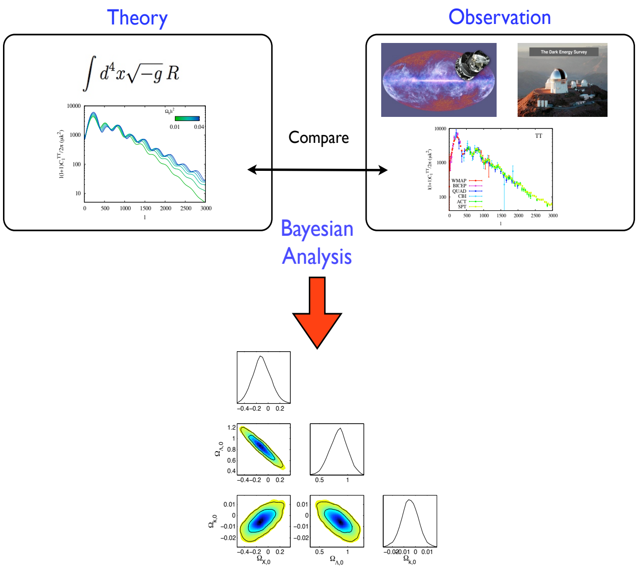
Updated Cosmology

with Python



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0.1 The Einstein Tensor

(jav: perhaps write a bibliography for each topic)

0.1.1 Christoffel symbols

The coefficients $\Gamma^a{}_{bc}$ are known as the **affine connection**, or traditionally called as the *Christoffel symbol* (of the second kind). It can be easily shown that $\Gamma^a{}_{bc}$ do not transform as the components of a tensor, however

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}, \tag{1}$$

is indeed a third-rank tensor, namely the **torsion tensor**. For convenience we can assume torsion free $T^a{}_{bc} = 0$, that is, the *affine connection is symmetric in its covariant indices*, i.e.

$$\Gamma^a{}_{bc} = \Gamma^a{}_{cb}. \tag{2}$$

Assumption 0.1.1:

Torsion free: $T^a{}_{bc} = 0 \quad \Rightarrow \quad \Gamma^a{}_{bc} = \Gamma^a{}_{cb}$.

We will use the *ansatz* that the covariant derivative of the metric tensor vanishes

$$g_{ab;c} = 0. \tag{3}$$

The covariant derivative (expressed by ∇ or ;) of a tensor is

$$A_{ab;c} = A_{ab,c} - \Gamma^d{}_{ac} A_{db} - \Gamma^d{}_{bc} A_{ad}.$$

By cyclically permuting the three indices of Eqn. (3), summing them all over, and using the covariant derivative of a tensor, we get

$$\begin{aligned} \Gamma^a{}_{bc} &= \frac{1}{2} g^{ad} (\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \\ &= \frac{1}{2} g^{ad} (g_{db,c} + g_{dc,b} - g_{bc,d}). \end{aligned} \tag{4}$$

Then, for the torsionless case, the quantity on the right hand side of Eqn. (4) is properly called the **metric connection** and often denoted by the symbol $\{{a \atop bc}\}$. In a torsionless manifold the affine and metric connections are equivalent.

The quantity Γ_{abc}, traditionally known as the **Christoffel symbol of the first kind**, is given by

$$\begin{aligned}\Gamma_{abc} &\equiv g_{ad}\Gamma^{d}{}_{bc} \\ &= \frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}).\end{aligned} \tag{5}$$

Adding Γ_{abc} to Γ_{bac} gives

$$g_{ab,c} = \Gamma_{abc} + \Gamma_{bac}, \tag{6}$$

which relates partial derivatives of the metric components to the connection coefficients. The contraction of the connection coefficients, leads to

$$\Gamma^{a}{}_{ab} = \partial_b \ln\sqrt{|g|} = \frac{1}{\sqrt{|g|}}\partial_b\sqrt{|g|}. \tag{7}$$

where the derivative of the determinant g of g_{ab} is

$$\begin{aligned}g_{,c} &= g g^{ab} g_{ab,c}, \\ &= 2g\Gamma^{a}{}_{ac}.\end{aligned} \tag{8}$$

HW1: Show that the components of $\Gamma^{a}{}_{bc}$ do not transform as the components of a tensor, but $T^{a}{}_{bc}$ do.

HW2: Prove (4), (6) and (8).

0.1.2 The curvature tensor

The **curvature tensor** (or the *Riemann-Christoffel tensor*) is defined in terms of the metric tensor g_{ab} and its first and second derivatives.

$$R^{a}{}_{bcd} \equiv \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} + \Gamma^{e}{}_{bd}\Gamma^{a}{}_{ec} - \Gamma^{e}{}_{bc}\Gamma^{a}{}_{ed}. \tag{9}$$

In a *flat* space-time, $\Gamma^{a}{}_{bc}$ and its derivatives are zero, and hence

$$R^{a}{}_{bcd} = 0. \tag{10}$$

The curvature tensor possesses a number of symmetries and satisfies certain identities, that are most easily derived in terms of its covariant components. An alternative way, and useful for this purpose, is the lowered version

$$R_{abcd} = g_{ae}R^{e}{}_{bcd}, \tag{11}$$

and after considerable algebra, we have

$$R_{abcd} = \frac{1}{2}\left(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac}\right) - g^{ef}\left(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbc}\right), \tag{12}$$

and the symmetries can be expressed as follow

$$\begin{aligned} R_{abcd} &= -R_{bacd}, \\ &= -R_{abdc}, \\ &= R_{cdab}. \end{aligned} \tag{13}$$

From the first set of symmetries, we notice that for $a = b$ or $c = d$ all the components of the Riemann tensor are zero. Then, we may easily deduce the *cyclic identity* (or **1st Bianchi identity**)

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{14}$$

which may be written as $R_{a[bcd]} = 0$. The conditions (13) and (14) reduce the number of independent components from N^4 to $N^2(N^2-1)/12$ (jav: do the math). In general, considering several dimensions, we have

No. of dimensions	2	3	4
No. of independent components of R_{abcd}	1	6	20

You can see from this table that in four dimensions the number of independent components is reduced from a possible 256 to 20.

Another useful relation we will need is the **2nd Bianchi identity**

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0, \tag{15}$$

and it can be written in the compact cyclic form $R_{ab[cd;e]}$.

HW: Show the validity of the expressions (13), (14) and (15).

0.1.3 Ricci tensor

From the symmetry properties, raising the index a and then contracting on the first two indices, gives

$$R^{a}{}_{acd} = 0. \tag{16}$$

Example 0.1.1: Show that $R^a{}_{acd}$=0.

Take the expression (11)

$$R_{abcd} = g_{ae}R^e{}_{bcd},$$

multiply $\times g^{ab}$ both sides

$$g^{ab}R_{abcd} = g^{ab}g_{ae}R^e{}_{bcd},$$

because the mood indices, we can interchanged $a \leftrightarrow b$ on the right hand side to get

$$g^{ba}\underbrace{g_{be}R^e{}_{acd}}_{\text{contraction}} = g^{ba}R_{bacd} = g^{ab}R_{bacd} = -g^{ab}R_{abcd},$$

where we have used the symmetry $g^{ba} = g^{ab}$ and the anti-symmetric relation in (13). Therefore $R^a{}_{acd} = 0$.

Contracting on the first and third indices, however, gives in general a non-zero result and this leads to a new tensor, the **Ricci tensor** (the trace of the Riemann tensor):

$$R_{ab} \equiv R^c{}_{acb} = g^{cd}R_{cadb}. \tag{17}$$

By raising the index a in the cyclic identity and contracting with d, one may easily show that the Ricci tensor is symmetric, $R^a{}_b = R_a{}^b$, and hence we can denote both by R^b_a. A further contraction gives the **Ricci scalar**, also known as the *curvature scalar*, which is the trace of the Ricci tensor:

$$R \equiv R^a{}_a = g^{ab}R_{ab} = g^{ab}g^{cd}R_{cadb}. \tag{18}$$

Example 0.1.2: Show R_{ab}=R_{ba}.

First, we write the cyclic expression (14) and multiply it by g^{cd} to get

$$g^{cd}R_{dacb} + g^{cd}R_{dcba} + g^{cd}R_{dbac} = 0,$$

notice the second term ($g^{cd}R_{dcba}$) vanishes by the first identity in (13). To then

$$\begin{aligned} g^{cd}R_{dacb} + g^{cd}R_{dbac} &= 0, \\ R^c{}_{acb} - g^{cd}R_{dbca} &= 0, \\ R^c{}_{acb} - R^c{}_{bca} &= 0, \\ R_{ab} &= R_{ba}. \end{aligned}$$

pb. The traceless part of the tensor R_{abcd} is defined by the equation

$$\begin{aligned} R_{abcd} &= \frac{1}{2}(g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}) \\ &-\frac{1}{6}(g_{ac}g_{bd} - g_{ad}g_{bc})R \\ &+C_{abcd}. \end{aligned} \tag{19}$$

The expressions in the first two lines have the symmetries of the curvature tensor, and C_{abcd} is called the *Weyl tensor*. The coefficients in the first two lines are chosen so the contraction of the Weyl tensor vanishes,

$$C^a{}_{bac} = 0.$$

The Weyl tensor will be helpful in simplifying the optical equation for the distortion in an inhomogeneous universe.

Another scalar that can be constructed from the Riemann tensor is the *Kretschmann scalar*:

$$K = R_{abcd}R^{abcd}. \tag{20}$$

For a Schwarzschild black hole of mass M, the Kretschmann scalar is

$$K = \frac{48G^2M^2}{c^4r^6},$$

which is not infinite at the event horizon $r = 2M$, hence we can tell immediately that the event horizon is a coordinate singularity not a real one. Likewise we can tell immediately that there is a real singularity at $r = 0$.

HW: Compute the Kretschmann scalar for a FRW spacetime.

0.1.4 Einstein tensor

Taking the Bianchi identities (15), and raising a and contracting with c (i.e. $\times g^{ac}$) gives

$$\nabla_e R_{bd} + \nabla_c R^c{}_{bde} + \nabla_d R^c{}_{bec} = 0, \tag{21}$$

which, on using the antisymmetry property in the third term, gives the Ricci tensor

$$\nabla_e R_{bd} + \nabla_c R^c{}_{bde} - \nabla_d R_{be} = 0. \tag{22}$$

If we now raise b and contract with e ($\times g^{be}$), we find

$$\nabla_e R^e{}_d + \nabla_c R^{ce}{}_{de} - \nabla_d R = 0, \tag{23}$$

Using the antisymmetry properties of Ricci tensor we may write the second term as

$$\nabla_c R^{ce}{}_{de} = \nabla_c R^{ec}{}_{ed} = \nabla_c R^{c}{}_{d} = \nabla_e R^{e}{}_{d}, \tag{24}$$

so the first and secod terms in (23) are identical and we obtain

$$2\nabla_e R^{e}{}_{d} - \nabla_d R = \nabla_e (2R^{e}{}_{d} - \delta^e_d R) = 0. \tag{25}$$

Finally, raising the index d $(\times g^{db})$, we obtain the important result

$$\nabla_a (R^{ab} - \frac{1}{2} g^{ab} R) = 0. \tag{26}$$

The term in parentheses is called the **Einstein tensor**

$$\boxed{G^{ab} \equiv R^{ab} - \frac{1}{2} g^{ab} R. \tag{27}}$$

It is clearly symmetric $G^{ab} = G^{ba}$ and thus possesses only one independent divergence $\nabla_a G^{ab}$ which vanishes.

HW: Show that $\nabla_a G^{ab} = 0$.

Hint: by contracting the second Bianchi identities, show that $R^{a}{}_{b;a} = \frac{1}{2} R_{;b}$.

The trace

$$\begin{aligned} g^{ab} G_{ab} &= g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R, \\ G &= R - \frac{1}{2} n R, \\ G &= \frac{2-n}{2} R. \end{aligned} \tag{28}$$

The special case of $n = 4$ dimensions gives the trace of the Einstein tensor as the negative of the Ricci tensor's trace, i.e. $G = -R$. Thus another name for the Einstein tensor is the *trace-reversed Ricci tensor*.

Example 0.1.3: The 2D sphere.

Let us get back to the metric of the surface of a sphere, with radius a, in spherical polar coordinates.

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2.$$

Then, we have the metric and its inverse

$$g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix} \qquad g^{ab} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/a^2\sin^2\theta \end{pmatrix}$$

Therefore, $g_{\theta\theta} = a^2$, $g_{\phi\phi} = a^2\sin^2\theta$, $g^{\theta\theta} = 1/a^2$, $g^{\phi\phi} = 1/a^2\sin^2\theta$. In two dimensions there are only six (2^3 minus the symmetric part) independent connection coefficients,

$$\Gamma^\theta{}_{\theta\theta}, \quad \Gamma^\theta{}_{\theta\phi}, \quad \Gamma^\theta{}_{\phi\phi}, \quad \Gamma^\phi{}_{\theta\theta}, \quad \Gamma^\phi{}_{\theta\phi}, \quad \Gamma^\phi{}_{\phi\phi}.$$

and the only derivative different from zero is $g_{\phi\phi,\theta} = 2a^2\sin\theta\cos\theta$. Therefore the Christoffel symbols are given by

$$\begin{aligned} \Gamma^\theta{}_{\theta\theta} &= 0, & \Gamma^\theta{}_{\theta\phi} &= 0, & \Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^\phi{}_{\phi\phi} &= 0 & \Gamma^\phi{}_{\theta\theta} &= 0, & \Gamma^\phi{}_{\theta\phi} &= \cot\theta. \end{aligned}$$

In two dimensions the Riemann tensor has only one independent non zero component, due to all the symmetries it satisfies: $R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi}$. And according to previous calculations:

$$R^\theta{}_{\phi\theta\phi} = \partial_\theta \Gamma^\theta{}_{\phi\phi} - \partial_\phi \Gamma^\theta{}_{\phi\theta} + \Gamma^i{}_{\phi\phi}\Gamma^\theta{}_{i\theta} - \Gamma^i{}_{\phi\theta}\Gamma^\theta{}_{i\phi}.$$

Since there are no ϕ derivatives, the second term vanishes. Since $\Gamma^\theta{}_{i\theta} = 0$, the first double Γ term vanishes too. Therefore, we get

$$\begin{aligned} R^\theta{}_{\phi\theta\phi} &= \partial_\theta \Gamma^\theta{}_{\phi\phi} - \Gamma^\phi{}_{\phi\theta}\Gamma^\theta{}_{\phi\phi} - \Gamma^\theta{}_{\phi\theta}\Gamma^\theta{}_{\theta\phi} \\ &= \partial_\theta(-\sin\theta\cos\theta) - (\cot\theta)(-\sin\theta\cos\theta) \\ &= \sin^2\theta - \cos^2\theta + \cos^2\theta = \sin^2\theta. \end{aligned}$$

Therefore $R_{\theta\phi\theta\phi} = g_{\theta i} R^i{}_{\phi\theta\phi} = \sin^2\theta$ and

$$R^\phi{}_{\theta\phi\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = \frac{1}{\sin^2\theta}\sin^2\theta = 1.$$

Example 0.1.4: The 2D sphere.

The Ricci tensor is then:

$$\begin{aligned}
R_{\theta\theta} &= R^i{}_{\theta i\theta} = R^\theta{}_{\theta\theta\theta} + R^\phi{}_{\theta\phi\theta} = 1. \\
R_{\phi\phi} &= R^i{}_{\phi i\phi} = R^\theta{}_{\phi\theta\phi} + R^\phi{}_{\phi\phi\phi} = \sin^2\theta. \\
R_{\theta\phi} &= R_{\phi\theta} = R^i{}_{\theta i\phi} = R^\theta{}_{\theta\theta\phi} + R^\phi{}_{\theta\phi\phi} = 0.
\end{aligned}$$

The Ricci scalar is:

$$R = g^{ab}R_{ab} = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{1}{a^2} + \left(\frac{1}{a^2\sin^2\theta}\right)(\sin^2\theta) = \frac{2}{a^2}.$$

The resulting Einstein tensor:

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 0,$$

for all $a, b \in [\theta, \phi]$. Thus the Gaussian curvature K (defined later) of a spherical surface is given by

$$K = \frac{R_{1212}}{|g|} = \frac{a^2\sin^2\theta}{a^4\sin^2\theta} = \frac{1}{a^2}.$$

Alternatives to GR

"In general relativity (GR) it is assumed, without empirical support, that torsion vanishes identically. Of course, one may claim that the experimental success of GR justifies the vanishing-torsion hypothesis. However, as it is argued below, all GR tests are compatible with a non-vanishing torsion, and, as a basic assumption of the theory, it is paramount to experimentally test it." [**?**] Furthermore, another important hypothesis is the fact that c, G, and α are constant, nevertheless, some important cosmological implications emerge when these are not. In this paper some models where all this standard assumptions may change are discussed.

Einstein-Cartan theory

The theory was first proposed by Élie Cartan in 1922 [**?**] and expounded in the following few years. This model states that GR must be extended in order to include an affine torsion, which in contrast with GR, allows the possibility of having an asymmetric Ricci tensor. Nowadays, this theory that extend the Riemannian geometry in that direction is better known as Riemann-Einstein-Cartan geometry and is determined by the following features:

- A specific choice of the metric tensor.
- A specific affine torsion tensor.
- Parallel transport must preserve lengths and angles as in the usual Riemannian geometry.

The corresponding equations of motion derived from the action variation are given by:

$$R_{ak} - \frac{1}{2} g_{ak} R = \frac{8\pi G}{c^4} P_{ak},$$

$$S^k_{ab} = \frac{8\pi G}{c^4} \sigma^k_{ab},$$

where σ^k_{ab} is the spin tensor of the source. $S^k_{ab} = T^k_{ab} + g^k_a T^m_{bm} - g^k_b T^m_{am}$ is the modified torsion tensor. And, finally T^k_{ab} is the affine torsion tensor that characterised these models. In the first equation one can see that it has the same structure than the usual Einstein's equations. Meanwhile, the second equation expresses the angular momentum conservation considering the spin-orbit interaction. Therefore, GR can be understood as a limit of the more general Riemann-Einstein-Cartan theory of gravity. Moreover, it is expected that this theory will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion [**?**].

Variable c and G

In contrast with what know from GR, the idea of a variable (non-constant) speed of light (VSL) has been considered over the years. Actually, Einstein itself thought seriously this idea in 1911 [**?**], where he assumed that clocks in a gravitational field run slower, whereby the corresponding frequencies are influenced by the gravitational potential. Later, in 1915 he concluded that light speed is constant when gravity does not have to be considered but that the speed of light cannot be constant in a gravitational field with variable strength [**?**].

Nowadays, this idea is still present but in cosmological models such as the alternative to inflation proposed by Jean-Pierre Petit, John Moffat, Andreas Albrecht and JoÃ£o Magueijo [**? ? ?**], where also the Newton's constant G is no longer a constant. In the minimally coupled theory one then simply replaces c by a field in a preferred frame. Hence, the action is given by

$$S = \int d^4x \left(\sqrt{-g} \left(\frac{\psi(R + 2\Lambda)}{16\pi G} + \mathcal{L}_{\mathrm{mat}} \right) + \mathcal{L}_{\psi} \right),$$

where $\psi(x^\mu) = c^4$. One can solve the cosmological field equations that define it for general power-law variations of "c" and "G". This allows us to determine the rate and sense of the changes required in "c" if the flatness, horizon, and cosmological constant problems are to be solved. The period when "c" varies is expected to happen only during the very early universe, therefore its observational remnants should be observable through the CMB fluctuations.

Variable α

It has been suggested by some astronomical observation that perhaps α, the fine structure constant, should not be strictly a constant, and this idea led to a serious consideration of a variable α by Jacob Bekenstein in 1981 [**?**]. However, one big consequence is that Maxwell's equations must be modified, but in order to test the viability of this conception, purely electromagnetic experiments are not enough. On the other hand, since the cosmological perspective, the framework predicts an $\frac{\dot{\alpha}}{\alpha}$ which can be compatible with the astronomical constraints; hence, these are too insensitive to determine any possible variability. In VSL theories a varying α is interpreted as $c \propto h \propto \alpha^{-1/2}$ and e is constant, Lorentz invariance is broken, and so by construction there is a preferred frame for the formulation of the physical laws.

Thus, so far, this idea cannot be ruled out with the experimental evidence, and more future tests are needed in order to determine its viability.

Some model with torsion and c, G and α varying has been discussed. Although most of these theories are not in contradiction with observational evidence, sometimes the framework looks more complicated than the ordinary one. Nevertheless, they are a valid and consistent approach that could be helpful to solve for example some cosmological problems.

Bibliography