$

$$\vec{u} \quad \rightarrow \quad a(t)H(t)\vec{x} + \vec{\mathrm{v}}. \tag{1.7}$$

$$\Phi \quad \rightarrow \quad \bar{\Phi}(t, \vec{x}) + \delta\Phi. \tag{1.8}$$

where we have introduced the density contrast δ as the fractional overdensity $= \delta\rho/\bar{\rho}$ perturbation, and $\delta\Phi$ as the perturbed gravitational potential.

0-order

At the background level, first consider a uniform expanding fluid ball satisfying Hubble's law $\vec{u} = \partial_t \vec{r} = H(t)\vec{r}$ [using $\vec{r} = a(t)\vec{x}$, from the perspective of the fluid equations].
Taking $\Phi = 0$ at $\vec{r} = 0$, we have from the Poisson equation (1.4), in spherical coordinates,

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) &= (4\pi G\rho - \Lambda) r^2, \\ \frac{\partial \Phi}{\partial r} &= \frac{1}{3}(4\pi G\rho - \Lambda) r, \\ \Rightarrow \Phi &= \frac{1}{6}(4\pi G\rho - \Lambda) r^2. \end{aligned}$$

The Euler equation (1.3) becomes

$$\begin{aligned}\frac{\partial H}{\partial t}\vec{r}+H^2\vec{r}\cdot\nabla_{\vec{r}}\vec{r} &= -\frac{1}{3}(4\pi G\rho-\Lambda)\vec{r},\\ \Rightarrow \frac{\partial H}{\partial t}+H^2 &= \frac{1}{3}(\Lambda-4\pi G\rho),\end{aligned}\tag{1.9}$$

with $\vec{r}\cdot\nabla_{\vec{r}}\vec{r}=\vec{r}$. This is the Newtonian limit of one of Friedmann equations.

Now the continuity equation (1.2)

$$\begin{aligned}\partial_t\rho+\nabla_{\vec{r}}\cdot[\rho(t)H(t)\vec{r}] &= 0,\\ \partial_t\rho+3\rho H &= 0,\end{aligned}$$

which has solution

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}+\frac{3}{a}\frac{\partial a}{\partial t}=0 \qquad \Rightarrow \rho\propto a^{-3},\tag{1.10}$$

the dilution of the mass density by expansion, as expected. Equations (1.9) y (1.10) have a first integral (exact) given by

$$-K=a^2\left(H^2-\frac{8\pi G}{3}\rho-\frac{1}{3}\Lambda\right).\tag{1.11}$$

HW: Show K is a constant, hence the solution

$$\begin{aligned}-\frac{\partial K}{\partial t} &= 2a^2H\left(H^2-\frac{8\pi G}{3}\rho-\frac{1}{3}\Lambda\right)+a^2\left(2H\frac{\partial H}{\partial t}-\frac{8\pi G}{3}\frac{\partial\rho}{\partial t}\right)\\ &= a^2\left[2H^3-\frac{16\pi G}{3}H\rho-\frac{2}{3}H\Lambda+2H\left(-H^2-\frac{4\pi G}{3}\rho+\frac{1}{3}\Lambda\right)+8\pi GH\rho\right]\\ &= 0.\end{aligned}$$

Therefore,

$$H^2-\frac{K}{a^2}=\frac{1}{3}(8\pi G\rho-\Lambda),\tag{1.12}$$

the intrinsic curvature of the surfaces of homogeneity .

Comoving coordinates

The position $\vec{r}=a(t)\vec{x}$, has velocity $\frac{d\vec{r}}{dt}=H(t)\vec{r}$, with $\vec{x}$ the comoving spatial coordinates

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)_{\vec{r}} &= \left(\frac{\partial}{\partial t}\right)_{\vec{x}}+\left(\frac{\partial\vec{x}}{\partial t}\right)_{\vec{r}}\cdot\nabla_{\vec{x}}\\ &= \left(\frac{\partial}{\partial t}\right)_{\vec{x}}-H(t)\vec{x}\cdot\nabla_{\vec{x}}.\end{aligned}\tag{1.13}$$

$\nabla_{\vec{x}}$ is the gradient respect to $\vec{x}$ at fixed t and $\nabla_{\vec{r}} = a^{-1}\nabla_{\vec{x}}$. For simplicity, now on we drop out the $\vec{x}$ index, i.e. $\nabla = \nabla_{\vec{x}}$.

Thus, the velocity field becomes

$$\vec{u} = \frac{d\vec{r}}{dt} = \frac{d(a\vec{x})}{dt} = aH\vec{x} + a\frac{d\vec{x}}{dt} = H\vec{r} + \vec{\mathrm{v}}, \tag{1.14}$$

the second term in the r.h.s. defines the peculiar velocity, which describes changes in the co-moving coordinates of the fluid elements in time.

1st order perturbation

The perturbed continuity equation, (1.2), at first order (on the new coordinate system)

$$\left[\frac{\partial}{\partial t} - H\vec{x}\cdot\nabla\right][\bar{\rho}(1+\delta)] + \frac{1}{a}\nabla\cdot[\bar{\rho}(1+\delta)(Ha\vec{x}+\vec{\mathrm{v}})] = 0,$$

$$\partial_t\bar{\rho} + \partial_t(\bar{\rho}\delta) - H\bar{\rho}\vec{x}\cdot\nabla\delta + \frac{\bar{\rho}}{a}\nabla\cdot[Ha\vec{x} + \vec{\mathrm{v}} + \delta Ha\vec{x}] + \frac{\bar{\rho}}{a}\nabla\cdot[\delta\vec{\mathrm{v}}] = 0,$$

$$\underbrace{\partial_t\bar{\rho} + 3\rho H}_{\mathrm{0th-order}} + \underbrace{(\partial_t\rho + 3\bar{\rho}H)\delta + \bar{\rho}\partial_t\delta + \frac{\bar{\rho}}{a}\nabla\cdot\vec{\mathrm{v}}}_{\mathrm{1st-order}} + \underbrace{\frac{\bar{\rho}}{a}\nabla\cdot(\delta\vec{\mathrm{v}})}_{\mathrm{2nd-order}} = 0.$$

Hence, in linear perturbation theory

$$\partial_t\delta + \frac{1}{a}\nabla\cdot\vec{\mathrm{v}} = 0. \tag{1.15}$$

HW: find the first order Euler and Poisson equations

$$\partial_t\vec{\mathrm{v}} + H\vec{\mathrm{v}} = -\frac{1}{a\bar{\rho}}\nabla\delta P - \frac{1}{a}\nabla\delta\Phi. \tag{1.16}$$

$$\nabla^2\delta\Phi = 4\pi G a^2\bar{\rho}\delta. \tag{1.17}$$

Scalar/vector decomposition

In linear theory, the escalar and vector parts decouple.

We can always decompose any vector $\vec{\mathrm{v}}$ as

$$\vec{\mathrm{v}} = \underbrace{\nabla\mathrm{v}}_{\mathrm{scalar}} + \underbrace{\vec{\mathrm{v}}_\perp}_{\mathrm{vector}},$$

with the property that $\nabla \cdot \vec{v}_\perp = 0$. From the continuity equation (1.15), we notice the vector part of $\vec{v}$ does not contribute to the *clumping* of the matter.
On the other hand,

- Since $\nabla \times \vec{v} = \nabla \times \vec{v}_\perp$, thus $\vec{v}_\perp$ describes the vorticity of the fluid.
- From perturbed Euler equation (1.16), taking the curl

$$\nabla \times \partial_t \vec{v} = \partial_t (\nabla \times \vec{v}_\perp) = -H \nabla \times \vec{v}_\perp .$$

where the first equality uses the continuity equation, and from the second we have that $\nabla \times \vec{v}_\perp$ decays as $1/a$ in a expanding universe.

Recalling $\nabla_{\vec{r}} = a^{-1}\nabla$, the physical vorticity is $\nabla_{\vec{r}} \times \vec{v} = a^{-1}\nabla \times \vec{v}_\perp$, then it falls as $1/a^2$, therefore **vector modes can be neglected**. For initial conditions from *inflation*, vector modes are *not excited* in the first place.

1.2 The Jean's length

The time derivate of the perturbed continuity equation (1.15)

$$\partial_t^2 \delta - \frac{1}{a} H \nabla \cdot \vec{v} + \frac{1}{a} \nabla \cdot \partial_t \vec{v} = 0,$$

using Euler (1.16) and Poisson (1.17)

$$\partial_t^2 \delta - \frac{1}{a} H \nabla \cdot \vec{v} - \frac{1}{a} \nabla \cdot \left(H \vec{v} + \frac{1}{a\bar{\rho}} \nabla \delta P + \frac{1}{a} \nabla \delta \Phi \right) = 0,$$

$$\Rightarrow \quad \partial_t^2 \delta - \frac{2}{a} H \nabla \cdot \vec{v} - \frac{1}{a^2 \bar{\rho}} \nabla^2 \delta P - \frac{1}{a^2} \nabla^2 \delta \Phi = 0.$$

Once again, using the perturbed Euler and Poisson equations

$$\Rightarrow \quad \partial_t^2 \delta + 2H \partial_t \delta - \frac{1}{a^2 \bar{\rho}} \nabla^2 \delta P - 4\pi G \bar{\rho} \delta = 0. \tag{1.18}$$

Jean's equation: the fundamental equation for the growth of structure in Newtonian theory. It shows the general competition between infall by gravitational attraction ($4\pi G \bar{\rho} \delta$) and pressure ($\nabla^2 \delta P$).

- Consider a barotropic fluid with equation of state $P = P(\rho)$

$$\Rightarrow \quad \delta P = \frac{\partial P}{\partial \bar{\rho}} \bar{\rho} \delta \equiv c_s^2 \bar{\rho} \delta,$$

where c_s^2 is the *speed of sound*.

If we move to the Fourier space, then $\nabla^2 \to -k^2$, and (1.18) becomes

$$\partial_t^2 \delta + 2H\partial_t \delta + w^2 \delta = 0, \tag{1.19}$$

with

$$w^2 = \frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\rho}.$$

We define the Jean's length associated to the wavenumber k_J, such that the frequency w is zero:

$$k_J \equiv \sqrt{\frac{4\pi G\bar{\rho}a^2}{c_s^2}} \qquad \text{hence} \qquad w^2 = \frac{c_s^2}{a^2}[k^2 - k_J^2].$$

- **Superhorizon**: $k \ll aH$. The solution has a *growing* and a *decaying* mode.
- **Subhorizon**: $k \gg aH$. The solution presents a wavelike behaviour with changing amplitude.

Static universe and no-gravity.

In a static universe ($H = 0$) and with no gravity ($\Phi = 0 \to \bar{\rho} = 0$), the Jean's equation (1.19) becomes

$$\partial_t^2 \delta + w^2 \delta = 0, \qquad \text{with} \qquad w^2 = \frac{c_s^2 k^2}{a^2}.$$

The solutions are *fluctuations with constant amplitude* given by

$$\delta_k = Ae^{iwt} + Be^{-iwt}.$$

Static universe with gravity.

In this case the the Jean's equation contains a source term

$$\partial_t^2 \delta + w^2 \delta = 0, \qquad \text{with} \qquad w^2 = \frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\rho}.$$

For large wavenumber (small scales), provided that $k > k_J$:

$\Rightarrow$ The Pressure support gives rise to acoustic oscillator (sound waves) in the fluid.

However for small wavenumber (large scales), provided that $k < k_J$ the system is unstable:

$\Rightarrow$ The fluctuations experience an exponential growth.

Introduction

As in many other areas of physics, perturbation theory has been a powerful tool to aboard several problems such as calculations of the power spectra of the cosmic microwave background (CMB), recombination and so on, where the most relevant terms are linear [**?**]. Therefore, several approaches to perturbation theory in cosmology have been proposed, and in this paper the Lagrangian perturbation theory is discussed and contrasted with the Eulerian perturbation theory.

Eulerian perturbation theory

In comoving coordinates, the characteristic equation for this formalism is given by

$$\partial_t^2\delta + 2\frac{\dot{a}}{a}\partial_t\delta = \frac{1}{a^2}\nabla[(1+\delta)\nabla\phi] + \frac{1}{a^2}\partial_\alpha\partial_\beta\left[\frac{P^{\alpha\beta}}{\rho_b} + (1+\delta)v^\alpha v^\beta\right]. \tag{1.20}$$

where $\delta \equiv \rho/\overline{\rho} - 1$, with ρ the particles density. Also, v is the mean velocity of a space phase element, ϕ is a solution for the Poisson's equation, and $P^{\alpha\beta}$ is called the pressure tensor. Then, in the Eulerian regime

$$\delta \ll 1 \quad \text{and} \quad \left(\frac{vt}{d}\right)^2 \approx \delta \ll 1,$$

where d is for the coherence length of the perturbation and t its dynamical time, $t \approx (G\rho)^{-1/2}$. Furthermore, In linear perturbation theory the fluid equations assume the form

$$\partial_t\delta + \nabla\cdot\mathbf{v}a = 0 \quad \& \quad \partial_t^2\delta + 2\frac{\dot{a}}{a}\partial_t\delta = \frac{\nabla^2 p}{\rho_b a^2} + 4\pi G\rho_b\delta.$$

with a the scale FLRW scale factor.

Lagrangian perturbation theory

This approach to perturbation theory was proposed by Moutarde *et al* in 1991 [**?**]. And as in the Eulerian case, this formalism is written in comoving coordinates, and the equivalent characteristic equation (1.20) is given by

$$J(\tau,\mathbf{q})\nabla_x\ddot{x} = \beta(\tau)[J(\tau,\mathbf{q}) - 1], \tag{1.21}$$

with x the particles position and $\mathbf{q}$ the corresponding momentum. $J^{-1} = \delta - 1$ and $\beta = \frac{6}{\tau^2+\Omega(k)}$. Notice that for any solution of (1.21), a divergence-free displacement field can be added, and this composition will be a solution as well. Then, in order to take this fact into account the following constraint is imposed

$$\nabla_x \times \dot{\mathbf{x}} = 0.$$

Finally, the equation to solve is obtained through $\Gamma \equiv \ddot{\mathbf{x}}$ as function of $\mathbf{q}$ as

$$\nabla_x \Gamma = J(\tau, \mathbf{q})^{-1} \sum_{i,j} \Gamma_{i,j} A_{ji},$$

where the A_{ij} are the cofactors of J.

Conclusion

An overview on the Lagrangian perturbation theory has been presented. In comparison with the Eulerian case, perturbative solutions are obtained by means of an iterative procedure. But in contrast, the expansion concerns the particles displacement field itself. Moreover, from Lagrangian perturbation theory is expected that its approach to non-linear regime might lead to finally understand deeply higher regimes. Then, it is hoped to obtain new and more precise results from this perturbation theory formalism in the future.

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