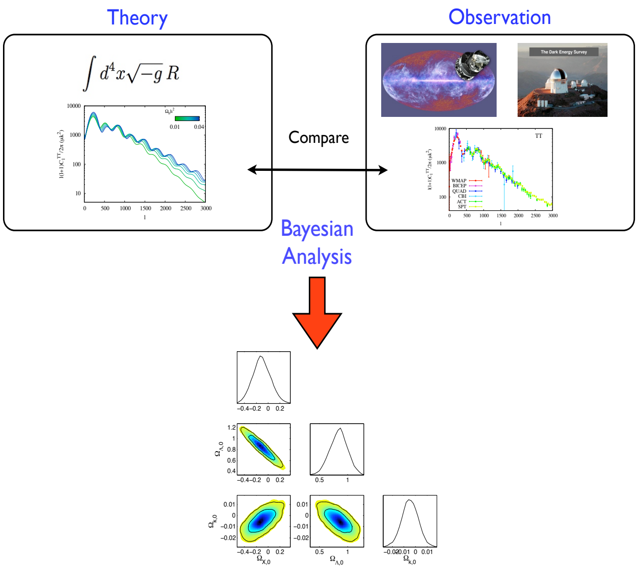
Updated Cosmology

with Python



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Homogeneous and Isotropic Universe

In this Chapter, we start by introducing the theoretical framework that underlies the standard model of modern cosmology: the concordance ΛCDM model. We briefly review the equations determining the evolution of an homogeneous and isotropic universe. For the sake of completeness, in the following chapters, we have included the inflationary model as a solution to some of the shortcomings of the Hot Big Bang model. Finally, the computation of the distance modulus, cosmic microwave background spectrum and matter power spectrum allows us to establish a connection between cosmological models and current (future) observations through statistical tools, described in the next chapter. Some reference books have been used throughout this short review, they include: Dodelson [7], Hobson et al. [9], Liddle and Lyth [11], Mukhanov [12]; as well as some lecture notes: Peiris [13], Pettini [14], Challinor [5].

The standard description of the dynamical properties of the universe is provided by the Einstein's theory of General Relativity (GR), which builds a connection between the geometry of the space-time and its matter-content, through fundamental quantities: the *metric* $g_{\mu\nu}$ and the *energy-momentum* tensor $T_{\mu\nu}$

$$\underbrace{G_{\mu\nu}[g_{\mu\nu}]}_{\text{Geometry}} = \frac{8\pi G}{c^4}\underbrace{T_{\mu\nu}}_{\text{Matter}}. \tag{1.1}$$

Einstein equations are very elegant, however they are indeed difficult to solve in the general case. Throughout this book we focus on the basic description of our Universe, an expanding

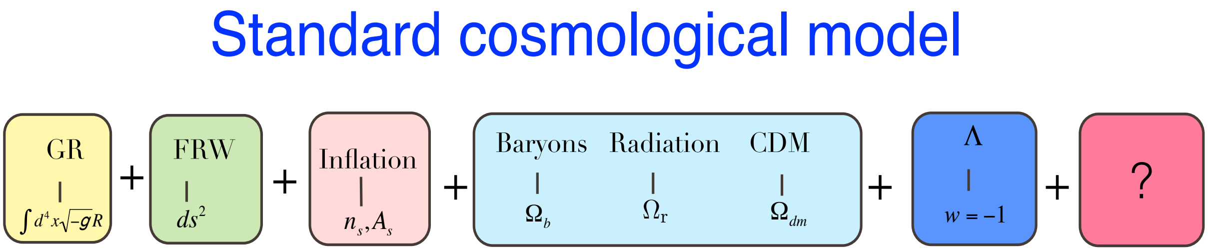

Figure 1.1: Main components of the Standard Cosmological Model.

homogeneous and isotropic described by the FLRW metric, and then move forward to the linear perturbation theory, as we shall see below.

1.1 The cosmological principle

In order to specify the global geometry of the universe, an essential assumption to do is the *Cosmological Principle*: when averaged over sufficiently large-scales at any particular time, the universe seems to be *Homogeneous* and *Isotropic* with a high accuracy. For instance, at scales larger than 150 Mpc, the distribution of galaxies over the celestial sphere does seem to justify the assumption of isotropy -i.e. independent of direction [6]. Moreover, the current uniformity in the temperature distribution of the *Cosmic Microwave Background* (CMB) radiation, to a few parts in 10^5, is the best observational evidence we have in support of an isotropic universe (see Figure 1.2; [10]). If isotropy is thus taken for granted and considering that our position in the universe is by no means preferred to any other, known as the *Copernican Principle*, then homogeneity -i.e. independent of position, follows from isotropy at every point.

Isotropy + Copernican Principle $\longrightarrow$ **Homogeneity.**

Isotropy states that the Universe looks the same in every direction.

Homogeneity states that the Universe looks the same at any point.

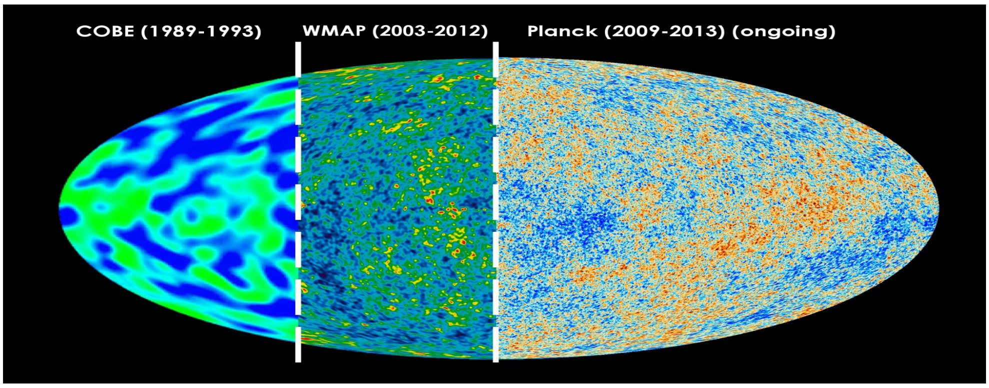

Figure 1.2: Anisotropies of the Universe seen by Planck satellite. Colours describe differences in temperatures, blue is cold and red hot, and they're about 10^{-6} (jav: use healpy to plot isotropy) www link.

It is worth noting that isotropy about every point automatically implies homogeneity. However, homogeneity does not necessarily imply isotropy. For example, a universe with a large-scale electric field that points out in one direction everywhere and has the same magnitude at every point would be homogeneous but not isotropic (see Figure 1.3).

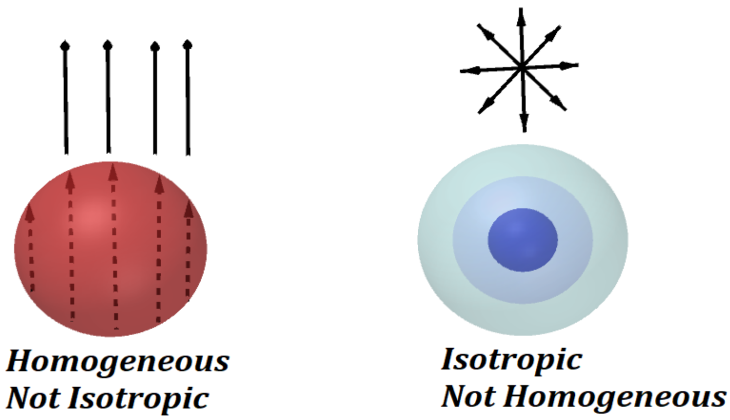

Figure 1.3: Illustration of how homogeneity and isotropy are not equivalent.

For those picky astrophysicist, who want to provide an explanation up to the most minimal detail, for instance the fingers of god (left of Figure 1.4) or the Stephen-Hawking letters (right of Figure 1.4) observed in the CMB, there exist models considering anisotropies in the Universe (not to explain these silly things though). Some examples of anisotropic models are the Bianchi [cite]. The type I models contain the Kasner metric, whereas, the Bianchi IX includes the Taub metric; the isotropy of the FLRW metric is a particular case of the types I, V, VII and IX. Inhomogeneous models: Lemaitre-Tolman-Bondi [8], Zsekeres [15]. For a review of the an inhomogeneous framework, see [3]. In the following sections we will present a brief introduction to these alternative descriptions of the Universe.

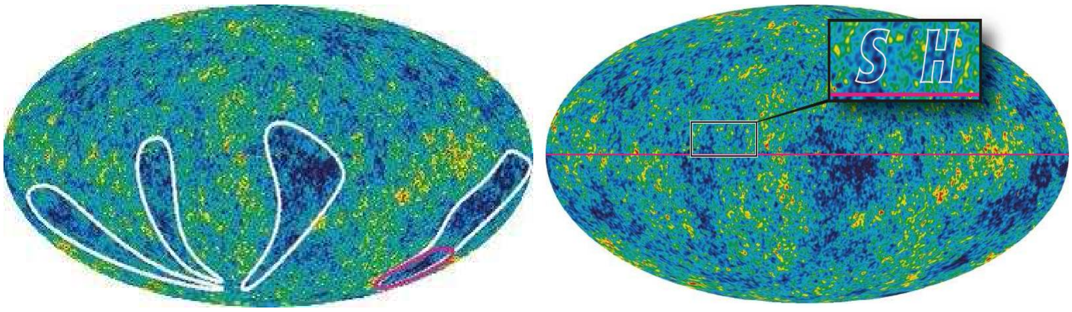

Figure 1.4: Anisotropies of the Universe seen by WMAP satellite.

HW: Describe some examples of non-homogeneous and non-isotropic universes

HW: The Anthropic Principle: Why the Universe is how it is?

Assumption 1.1.1:

Homogeneity and Isotropy.

Anthropic Principle

Why the Universe is how it is? This doubt has triggered a big debate among the scientific community during the last decades. However, even though the above question looks innocent, some people are still skeptical about the answers.

Despite the idea had been used previously by R. H. Dicke, the phrase **anthropic principle** was first attributed to the theoretical physicist Brandon Carter in 1974, who established the statement: “Although our situation is not necessarily central, it is inevitably privileged to some extent”[4]. In general, one can think about the anthropic principle as the idea that the behaviour and evolution of the Universe must be compatible with the conscious life we are observing and measuring. In other words, **the Universe has specific physical constants because they are the necessary ones to permit life** (jav: constant books cite[?]). However, these ideas have gone further, generating two variations of this concept [2]:

- Weak anthropic principle: “The observed values of all physical and cosmological quantities are not equally probable but they take on values restricted by the requirement that there exist sites where carbon-based life can evolve and by the requirements that the universe be old enough for it to have already done so”.
- Strong anthropic principle: “The Universe must have those properties which allow life to develop within it at some stage in its history”.

However, from the weak anthropic principle one can notice that this is very restrictive since it does not allow different kind of life apart of carbon-based, such as the possibility of silicon-based life which has been considered since a few years ago [1]. Meanwhile, the strong version can be criticised based upon the fact that it cannot be tested.
The philosophical conception that tries to explain why the universe is determined to be in the way we know it, could be summarised in a shallow way as follows: **The world is necessarily as it is because there are beings who wonder why it is so**. Even though this idea has received enough criticism and skepticism, it is the best explanation to some fundamental problems in physics, for instance, why the cosmological constant or the electroweak scale have that precise value. Therefore, until a more fundamental explanation to this kind of problems is proposed, the anthropic principle will be the most accepted answer.

1.2 The Geometry

The left hand side of Einstein's equations (1.1) contain, what is called the metric tensor $g_{\mu\nu}$ which describes the geometry of the space time. Let us start by defining this quantity.

1.2.1 The spatial metric g_{ab}

Consider two infinitesimally separated points P and Q in the manifold[1], Figure 1.5, with coordinates x^a and $x^a + dx^a$ respectively ($a = 1, 2, ..., N$), with N being the dimension of the space.

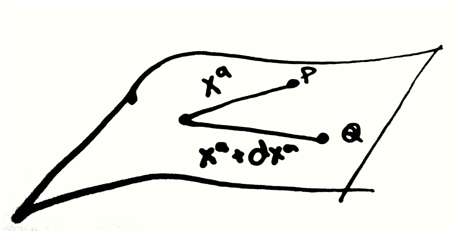

Figure 1.5: Points in space.

The local geometry of the manifold at the point P is determined by defining the squared *invariant distance* or *interval* (ds) between P and Q. In general, the interval is a function represented by

$$ds^2 = f(x^a, dx^a). \tag{1.2}$$

For developing general relativity we are interested in an expression of the form[2]

$$ds^2 = g_{ab}(x)dx^a dx^b, \tag{1.3}$$

where $g_{ab}(x)$ are the components of the *metric tensor* field, in our chosen coordinate system. Manifolds with a geometry express in the form of Eqn. (1.3) are called *Riemannian manifolds*. However, strictly speaking, the manifold is only Riemannian if $ds^2 > 0$, and because here ds^2 can be either positive, negative or zero, then the manifold should properly be called *pseudo-Riemannian* (we simply referred to it as Riemannian).

[1] In general terms, an n-dimensional manifold is an space that locally looks like $\mathbb{R}^n$.

[2] A consequence of the equivalence principle, is that it restricts the possible geometry of the curved space-time to a pseudo-Riemannian one.

The metric functions $g_{ab}(x)$ correspond to the elements of a position dependent squared matrix. These metric functions can be chosen such that the matrix is symmetric, that is $g_{ab}(x) = g_{ba}(x)$. Let us assume for a moment the matrix is a generic one, and hence it can be decomposed as the sum of a symmetric part and anti-symmetric component:

$$g_{ab}(x) = \underbrace{\frac{1}{2}[g_{ab}(x) + g_{ba}(x)]}_{\text{symmetric}} \quad + \quad \underbrace{\frac{1}{2}[g_{ab}(x) - g_{ba}(x)]}_{\text{anti-symmetric}}. \tag{1.4}$$

The contribution to the interval ds^2 from the antisymmetric part would be $\frac{1}{2}[g_{ab}(x) - g_{ba}(x)]dx^a dx^b$ which vanishes identically when the components a and b are interchanged. Therefore the antisymmetric part of g_{ab} can be safely neglected.

Qz: How many independent variables have an N-dimensional symmetry matrix?

In a N-dimensional Riemannian manifold there are $\frac{1}{2}N(N+1)$ independent metric functions $g_{ab}(x)$. And since, in general, there are N arbitrary coordinate transformations, thus there are only $\frac{1}{2}N(N+1) - N = \frac{1}{2}N(N-1)$ independent degrees of freedom associated with $g_{ab}(x)$. For the particular case of a **diagonal metric**, i.e. $g_{ab}(x) = 0$ for $a \neq b$, the line element takes the form

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots + g_{NN}(dx^N)^2. \tag{1.5}$$

Such system of coordinates is called *orthogonal*, since any pair of coordinate curves cross at right angles.

Example 1.2.1: A trivial example

The **Euclidean metric** is the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that assigns to any two vectors in Euclidean n-space $\vec{x} = (x^1, \ldots, x^n)$ and $\vec{y} = (y^1, \ldots, y^n)$ the number $d(\vec{x}, \vec{y}) = \sqrt{(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2}$, that is, the *standard* distance between any two vectors in $\mathbb{R}^n$. The Euclidean metric (in a 3-dimensional space) is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

giving the line element

$$ds^2 = g_{ij} dx^i dx^j = (dx)^2 + (dy)^2 + (dz)^2.$$

Examples of line elements:

- The 3-**Euclidean** spacetime

$$ds^2 = dx^2 + dy^2 + dz^2.$$

- The **Minkowski** spacetime, or the special relativistic line element

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

- The **3-Sphere** $\mathbb{S}^3$ line element (with radius a)

$$ds^2 = \frac{a^2}{a^2 - r^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

- The **Friedmann-Lemaitre-Robertson-Walker** line element

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

- The locally rotationally symmetric (LRS) **Bianchi type-I** metric

$$ds^2 = c^2dt^2 - S^2\left[e^{\frac{4}{\sqrt{6}}\varphi}\mathrm{d}x^2 + e^{-\frac{2}{\sqrt{6}}\varphi}(\mathrm{d}y^2 + \mathrm{d}z^2)\right].$$

- The **Kerr-Newman** line element that describes the geometry of spacetime for a rotating charged black hole with mass M, charge Q and angular momentum J, is

$$\begin{aligned} ds^2 &= \left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 - (c\ dt - a\sin^2\theta d\phi)^2\frac{\Delta}{\rho^2} \\ &+((r^2+a^2)d\phi - ac\ dt)^2\frac{\sin^2\theta}{\rho^2}, \end{aligned}$$

with

$$a = \frac{J}{Mc}, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - r_s r + a^2 + r_Q^2.$$

Here r_s is the Schwarzschild radius of the massive body, related to its total mass-equivalent M by $r_s = 2GM/c^2$, and r_Q the length-scale corresponding to the electric charge Q of the mass $r_Q^2 = \frac{Q^2G}{4\pi\epsilon_0 c^4}$.

Here some particular examples for the Kerr-Newman metric

	$J=0$	Rotating ($J \neq 0$)
$Q=0$	Schwarzschild	Kerr
Charged ($Q \neq 0$)	Reissner-Nordström	Kerr-Newman

The 3-sphere example

Let us find the metric for a *3-sphere* $\mathbb{S}^3$ embedded in *four-dimensional* Euclidean space $\mathbb{R}^4$. First of all, the four-dimensional Euclidean space is described by the line element

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2, \tag{1.6}$$

however, we limit ourselves to move over the 3-dimensional-space restricted by the radius a

$$x^2 + y^2 + z^2 + w^2 = a^2. \tag{1.7}$$

Differentiating both sides of the equation, yields to

$$2xdx + 2ydy + 2zdz + 2wdw = 0, \tag{1.8}$$

and substituting dw in (1.6) follows the line element:

$$ds^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{a^2 - (x^2 + y^2 + z^2)}. \tag{1.9}$$

Now, to get a better sense of the metric, transforming to spherical polar coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \tag{1.10}$$

we obtain an alternative form for the line element (1.6)

$$\boxed{ds^2 = \frac{a^2}{a^2 - r^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.} \tag{1.11}$$

Notice that taking the limit for the radius of the sphere $a \to \infty$, the metric gets the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{1.12}$$

which is the ordinary Euclidean 3-dimensional space $ds^2 = dx^2 + dy^2 + dz^2$ but now written in spherical coordinates. The line element (1.11) therefore describes a non-Euclidean three-dimensional space.

Now that we have a description of the geometry of the space, we can compute physical quantities such as length, area and volume.

1.2.2 Lengths, areas and volumes

The lengths of curves follow immediately from the line element (ds^2). Suppose that the points A and B are joined by some path; then the length of this curve is given by

$$L_{AB} = \int_A^B ds = \int_A^B |g_{ab} dx^a dx^b|^{1/2}, \tag{1.13}$$

where the integral is evaluated along the curve. If the equation of the curve $x^a(u)$ is given in terms of some affine parameter u, as shown in Figure 1.6, then

$$L_{AB} = \int_{u_A}^{u_B} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{1/2} du, \tag{1.14}$$

where u_A and u_B are the values of the parameter u at the endpoints of the curve.

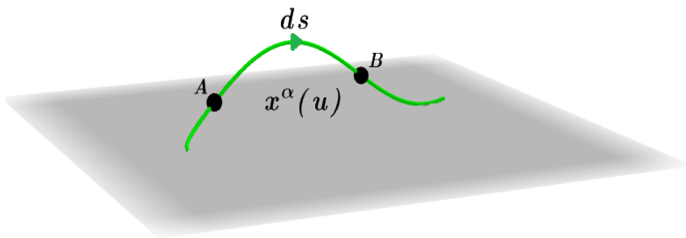

Figure 1.6: Two points describe a path defined by the line element $ds = |g_{ab} dx^a dx^b|^{1/2}$.

On the other hand, the proper lengths of two line segments will be $\sqrt{g_{11}} dx^1$ and $\sqrt{g_{22}} dx^2$ respectively. Thus the element of area is (see Figure 1.7)

$$dA = \sqrt{|g_{11} g_{22}|} dx^1 dx^2. \tag{1.15}$$

We can go even further and define a higher-dimensional *volume elements* in a similar way until we reach the N-dimensional volume element

$$d^N V = \sqrt{|g_{11} g_{22} \cdots g_{NN}|} dx^1 dx^2 \cdots dx^N. \tag{1.16}$$

In the general case, where the coordinates are not orthogonal, the volume elements may be rewritten as (with $g = det[g_{ij}]$),

$$d^N V = \sqrt{|g|} dx^1 dx^2 \cdots dx^N. \tag{1.17}$$

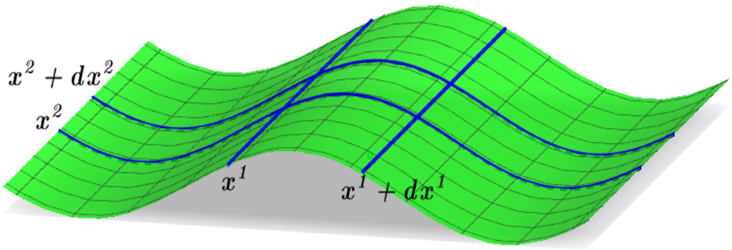

Figure 1.7: Element of Area. Orthogonal coordinates.

The Two-sphere

Let us consider the two-dimensional geometry of the surface of a sphere in terms of its radius ($\rho \ \epsilon \ [0, a]$) and zenith angle ($\phi \ \epsilon \ [0, 2\pi]$), assuming it is embedded in a three-dimensional Euclidean space (see Figure 1.8).

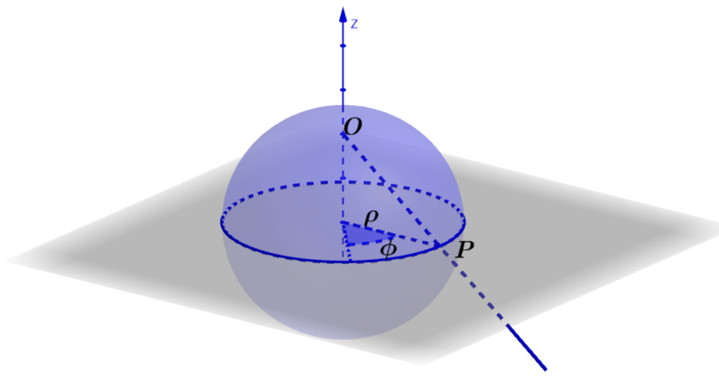

Figure 1.8: 2D - Sphere.

This can be seen as eliminating one dimension in (1.11) and making the substitutions

$$x = \rho \cos \phi, \qquad y = \rho \sin \phi, \tag{1.18}$$

with some algebra we obtain

$$ds^2 = \frac{a^2}{a^2 - \rho^2} d\rho^2 + \rho^2 d\phi^2. \tag{1.19}$$

Note that the line element contains a '*hidden symmetry*', namely our freedom to choose an arbitrary point on the sphere as the origin $\rho = 0$. There is also a *coordinate singularity*, which has resulted simply from choosing coordinates with a restricted domain of validity. From Eqn. (1.19) we see that this coordinate system is orthogonal, with $g_{\rho\rho} = a^2/(a^2 - \rho^2)$ and $g_{\phi\phi} = \rho^2$. From (1.13), the distance *in the surface* D from the centre to a particular radius $\rho = R$, at constant ϕ, is

$$D = \int_0^R \frac{a}{(a^2 - \rho^2)^{1/2}} d\rho = a \sin^{-1}\left(\frac{R}{a}\right), \tag{1.20}$$

while the circumference of the circle is given by

$$C = \int_0^{2\pi} R d\phi = 2\pi R. \tag{1.21}$$

Similarly, from (1.15) we have the area within the circumference C

$$A = \int_0^{2\pi} \int_0^R \frac{a}{(a^2 - \rho^2)^{1/2}} \rho d\rho d\phi = 2\pi a^2 \left[1 - \left(1 - \frac{R^2}{a^2}\right)^{1/2}\right]. \tag{1.22}$$

Writing the circumference C and the area A in terms of the distance D, we have

$$C = 2\pi a \sin\left(\frac{D}{a}\right) \qquad \text{and} \qquad A = 2\pi a^2 \left[1 - \cos\left(\frac{D}{a}\right)\right]. \tag{1.23}$$

As D increases, the circumference of the circle C do so until the point where $D = \pi a/2$ and then C becomes smaller as D increases.

The total area, by symmetry, is given by

$$A_{\rm tot} = 2 \int_0^{2\pi} \int_0^a \frac{a}{(a^2 - r^2)^{1/2}} r dr d\phi = 4\pi a^2. \tag{1.24}$$

HW: hb.

1.- Compute the Volume of the 3D sphere [Eqn 1.11]. Hint:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{ar^2 \sin\theta}{(a^2 - r^2)^{1/2}} dr d\theta d\phi,$$

[the total volume $V_{\rm tot}$ is found when $R = a$].

2.- For this 3D-sphere, show that the line element can be written in the form

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)].$$

Then, calculate the area of the 2D sphere defined by $\chi = \chi_0$. Also find the total volume of the 3D space.

3.-By identifying a suitable coordinate transformation, show that the line element

$$ds^2 = (c^2 - a^2t^2)dt^2 - 2at\ dt\ dx - dx^2 - dy^2 - dz^2.$$

can be reduced to the Minkowski line element [a is a constant].

Python

In [1]:
```
from sympy import *
init_printing()
```

Distance in the surface

$$D = \int_0^R \frac{a}{(a^2-\rho^2)^{\frac{1}{2}}} d\rho$$

In [2]:
```
a, rho, R = symbols('a, \\rho, R', positive=True)
half = Rational(1, 2)
D = a / (a**2 - rho**2)**half

integrate(D, (rho, 0, R))
```

Out[2]:

$$a \operatorname{asin}\left(\frac{R}{a}\right)$$

Circumference

$$C = \int_0^{2\pi} R d\phi$$

In [3]:
```
phi = symbols ('\\phi')
C   = R

integrate(C,(phi,0,2*pi))
```

Out[3]:

$$2\pi R$$

Area

$$A = \int_0^{2\pi}\int_0^R \frac{a}{(a^2-\rho^2)^{\frac{1}{2}}} \rho d\rho d\phi$$

In [4]:
```
A = a / (a**2 - rho**2)**half*rho

simplify(integrate(A,(rho,0,R),(phi,0,2*pi)))
```

Out[4]:

$$2\pi a\left(a - \sqrt{-R^2+a^2}\right)$$

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