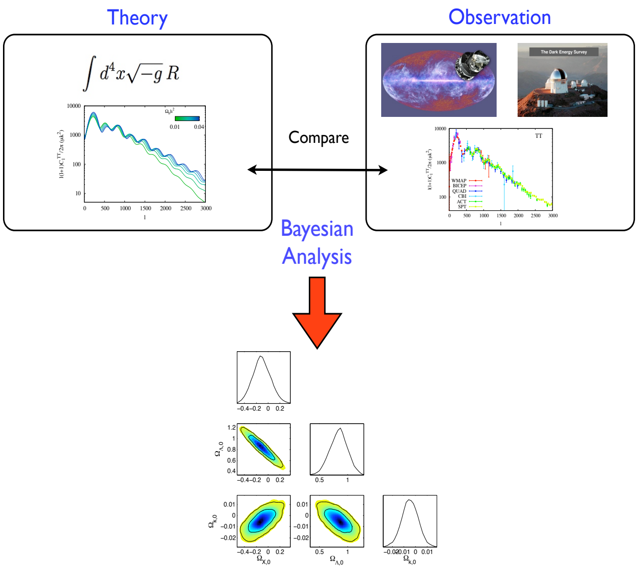
Updated Cosmology

with Python



José-Alberto Vázquez

ICF-UNAM / Kavli-Cambridge

In progress

August 12, 2017

Homework 5.b

1) The most general spherically symmetric metric can be written as

$$ds^2 = -e^{2F(r,t)}dt^2 + e^{2H(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

This metric is very important as it underlies the theory of both homogeneous cosmological models and spherically symmetric models for massive stars and black holes. The functions $F(r,t)$ and $H(r,t)$ are determined by the material content of the space-time as described by the energy–momentum tensor and by the boundary conditions defining the problem.

Compute the components of the Einstein tensor.

$$\begin{aligned}
G_{00} &= e^{-2H}\left(\frac{2}{r}H' - \frac{1}{r^2}\right) + \frac{1}{r^2},\\
G_{11} &= e^{-2H}\left(\frac{2}{r}F' + \frac{1}{r^2}\right) - \frac{1}{r^2},\\
G_{22} = G_{33} &= e^{-2H}\left(F'' + F'^2 - H'F' + \frac{1}{r}(F' - H')\right)\\
&\quad - e^{-2F}\left(\ddot{H} + \dot{H}^2 - \dot{H}\dot{F}\right),\\
G_{01} = G_{10} &= \frac{2}{r}\dot{H}\, e^{-(F+H)}.
\end{aligned}$$

2) The Bianchi models are a large family of homogeneous but anisotropic cosmological models. We consider the homogenous and anisotropic space-time described by Bianchi type-III metric in the form

$$ds^2 = dt^2 - A(t)^2 dx^2 - B(t)^2 e^{-2\alpha x} dy^2 - C(t)^2 dz^2, \tag{2}$$

where $A(t)$, $B(t)$ and $C(t)$ are the scale factors (metric tensors) and functions of the cosmic time t, and α is a constant. (Bianchi type-I metric can be recovered by choosing $\alpha = 0$). Here, we assume an anisotropic fluid whose energy-momentum tensor is in diagonal form:

$$T_\nu{}^\mu = \mathrm{diag}[1, -w_x, -w_y, -w_z] = \mathrm{diag}[1, -w, -(w+\gamma), -(w+\delta)]\rho, \tag{3}$$

where ρ is the energy density of the fluid, w_x, w_y and w_z are the directional EoS parameters on the x, y and z axes respectively; w is the deviation-free EoS parameter of the fluid. δ and γ are not necessarily constants and can be functions of the cosmic time t.

a) Compute the components of the Einstein's field equations

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{\alpha^2}{A^2} = \rho, \tag{5}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = -w\rho, \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} = -(w+\delta)\rho, \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{\alpha^2}{A^2} = -(w+\gamma)\rho, \tag{8}$$

$$\alpha\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0 \tag{9}$$

where the overdot denotes derivation with respect to the cosmic time t.

b) Show the solution of Eqn. (9) gives $B = c_1 A$, where c_1 is the positive constant of integration.

c) Substitute this solution into (7), and subtract the result from (6), to show that the skewness parameter on the y axis is null, i.e. $\delta = 0$, which means that the directional EoS parameters, hence the pressures, on the x and y axes are equal.

The directional Hubble parameters in the directions of x, y and z for the Bianchi type-III metric may be defined as follows,

$$H_x \equiv \frac{\dot{A}}{A}, \quad H_y \equiv \frac{\dot{B}}{b}, \quad H_z \equiv \frac{\dot{C}}{C}. \tag{4}$$

and the mean Hubble parameter is given by

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{5}$$

By solving the set of field equations, the difference between the expansion rates on x and z axes could be found $H_x - H_z$, [see Gen Relativ Gravit (2010) 42:763–775].

3) The action of a massive scalar field, with potential $V(\phi)$, is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{6}$$

a) Show that the corresponding field equation for ϕ, obtained from the Euler-Lagrange equations, reads as

$$\Box^2\phi + \frac{dV}{d\phi} = 0. \tag{7}$$

where the $d'Alembertian$ is : $\Box^2 = \partial_a\partial^a$.