Appendix A

The symplectic groups, their parametrization and cover

ABSTRACT: In this appendix we summarize the parametrization and some of the properties of connectivity and covering of the symplectic groups. This material was developed by Valentin Bargmann in his early work on the three-dimensional Lorentz group and on Hilbert spaces of analytic functions, and has been shown to be particularly relevant for Lie optics. Wave optics —at least in its paraxial approximation— seems to work with the *double* cover of the symplectic group of geometric first-order optics. This is strongly reminiscent of the double cover which the spin group affords over the classical rotation group, and makes necessary a closer acquaintance with the *metaplectic* groups.

A.1 Rank one: $SL(2,\Re)$, $Sp(2,\Re)$, $SU(1,1)$, $SO(2,1)$, and $\overline{Sp(2,\Re)}$

Lie groups of rank one present several accidental homomorphisms. Among the compact groups, the three-dimensional rotation group $SO(3)$ and the two-dimensional special unitary group $SU(2)$ are probably the most famous pair of homomorphic groups, the latter covering the former twice and allowing the description of phenomena such as spin. Non-compact groups of rank one present a fourfold such homomorphism: the group of 2×2 real matrices $SL(2,\Re)$, is isomorphic to the two-dimensional symplectic group $Sp(2,\Re)$ and to the two-dimensional pseudo-unitary group $SU(1,1)$, and covers twice the three-dimensional pseudo-orthogonal group $SO(2,1)$. These groups are themselves infinitely connected, and possess a common universal cover $\overline{Sp(2,\Re)}$. A particularly relevant group for Lie optics is the *metaplectic* group, $Mp(2,\Re)$; it covers $Sp(2,\Re)$ twice. We start with $SL(2,\Re)$ and relate to it all other homomorphic groups.

A.1.1 $SL(2,\Re)$

We denote by $SL(2,\Re)$ the set of 2×2 real, unimodular matrices

$$\mathbf{g} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det \mathbf{g} = ad - bc = 1, \tag{1}$$

with the group product defined as ordinary matrix multiplication. It is a three-parameter noncompact semisimple group, connected and infinitely-connected. The latter facts are not obvious, and will be further elaborated in Section A.2, below.

A.1.2 *Sp(2,ℜ)*

The set of 2×2 matrices (1) is, at the same time, a set of ***symplectic***[1] matrices, ***i.e.***, they satisfy

$$\mathbf{g}\mathbf{M}_{Sp(2)}\mathbf{g}^\top = \mathbf{M}_{Sp(2)}, \qquad \mathbf{M}_{Sp(2)} := \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \tag{2}$$

where $\mathbf{g}^\top$ is the transpose of $\mathbf{g}$ and $\mathbf{M}_{Sp(2)}$ is the symplectic *metric* matrix. The unimodularity condition in (1) implies the validity of (2), as may be verified through elementary algebra. The set of matrices satisfying (2) constitute the *group*, denoted *Sp(2,ℜ)*, of two-dimensional real symplectic matrices.

A.1.3 *SU(1,1)*

The above two groups are isomorphic to a third one, the group of complex *"1+1"* unimodular pseudo-unitary matrices, denoted *SU(1,1)*, whose elements $\mathbf{u}$ satisfy

$$\mathbf{u}\mathbf{M}_{E(1,1)}\mathbf{u}^\dagger = \mathbf{M}_{E(1,1)}, \qquad \mathbf{M}_{E(1,1)} := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

where $\mathbf{u}^\dagger := \mathbf{u}^{\top *}$ is the *adjoint* (transpose, complex conjugate) of $\mathbf{u}$, and $\mathbf{M}_{E(1,1)}$ the pseudo-euclidean metric matrix.

It is easy to show that the most general matrix $\mathbf{u} \in$ *SU(1,1)* has the form

$$\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1. \tag{4}$$

The elements of the *SL(2,ℜ)* = *Sp(2,ℜ)* and *SU(1,1)* groups realized as 2×2 matrices, are related through a *similarity transformation*, an outer isomorphism by the complex unitary matrix

$$\mathbf{W} := \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_4^{-1} & \omega_4^{-1} \\ -\omega_4 & \omega_4 \end{pmatrix} = \mathbf{W}^{\dagger -1}, \qquad \omega_4 := e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i). \tag{5}$$

We display this isomorphism explicitly in terms of the group parameters as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{g}(\mathbf{u}) = \mathbf{W}\mathbf{u}\mathbf{W}^{-1} = \begin{pmatrix} \mathrm{Re}\,(\alpha+\beta) & -\mathrm{Im}\,(\alpha-\beta) \\ \mathrm{Im}\,(\alpha+\beta) & \mathrm{Re}\,(\alpha-\beta) \end{pmatrix}, \tag{6a}$$

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \mathbf{u}(\mathbf{g}) = \mathbf{W}^{-1}\mathbf{g}\mathbf{W} = \frac{1}{2}\begin{pmatrix} (a+d)-i(b-c) & (a-d)+i(b+c) \\ (a-d)-i(b+c) & (a+d)+i(b-c) \end{pmatrix}. \tag{6b}$$

Since $\mathbf{W}^{-1}\mathbf{M}_{Sp(2)}\mathbf{W} = -i\,\mathbf{M}_{E(1,1)}$, (2) for $\mathbf{g}$ is equivalent to (3) for $\mathbf{u}$.

Of course, any other matrix $\mathbf{W}' = \mathbf{g}_0\mathbf{W}$ or $\mathbf{W}\mathbf{u}_0$, for fixed $\mathbf{g}_0 \in$ *Sp(2,ℜ)* or $\mathbf{u}_0 \in$ *SU(1,1)*, may be used to define equivalent isomorphisms between the two groups. What makes (5) particularly convenient for us is that it establishes the appropriate link between the *standard* realizations of the groups.

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[1] SYMPLECTIC (simplektik), ***adjective and substantive***, **first appearance: 1839.** [*Adaptation from Greek συμπλεκτικός, formed on σύν* SYM- +πλέκειν, TO TWINE, PLAIT, WEAVE.] **A.** adjective **Epithet of a bone of the suspensorium in the skull of fishes, between the hyomandibular and the quadrate bones. B.** substantive **The symplectic bone. —*The Oxford Universal Dictionary on Historical Principles,* third ed., 1955. The use of this name for the Cartan C–family of semisimple groups is due to Hermann Weyl, in *The theory of groups and quantum mechanics,* 2nd ed. (Dover, New York, 1930).**

A.1.4 *SO(2,1)*

For every real 2×2 matrix $\mathbf{g}$ in (1), we construct the real 3×3 matrix

$$\mathbf{\Gamma}(\mathbf{g}) := \begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & -cd-ab \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & cd-ab \\ -bd-ac & bd-ac & ad+bc \end{pmatrix}. \tag{7}$$

It has the properties

$$\mathbf{\Gamma}\mathbf{M}_{E(1,2)}\mathbf{\Gamma}^{\top} = \mathbf{M}_{E(1,2)}, \qquad \mathbf{M}_{E(1,2)} := \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{8a}$$

$$\det \mathbf{\Gamma} = 1, \tag{8b}$$

$$\mathbf{\Gamma}(\mathbf{g}_1)\mathbf{\Gamma}(\mathbf{g}_2) = \mathbf{\Gamma}(\mathbf{g}_1\mathbf{g}_2), \qquad \mathbf{\Gamma}(\mathbf{1}) = \mathbf{1}, \quad \mathbf{\Gamma}(\mathbf{g}^{-1}) = \mathbf{\Gamma}(\mathbf{g})^{-1}. \tag{8c}$$

The matrices $\mathbf{\Gamma}$ satisfying (8*a*) are called *"1+2"* pseudo-orthogonal matrices, and the statement of unimodularity in (8*b*) reduces the matrices under consideration to a set connected to the identity. They form a group denoted *SO(2,1)*. The rows and columns numbering the elements $\Gamma_{\mu,\nu}$ range here over μ, $\nu = 0, 1, 2$; the value 0 corresponds to the first row, *i.e.*, the time-like coordinate.

Since $\mathbf{\Gamma}(\mathbf{g})$ is quadratic in the parameters of $\mathbf{g}$, and $\mathbf{\Gamma}(\mathbf{g}) = \mathbf{\Gamma}(-\mathbf{g})$, (7) defines a 2:1 homomorphism between *Sp(2,ℜ)* and *SO(2,1)*, with kernel $\{-\mathbf{1}, \mathbf{1}\} = \mathfrak{Z}_2 \subset Sp(2,\Re)$. The *SO(2,1)* matrix corresponding to any *pair of* given *SU(1,1)* matrices may be obtained through (6) and (7). It is

$$\mathbf{\Gamma}(\mathbf{u}) = \begin{pmatrix} |\alpha|^2+|\beta|^2 & 2\,\mathrm{Re}\,\alpha\beta^* & 2\,\mathrm{Im}\,\alpha\beta^* \\ 2\,\mathrm{Re}\,\alpha\beta & \mathrm{Re}\,(\alpha^2+\beta^2) & \mathrm{Im}\,(\alpha^2-\beta^2) \\ -2\,\mathrm{Im}\,\alpha\beta & -\mathrm{Im}\,(\alpha^2+\beta^2) & \mathrm{Re}\,(\alpha^2-\beta^2) \end{pmatrix}, \tag{9a}$$

and conversely,

$$\alpha = \pm\sqrt{\tfrac{1}{2}(\Gamma_{11}+\Gamma_{12}) + \tfrac{1}{2}i(\Gamma_{12}-\Gamma_{21})}, \qquad \beta = \frac{1}{2\alpha}(\Gamma_{10} - i\Gamma_{20}). \tag{9b}$$

A.2 Connectivity

The connectivity of a three-dimensional manifold is, when multiple, a challenge to the mind. If we are to picture this intuitively, we must build up a proper representation of the manifold and follow one-parameter lines. If these lines turn out to be circles, then the covering of these by the real line becomes plausible within the group. As is so often the case, the complex plane is useful and *SU(1,1)* is, among the groups of last section, the easiest to analyze.

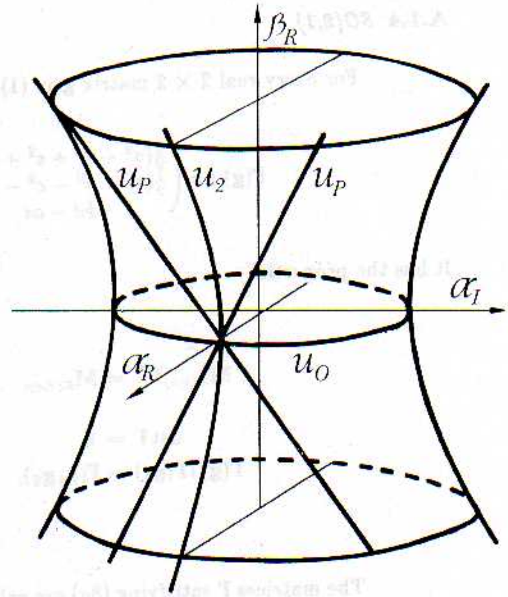

Figure. The $SU(1,1) \simeq Sp(2,\Re)$ group manifold is three-dimensional, connected and infinitely connected. It is $\Re^3$ pierced by a one-sheeted equilateral hyperboloid. Three representatives of one-parameter subgroups are drawn: $\mathbf{u}_0(\tau)$ (circle), $\mathbf{u}_2(\tau)$ (one branch of a hyperbola), and $\mathbf{u}_{P\pm}(\tau)$ (straight lines), lying on the $\beta_I = 0$ hyperboloid.

A.2.1 The connectivity of *SU(1,1)*

The connectivity properties of *Sp(2,ℜ)* and its isomorphic groups is seen best in terms of the *SU(1,1)* parameters (4). We write the real and imaginary parts of α and β as $\alpha = \alpha_R + i\alpha_I$ and $\beta = \beta_R + i\beta_I$, so the unimodularity condition reads $\alpha_R^2 + \alpha_I^2 - \beta_R^2 = 1 + \beta_I^2 \geq 1$. For fixed β_I, the remaining three parameters are constrained to a one-sheeted revolution hyperboloid with its circular waist in the α-plane. As we let β_I range over $\Re$, we fill twice a region of $\Re^3$-space which is bounded by the $\beta_I = 0$ equilateral revolution hyperboloid. See the figure above. Topologically, the group manifold of *Sp(2,ℜ)* is $\mathfrak{S}_1 \times \Re^2$, the circle times the cartesian plane.

The points $(\alpha_R, \alpha_I, \beta_R)$ in $\Re^3$ which do *not* belong to the group manifold as described above, form a simply connected tubular region. Any plane for which this region has an elliptic section is an infinitely connected space (the plane minus the unit disk), and hence so is a foliation of $\Re^3$ by such planes. One-parameter subgroups of *SU(1,1)* are lines in $\Re^3$ which must pass through the group identity: the point $(\alpha_R = 1, \alpha_I = 0, \beta_R = 0)$ on the circular waist.

The class representatives of the three nonequivalent subgroup are:

Elliptic: $\mathbf{u}_0(\tau)$, $\tau \in \Re \,(\text{mod } 4\pi)$ given by $\begin{cases} \alpha_R = \cos\frac{1}{2}\tau, \\ \alpha_I = \sin\frac{1}{2}\tau, \\ \beta_R = 0. \end{cases}$

Hyperbolic: $\mathbf{u}_2(\tau)$, $\tau \in \Re$ given by $\begin{cases} \alpha_R = \cosh\frac{1}{2}\tau, \\ \alpha_I = 0, \\ \beta_R = -\sinh\frac{1}{2}\tau. \end{cases}$

Parabolic: $\mathbf{u}_{P\pm}(\tau)$, $\tau \in \Re$ given by $\begin{cases} \alpha_R = 1, \\ \alpha_I = \frac{1}{2}\tau, \\ \beta_R = \pm\frac{1}{2}\tau. \end{cases}$

The elliptic subgroup is the circular waist in the figure, the hyperbolic subgroup is one of the two branches of the equilateral hyperbola on the $\alpha_I = 0$ plane, while the parabolic ones are the straight lines contained in the hyperboloid which pass through the identity parallel to the α_I–β_R plane. The only subgroup class with a nontrivial covering is thus the elliptic one.

A.2.2 The covering group of *SU(1,1)*

Following Bargmann's pivotal work in reference [1], we now present the simply-connected universal covering group of *SU(1,1)*, making use of the (maximal) compact subgroup *U(1) = SO(2)*. This corresponds to the Iwasawa decomposition which factorizes globally a noncompact semisimple group into its maximal compact subgroup, times a solvable subgroup.

We write $\mathbf{u} \in SU(1,1)$ in (4), in the form

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda \end{pmatrix}, \qquad \omega \in \mathfrak{S}_1,\ \lambda > 0,\ \mu \in \mathfrak{C}, \tag{10a}$$

i.e., ω and λ are the argument and absolute value of α:

$$\begin{aligned} \omega &:= \arg \alpha = \tfrac{1}{2} i \ln(\alpha^* \alpha^{-1}), \\ \lambda &:= |\alpha| > 0, \\ \mu &:= e^{-i\omega}\beta = \sqrt{\frac{\alpha^*}{\alpha}}\,\beta, \end{aligned} \tag{10b}$$

with all multivalued functions evaluated on the *principal sheet*. Conversely, of course, $\alpha = e^{i\omega}\lambda$ and $\beta = e^{i\omega}\mu$. We note that $|\alpha|^2 - |\beta|^2 = \lambda^2 - |\mu|^2 = 1$, so $|\mu| < \lambda$.

This parametrization will be generalized to *Sp(2N,ℜ)* following Bargmann [2] in Section **A.4**. Actually, in the first treatment of *SU(1,1)*, Bargmann [1] used in 1947 an equivalent set of parameters

$$\begin{gathered} (\omega, \gamma), \qquad \omega \in \mathfrak{S}_1, \quad \gamma \in \mathfrak{C}, \quad |\gamma| < 1 \\ \gamma := \frac{\mu}{\lambda} = \frac{\beta}{\alpha}, \qquad \lambda = \frac{1}{\sqrt{1 - |\gamma|^2}}, \quad \mu = \frac{\gamma}{\sqrt{1 - |\gamma|^2}}. \end{gathered} \tag{10c}$$

In some respects this parametrization is more convenient, but it does not generalize easily to N dimensions. We shall here prefer the former and call $\{\omega, \lambda, \mu\}$ the *Bargmann parameters* of *SU(1,1)*, writing $\mathbf{u}\{\omega, \lambda, \mu\}$ when they are used.[2]

The *Sp(2,ℜ) = SL(2,ℜ)* parameters (1) are expressed in terms of the Bargmann parameters through

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \lambda + \operatorname{Re}\mu & \operatorname{Im}\mu \\ \operatorname{Im}\mu & \lambda - \operatorname{Re}\mu \end{pmatrix}. \tag{11a}$$

This displays the global decomposition of any nonsingular matrix into the product of an orthogonal and a positive definite symmetric matrix. Conversely,

$$\omega = \arg[(a+d) - i(b-c)], \qquad \mu = e^{-i\omega}[(a-d) + i(b+c)]. \tag{11b}$$

[2] Although λ is a redundant parameter (since $\lambda = \sqrt{1 + |\mu|^2}$), it will be kept for the sake of easy comparison with the N-dimensional case.

A.2.3 The covering group $\overline{Sp(2,\mathfrak{R})}$

Through matrix multiplication $\mathbf{u}\{\omega,\lambda,\mu\} = \mathbf{u}\{\omega_1,\lambda_1,\mu_1\}\mathbf{u}\{\omega_2,\lambda_2,\mu_2\}$ of either (10) or (11) we obtain

$$\omega = \omega_1 + \omega_2 + \arg\nu, \tag{12a}$$
$$\lambda = \lambda_1\,|\nu|\,\lambda_2, \tag{12b}$$
$$\mu = e^{-i\arg\nu}\,[\lambda_1\mu_2 + e^{-2i\omega_2}\mu_1\lambda_2], \tag{12c}$$

where

$$\nu := 1 + e^{-2i\omega_2}\lambda_1^{-1}\,\mu_1\,\mu_2^*\,\lambda_2^{-1}, \quad |\nu - 1| < 1. \tag{12d}$$

The last inequality stems from $|\mu_i/\lambda_i| < 1$, and implies that ν is within a circle of radius less than one, centered at $\nu = 1$; hence $\nu \neq 0$.

The group unit is $\mathbf{1} = \mathbf{u}\{\omega = 0, \lambda = 1, \mu = 0\}$ and the inverse is given by $\mathbf{u}\{\omega,\lambda,\mu\}^{-1} = \mathbf{u}\{-\omega,\lambda,-e^{2i\omega}\mu\}$. Note that the subset of $\mathbf{u}$'s given by $\mathbf{u}\{\omega = 0,\lambda,\mu\}$ are naturally coset representatives of *U(1)\SU(1,1)*, but do *not* constitute a group.

From (10) and (11) it is clear that *SU(1,1)* and $Sp(2,\mathfrak{R}) = SL(2,\mathfrak{R})$ are described when ω is counted modulo 2π, that is $\omega \in \mathfrak{S}_1$, *i.e.* $\omega \equiv \omega(\text{mod } 2\pi)$. If we drop this identification and consider $\omega \in \mathfrak{R}$ *with no modular condition*, defining the composition law through (12), we describe a *covering* of *SU(1,1)* whose elements we denote by $\overline{u}\{\omega,\lambda,\mu\}$ ($\omega \in \mathfrak{R}$, $\mu \in \mathfrak{C}$). The manifold of this group is $\mathfrak{R} \times \mathfrak{C} = \mathfrak{R}^3$, and this is simply connected. The composition rule (12) for $\overline{u}\{\omega,\lambda,\mu\}$ thus describes the ***universal covering*** group $\overline{Sp(2,\mathfrak{R})} = \overline{SL(2,\mathfrak{R})} = \overline{SU(1,1)}$ of $Sp(2,\mathfrak{R}) = SL(2,\mathfrak{R}) = SU(1,1)$.

A.2.4 The metaplectic group $Mp(2,\mathfrak{R})$

The center of $\overline{Sp(2,\mathfrak{R})}$ is the set of elements $\mathfrak{Z}_\infty = \{\overline{u}\{n\pi,1,0\},\ n \in \mathfrak{Z}\}$, so that the pseudo-orthogonal group is $SO(2,1) = \overline{Sp(2,\mathfrak{R})}/\{\overline{u}\{n\pi,1,0\},\ n \in \mathfrak{Z}\}$, and the symplectic group is $Sp(2,\mathfrak{R}) = \overline{Sp(2,\mathfrak{R})}/\{\overline{u}\{2n\pi,1,0\},\ n \in \mathfrak{Z}\}$. Various M-fold coverings of $Sp(2,\mathfrak{R})$ may be obtained from the universal cover $\overline{Sp(2,\mathfrak{R})}$, modulo $\{\overline{u}\{2Mn\pi,1,0\},\ n \in \mathfrak{Z}\}$. In particular, we are interested in the *two-fold cover* of $Sp(2,\mathfrak{R})$, the ***metaplectic*** group

$$Mp(2,\mathfrak{R}) = \overline{Sp(2,\mathfrak{R})}/\{\overline{u}\{n\pi,1,0\},\ n \in \mathfrak{Z}\}. \tag{13}$$

Its elements will be written $\tilde{g}(\omega,\lambda,\mu)$, with $\mu \in \mathfrak{C}$, $\omega \equiv \omega(\text{mod } 4\pi)$. The 2:1 mapping from $Mp(2,\mathfrak{R})$ to $Sp(2,\mathfrak{R})$ is given by (11) [(10) on *SU(1,1)*], and assigns the same image to $\tilde{g}(\omega,\lambda,\mu)$ and $\tilde{g}(\omega+2\pi,\lambda,\mu)$.

Neither the metaplectic group nor its covers have a *matrix* representation (by *finite*-dimensional matrices, that is). They *do* have representations which are infinite-dimensional, as by integral kernels. This fact accounts for some of the difficulty we encounter when working with covers of $Sp(2,\mathfrak{R})$.

Single-valued functions on covering groups may give rise to multivalued functions on the original group. The phase of the canonical transform[3] kernel $\theta_{\mathbf{g}}$ is a prime example of a single-valued function $\theta_{\tilde{g}}$ on $Mp(2,\mathfrak{R})$, yielding a two-valued function on $Sp(2,\mathfrak{R})$.

[3] See, for example, K.B. Wolf, *Integral Transforms in Science and Engineering*, (Plenum Publ. Corp., New York, 1979), part 4.

A.3 Subgroups

We list below some useful one-parameter subgroups $\mathbf{g}(\tau)$ of $Sp(2,\Re) = SL(2,\Re)$, together with their counterparts $\mathbf{u} \in SU(1,1)$, $\mathbf{\Gamma} \in SO(2,1)$, and one $\overline{u} \in \overline{Sp(2,\Re)}$:

A.3.1 Elliptic subgroup

$$\mathbf{g}_0(\tau) = \begin{pmatrix} \cos\frac{1}{2}\tau & -\sin\frac{1}{2}\tau \\ \sin\frac{1}{2}\tau & \cos\frac{1}{2}\tau \end{pmatrix}, \qquad \mathbf{u}_0(\tau) = \begin{pmatrix} e^{i\tau/2} & 0 \\ 0 & e^{-i\tau/2} \end{pmatrix},$$
$$\mathbf{\Gamma}_0(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau \\ 0 & -\sin\tau & \cos\tau \end{pmatrix}, \qquad \overline{u}_0\{\tfrac{1}{2}\tau, 1, 0\}. \tag{14a}$$

A.3.2 Hyperbolic subgroups

$$\mathbf{g}_1(\tau) = \begin{pmatrix} \cosh\frac{1}{2}\tau & -\sinh\frac{1}{2}\tau \\ -\sinh\frac{1}{2}\tau & \cosh\frac{1}{2}\tau \end{pmatrix}, \qquad \mathbf{u}_1(\tau) = \begin{pmatrix} \cosh\frac{1}{2}\tau & -i\sinh\frac{1}{2}\tau \\ i\sinh\frac{1}{2}\tau & \cosh\frac{1}{2}\tau \end{pmatrix},$$
$$\mathbf{\Gamma}_1(\tau) = \begin{pmatrix} \cosh\tau & 0 & \sinh\tau \\ 0 & 1 & 0 \\ \sinh\tau & 0 & \cosh\tau \end{pmatrix}, \qquad \overline{u}_1\{0, \cosh\tfrac{1}{2}\tau, -i\sinh\tfrac{1}{2}\tau\}; \tag{14b}$$

$$\mathbf{g}_2(\tau) = \begin{pmatrix} e^{-\tau/2} & 0 \\ 0 & e^{\tau/2} \end{pmatrix}, \qquad \mathbf{u}_2(\tau) = \begin{pmatrix} \cosh\frac{1}{2}\tau & -\sinh\frac{1}{2}\tau \\ -\sinh\frac{1}{2}\tau & \cosh\frac{1}{2}\tau \end{pmatrix},$$
$$\mathbf{\Gamma}_2(\tau) = \begin{pmatrix} \cosh\tau & -\sinh\tau & 0 \\ -\sinh\tau & \cosh\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \overline{u}_2\{0, \cosh\tfrac{1}{2}\tau, -\sinh\tfrac{1}{2}\tau\}. \tag{14c}$$

A.3.3 Parabolic subgroups

$$\mathbf{g}_+(\tau) = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{u}_+(\tau) = \begin{pmatrix} 1+i\frac{1}{2}\tau & -i\frac{1}{2}\tau \\ i\frac{1}{2}\tau & 1-i\frac{1}{2}\tau \end{pmatrix},$$
$$\mathbf{\Gamma}_+(\tau) = \begin{pmatrix} 1+\frac{1}{2}\tau^2 & -\frac{1}{2}\tau^2 & \tau \\ \frac{1}{2}\tau^2 & 1-\frac{1}{2}\tau^2 & \tau \\ \tau & -\tau & 1 \end{pmatrix}, \qquad \overline{u}_+\{\arg[1+i\tfrac{1}{2}\tau], |1+i\tfrac{1}{2}\tau|, e^{-i(\omega+\pi/2)}\tfrac{1}{2}\tau\}; \tag{14d}$$

$$\mathbf{g}_-(\tau) = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}, \qquad \mathbf{u}_-(\tau) = \begin{pmatrix} 1+i\frac{1}{2}\tau & i\frac{1}{2}\tau \\ -i\frac{1}{2}\tau & 1-i\frac{1}{2}\tau \end{pmatrix},$$
$$\mathbf{\Gamma}_-(\tau) = \begin{pmatrix} 1+\frac{1}{2}\tau^2 & \frac{1}{2}\tau^2 & -\tau \\ -\frac{1}{2}\tau^2 & 1-\frac{1}{2}\tau^2 & \tau \\ -\tau & -\tau & 1 \end{pmatrix}, \qquad \overline{u}_-\{\arg[1+i\tfrac{1}{2}\tau], |1+i\tfrac{1}{2}\tau|, e^{-i(\omega-\pi/2)}\tfrac{1}{2}\tau\}. \tag{14e}$$

In all but the first (the elliptic subgroup), the correspondence between $Sp(2,\Re)$ and $SO(2,1)$ is one-to-one.

A.3.4 Conjugation and trace

All one-parameter subgroups can be obtained through similarity conjugation out of $\mathbf{g}_0(\tau)$, $\mathbf{g}_2(\tau)$, and $\mathbf{g}_+(\tau)$. In the above list,

$$\mathbf{g}_0(\tfrac{1}{2}\pi)\,\mathbf{g}_2(\tau)\,\mathbf{g}_0(\tfrac{1}{2}\pi)^{-1} = \mathbf{g}_1(\tau) \quad \text{and} \quad \mathbf{g}_0(\pi)\,\mathbf{g}_+(\tau)\,\mathbf{g}_0(\pi)^{-1} = \mathbf{g}_-(\tau).$$

The two parabolic subgroups listed at the end of **A.2.1** and displayed in the figure, are related to those of **A.2.3** through

$$\mathbf{g}_0(\pm\tfrac{1}{2}\pi)\,\mathbf{g}_+(\tau)\,\mathbf{g}_0(\pm\tfrac{1}{2}\pi)^{-1} = \mathbf{g}_{P\pm}(\tau).$$

Under similarity conjugation, the 2×2 *trace* of the matrices, $T := \operatorname{tr}\mathbf{u} = 2\operatorname{Re}\alpha = a + d$, is left invariant. For the three subgroup cases, we have $T_0(\tau) = 2\cos\frac{1}{2}\tau \in [-2, 2]$, $T_2(\tau) = 2\cosh\frac{1}{2}\tau \in [2, \infty)$, and $T_P(\tau) = 2$. [Note $T(\tau = 0) = 2$ in all cases.] If these subgroups are drawn as lines in the group manifold of the figure, the elliptic subgroups will be represented by plane ellipses —in any plane containing the α_R axis— passing through the identity $\mathbf{1}$ ($\alpha_R = 1$, $\alpha_I = 0$, $\beta_R = 0$) and $-\mathbf{1}$ ($\alpha_R = -1$, $\alpha_I = 0$, $\beta_R = 0$), with foci on the α_I–β_R plane. The hyperbolic subgroups will be represented by one branch of plane hyperbolæ with foci on the α_R axis. The parabolic subgroups appear as straight lines in the $\alpha_R = 2$ plane bounded by the $(P\pm)$-intercepts with the equilateral hyperboloid.

Conversely, any *SU(1,1)* $\simeq$ *Sp(2,ℜ)* group element (different from $\mathbf{1}$ or $-\mathbf{1}$) whose trace $T = \operatorname{tr}\mathbf{u}$ is in $(-2, 2)$, $\{2\}$, or $(2, \infty]$, may be placed on a one-parameter elliptic, parabolic, or hyperbolic subgroup, respectively. If $T \le -2$, no such subgroup can be found, but one may write $\mathbf{u} = (-\mathbf{1})\mathbf{u}'$ and place $\mathbf{u}'$ on a one-parameter subgroup as before. In *SO(2,1)*, all elements $\mathbf{\Gamma}$ may be placed on one-parameter subgroups connected to the identity.

We also have the subset (*not* a subgroup) given by

$$\mathbf{g}_\gamma = \begin{pmatrix} \lambda + \operatorname{Re}\mu & \operatorname{Im}\mu \\ \operatorname{Im}\mu & \lambda - \operatorname{Re}\mu \end{pmatrix}, \qquad \mathbf{u}_\gamma = \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda \end{pmatrix},$$
$$\mathbf{\Gamma}_\gamma = \frac{1}{1 - |\gamma|^2}\begin{pmatrix} \lambda^2 + |\mu|^2 & 2\lambda\operatorname{Re}\mu & -2\lambda\operatorname{Im}\mu \\ 2\lambda\operatorname{Re}\mu & \lambda^2 + \operatorname{Re}\mu^2 & -\operatorname{Im}\mu^2 \\ -2\lambda\operatorname{Im}\mu & -\operatorname{Im}\mu^2 & \lambda^2 - \operatorname{Re}\mu^2 \end{pmatrix}, \qquad \overline{u}_\gamma\{0, \lambda, \mu\}. \tag{14f}$$

A.4 The general case of rank N

For rank $N = 1$, we saw, *Sp(2N,ℜ)* is homomorphic to the lowest-dimensional counterparts of two other Cartan-classified families, to a total of four groups. For $N \ge 2$, the only accidental homomorphism occurs for *Sp(4,ℜ)* $\simeq$ *SO(3,2)*, and is 2:1. We will now give Bargmann's treatment [2] of the covering of the *general* symplectic group *Sp(2N,ℜ)*.

A.4.1 *Sp(2N,ℜ)*

The group *Sp(2N,ℜ)* is defined as the set of real $2N \times 2N$ matrices $\mathbf{g}$ obeying the $2N$-dimensional version of (2):

$$\mathbf{g}\mathbf{M}_{Sp(2N)}\mathbf{g}^\top = \mathbf{M}_{Sp(2N)}, \qquad \mathbf{M}_{Sp(2N)}^\top = -\mathbf{M}_{Sp(2N)}, \qquad \det\mathbf{M}_{Sp(2N)} \ne 0. \tag{15a}$$

When we write the matrices involved in terms of $N \times N$ submatrices, we may choose

$$\mathbf{g} := \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \qquad \mathbf{M}_{Sp(2N)} := \begin{pmatrix} \mathbf{0} & +\mathbf{1}_N \\ -\mathbf{1}_N & \mathbf{0} \end{pmatrix}, \tag{15b}$$

where $\mathbf{1}_N$ is the N-dimensional unit matrix. This leads to the following relations between the $N \times N$ submatrices:

$$\mathbf{A}\mathbf{B}^\top = \mathbf{B}\mathbf{A}^\top, \quad \mathbf{A}\mathbf{C}^\top = \mathbf{C}\mathbf{A}^\top, \quad \mathbf{B}\mathbf{D}^\top = \mathbf{D}\mathbf{B}^\top, \quad \mathbf{C}\mathbf{D}^\top = \mathbf{D}\mathbf{C}^\top, \tag{15c}$$

$$\mathbf{A}\mathbf{D}^\top - \mathbf{B}\mathbf{C}^\top = \mathbf{1}_N. \tag{15d}$$

The inverse of a symplectic matrix may thus be written as

$$\mathbf{g}^{-1} = \mathbf{M}_{Sp(2N)}\mathbf{g}^\top \mathbf{M}_{Sp(2N)} = \begin{pmatrix} \mathbf{D}^\top & -\mathbf{B}^\top \\ -\mathbf{C}^\top & \mathbf{A}^\top \end{pmatrix}. \tag{16}$$

The number of independent parameters of $Sp(2N,\Re)$ is $2N^2 + N$.

A.4.2 The Bargmann form for $Sp(2N,\Re)$

In order to explore the connectivity properties of the $Sp(2N,\Re)$ manifold and parametrize its covering group, we shall present a generalization of the $Sp(2,\Re) = SU(1,1)$ isomorphism. Although clearly $Sp(2N,\Re)$ is not *isomorphic* to any pseudo-unitary group, its *inclusion* in $U(N,N)$ will display the connectivity properties through its unitary $U(N)$ maximal compact subgroup, generalizing the role of $U(1) = SO(2)$ in $Sp(2,\Re)$.

We construct first the $2N \times 2N$ matrix $\mathbf{W}_N = \mathbf{W} \otimes \mathbf{1}_N$, where $\mathbf{W} = \mathbf{W}_1$ is the 2×2 matrix (5) which gives the $N \times N$ block coefficients. Taking now $\mathbf{g}$ from (15), we write

$$\begin{aligned} \mathbf{u}(\mathbf{g}) &:= \mathbf{W}_N^{-1}\mathbf{g}\mathbf{W}_N \\ &= \frac{1}{2}\begin{pmatrix} [\mathbf{A}+\mathbf{D}] - i[\mathbf{B}-\mathbf{C}] & [\mathbf{A}-\mathbf{D}] + i[\mathbf{B}+\mathbf{C}] \\ [\mathbf{A}-\mathbf{D}] - i[\mathbf{B}+\mathbf{C}] & [\mathbf{A}+\mathbf{D}] + i[\mathbf{B}-\mathbf{C}] \end{pmatrix} =: \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\beta}^* & \boldsymbol{\alpha}^* \end{pmatrix}. \end{aligned} \tag{17}$$

The symplecticity property of $\mathbf{g}$ becomes thus

$$\begin{aligned} \mathbf{u}\mathbf{M}_{E(N,N)}\mathbf{u}^\dagger &= \mathbf{M}_{E(N,N)}, \\ \mathbf{M}_{E(N,N)} &:= i\mathbf{W}_N^{-1}\mathbf{M}_{Sp(2N)}\mathbf{W}_N = \begin{pmatrix} +\mathbf{1}_N & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_N \end{pmatrix}. \end{aligned} \tag{18a}$$

This condition alone would define $\mathbf{u}$ as a pseudo-unitary $U(N,N)$ matrix, but the restriction (17) stemming from the reality of $\mathbf{g}$, makes $\boldsymbol{\alpha}^*$ the complex conjugate of $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}^*$ that of $\boldsymbol{\beta}$, restricting $\mathbf{u}$ to $Sp(2N,\Re) \subset U(N,N)$.

The $N \times N$ submatrices of the Bargmann-form $Sp(2N,\Re)$ matrices obey

$$\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger - \boldsymbol{\beta}\boldsymbol{\beta}^\dagger = \mathbf{1}, \qquad \boldsymbol{\alpha}^\dagger\boldsymbol{\alpha} - \boldsymbol{\beta}^\top\boldsymbol{\beta}^* = \mathbf{1}, \tag{18b}$$

$$\boldsymbol{\alpha}\boldsymbol{\beta}^\top - \boldsymbol{\beta}\boldsymbol{\alpha}^\top = \mathbf{0}, \qquad \boldsymbol{\alpha}^\top\boldsymbol{\beta}^* - \boldsymbol{\beta}^\dagger\boldsymbol{\alpha} = \mathbf{0}. \tag{18c}$$

Since $\mathbf{1} + \boldsymbol{\beta}\boldsymbol{\beta}^\dagger$ is a positive definite matrix, $\boldsymbol{\alpha}$ has an inverse. From the last equations, $\boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$ and $\boldsymbol{\beta}^*\boldsymbol{\alpha}^{-1}$ are shown to be symmetric. The inverse follows:

$$\mathbf{u}^{-1} = \mathbf{M}_{E(N,N)}\mathbf{u}^\dagger\mathbf{M}_{E(N,N)}^{-1} = \begin{pmatrix} \boldsymbol{\alpha}^\dagger & -\boldsymbol{\beta}^\top \\ -\boldsymbol{\beta}^\dagger & \boldsymbol{\alpha}^\top \end{pmatrix}. \tag{19}$$

Finally, corresponding to (6a), the mapping inverse to (17) is

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \mathbf{g}(\mathbf{u}) = \mathbf{W}_N\mathbf{u}\mathbf{W}_N^{-1} = \begin{pmatrix} \mathrm{Re}\,(\boldsymbol{\alpha}+\boldsymbol{\beta}) & -\mathrm{Im}\,(\boldsymbol{\alpha}-\boldsymbol{\beta}) \\ \mathrm{Im}\,(\boldsymbol{\alpha}+\boldsymbol{\beta}) & \mathrm{Re}\,(\boldsymbol{\alpha}-\boldsymbol{\beta}) \end{pmatrix}. \tag{20}$$

A.4.3 The subgroup $U(N) \subset Sp(2N,\Re)$

The maximal compact subgroup of $Sp(2N,\Re)$ is $U(N)$. This fact may be seen knowing that the maximal compact subgroup of $GL(2N,\mathbb{C})$ —the group of complex $2N \times 2N$ matrices— is $U(2N)$; this is the weakest restriction which puts an upper bound to the norm of this row and column vectors.[4] The intersection of $U(2N)$ with the Bargmann form of $Sp(2N,\Re)$ is the set of matrices satisfying both $\mathbf{u}_0\mathbf{u}_0^\dagger = \mathbf{1}$ and (18), which have therefore vanishing off-diagonal blocks and conjugate diagonal ones, *i.e.*,

$$\mathbf{u}_0 = \begin{pmatrix} \boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}^* \end{pmatrix}, \qquad \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger = \mathbf{1}. \tag{21a}$$

The set of $\mathbf{u}_0$'s thus constitutes a $U(N)$ group. In the real form (15) of $Sp(2N,\Re)$, this N^2-parameter subgroup is the set of matrices

$$\mathbf{g}_0 = \begin{pmatrix} \operatorname{Re}\boldsymbol{\alpha} & -\operatorname{Im}\boldsymbol{\alpha} \\ \operatorname{Im}\boldsymbol{\alpha} & \operatorname{Re}\boldsymbol{\alpha} \end{pmatrix} = \mathbf{g}_0^{\top -1}, \qquad \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger = \mathbf{1}. \tag{21b}$$

All these matrices are *orthogonal* $2N \times 2N$ matrices, but not the most general ones, since the group $O(2N)$ has $2N^2 - N$ parameters.

A.4.4 The $Sp(2N,\Re)$ manifold

A well known theorem in matrix theory states that any real matrix $\mathbf{R}$ may be decomposed into the product of an orthogonal $\mathbf{Q}$ and a symmetric positive definite matrix $\mathbf{S}$, uniquely, as $\mathbf{R} = \mathbf{QS}$. Additionally, Bargmann [2, §2.3] shows that if $\mathbf{R} \in Sp(2N,\Re)$, then also $\mathbf{Q}$ and $\mathbf{S}$ belong to this group. Through $\mathbf{W}_N$ [Eq. (17)] the matrices $\mathbf{Q}$ and $\mathbf{S}$ map onto unitary and hermitian positive definite ones. Restriction to the Bargmann form of $Sp(2N,\Re)$ in (17) details that $\mathbf{u}(\mathbf{Q})$ is given by $\boldsymbol{\alpha} \in U(N)$ and $\boldsymbol{\beta} = \mathbf{0}$ [*i.e.*, as in (21*a*), rather than simply a phase as in (14*a*)], and $\mathbf{u}(\mathbf{S})$ with $\boldsymbol{\alpha} = \boldsymbol{\alpha}^\dagger$ and $\boldsymbol{\beta} = \boldsymbol{\beta}^\top$, obeying (18). The former set of matrices is an N^2-dimensional real manifold with the topology of $U(N)$, while the real dimension of the latter is $N^2 + N$ with the euclidean topology of $\Re^{N^2+N}$. This last fact may be seen either through counting N^2 parameters for hermitian and $N^2 + N$ for symmetric complex matrices, minus N^2 conditions from the two independent equations in (18); or, succintly [2],

$$\mathbf{u}(\mathbf{Q}) = \begin{pmatrix} \boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}^* \end{pmatrix}, \qquad \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger = \mathbf{1}, \tag{22a}$$

$$\mathbf{u}(\mathbf{S}) = \exp\begin{pmatrix} \mathbf{0} & \boldsymbol{\xi} \\ \boldsymbol{\xi}^* & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\xi} = \boldsymbol{\xi}^\top. \tag{22b}$$

Since $\boldsymbol{\alpha} \in U(N) \Rightarrow |\det \boldsymbol{\alpha}| = 1$, the group $U(N)$ is the direct product of the compact group of *unimodular* unitary matrices $SU(N)$, times the $U(1)$ group of determinant phases $e^{i\theta}$, $\theta \in \mathfrak{S}_1$ (the circle). Topologically, thus,

$$Sp(2N,\Re) \sim U(1) \times SU(N) \times \Re^{N^2+N}. \tag{23}$$

Since both $SU(N)$ and $\Re^{N^2+N}$ are simply connected, the connectivity of $Sp(2N,\Re)$ is that of $U(1) \sim \mathfrak{S}_1$, *i.e.*, connected and infinitely connected. This is the generalization of the $Sp(2,\Re)$ case presented in **A.1.2**; there, the $SU(1) = \{1\}$ factor was absent.

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[4]The fact that $U(N)$ is the maximal compact subgroup of $Sp(2N,\Re)$ is quite clear, otherwise, if we recall that we realize the latter as generated by quadratic monomials in the N-dimensional oscillator raising and lowering operators; the symmetry group of the system is generated by the N^2 mixed products

A.4.5 The Bargmann parameters for *Sp(2N,ℜ)*

We shall now generalize the Bargmann parametrization (10) of *SU(1,1)* to *Sp(2N,ℜ)* in its *pseudo-unitary* form (17); then, through covering $\mathfrak{S}_1$ by ℜ, we shall parametrize the universal covering group $\overline{Sp(2N,R)}$ of *Sp(2N,R)*. We write **u** in (17) as

$$\mathbf{u}\{\omega,\boldsymbol{\lambda},\boldsymbol{\mu}\} = \begin{pmatrix} e^{i\omega}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & e^{-i\omega}\mathbf{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} & \boldsymbol{\mu} \\ \boldsymbol{\mu}^* & \boldsymbol{\lambda}^* \end{pmatrix}, \qquad \omega \in \mathfrak{S}_1, \quad \det\boldsymbol{\lambda} > 0, \tag{24a}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy Eqs. (18*b*) and (18*c*) with $\boldsymbol{\alpha} \mapsto \boldsymbol{\lambda}$, $\boldsymbol{\beta} \mapsto \boldsymbol{\mu}$. These are the Bargmann parameters for *Sp(2N,ℜ)*. The crux of the matter is to separate the *U(1)* factor in (23) into a single phase parameter $\omega \in \mathfrak{S}_1$, so that (10*b*) is generalized to

$$\omega = \frac{1}{N}\arg\det\boldsymbol{\alpha}, \qquad e^{iN\omega} = \frac{\det\boldsymbol{\alpha}}{|\det\boldsymbol{\alpha}|}, \tag{24b}$$

$$\boldsymbol{\lambda} = e^{-i\omega}\boldsymbol{\alpha}, \qquad \det\boldsymbol{\lambda} = |\det\boldsymbol{\alpha}| > 0, \tag{24c}$$

$$\boldsymbol{\mu} = e^{-i\omega}\boldsymbol{\beta} \tag{24d}$$

Here, *unlike* the $(N = 1)$-dimensional case, $\boldsymbol{\lambda}$ is *not* a redundant parameter.

The product for the *Sp(2N,ℜ)* Bargmann parameters, $\mathbf{u}\{\omega,\boldsymbol{\lambda},\boldsymbol{\mu}\} = \mathbf{u}\{\omega_1,\boldsymbol{\lambda}_1,\boldsymbol{\mu}_1\}\,\mathbf{u}\{\omega_2,\boldsymbol{\lambda}_2,\boldsymbol{\mu}_2\}$, is obtained straightforwardly and yields

$$\begin{aligned}\omega &= \frac{1}{N}\ \arg\det[e^{i(\omega_1+\omega_2)}\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2 + e^{i(\omega_1-\omega_2)}\boldsymbol{\mu}_1\boldsymbol{\mu}_2^*] \\ &= \frac{1}{N}\ \arg[e^{iN(\omega_1+\omega_2)}\det\boldsymbol{\lambda}_1\ \det\boldsymbol{\nu}\ \det\boldsymbol{\lambda}_2] \\ &= \omega_1+\omega_2+\omega_\nu,\end{aligned} \tag{25a}$$

$$\boldsymbol{\lambda} = e^{-i\omega_\nu}\boldsymbol{\lambda}_1\,\boldsymbol{\nu}\,\boldsymbol{\lambda}_2, \tag{25b}$$

$$\boldsymbol{\mu} = e^{-i\omega_\nu}(\boldsymbol{\lambda}_1\boldsymbol{\mu}_2 - e^{-2i\omega_2}\boldsymbol{\mu}_1\boldsymbol{\lambda}_2^*), \tag{25c}$$

where the role of ν in (12*d*) is taken by the nonsingular matrix $\boldsymbol{\nu}$:

$$\boldsymbol{\nu} := \mathbf{1} + e^{-2i\omega_2}\boldsymbol{\lambda}_1^{-1}\,\boldsymbol{\mu}_1\,\boldsymbol{\mu}_2^*\,\boldsymbol{\lambda}_2^{-1}, \tag{26a}$$

$$\omega_\nu := \frac{1}{N}\ \arg\det\boldsymbol{\nu}. \tag{26b}$$

The nonsingularity of the matrix $\boldsymbol{\nu}$, necessary for a proper definition of the argument ω_ν, can be proven through noting that the operator norms $[\mathbf{v}^\dagger\mathbf{A}^\dagger\mathbf{A}\mathbf{v} \le |\mathbf{A}|^2\mathbf{v}^\dagger\mathbf{v}$ for an arbitrary vector $\mathbf{v}]$ of the symmetric matrices $\boldsymbol{\lambda}_i^{-1}\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_i^*\boldsymbol{\lambda}_i^{-1}$ are bounded by $0 \le 1 - |\boldsymbol{\lambda}_i|^2 < 1$, $i = 1, 2$. Consequently, $|\boldsymbol{\nu} - \mathbf{1}| < 1$.

The *Sp(2N,ℜ)* matrices (15), written through (20) in terms of the Bargmann parameters, read

$$\mathbf{g}\{\omega,\boldsymbol{\lambda},\boldsymbol{\mu}\} = \begin{pmatrix} \cos\omega\mathbf{1} & -\sin\omega\mathbf{1} \\ \sin\omega\mathbf{1} & \cos\omega\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,(\boldsymbol{\lambda}+\boldsymbol{\mu}) & -\mathrm{Im}\,(\boldsymbol{\lambda}-\boldsymbol{\mu}) \\ \mathrm{Im}\,(\boldsymbol{\lambda}+\boldsymbol{\mu}) & \mathrm{Re}\,(\boldsymbol{\lambda}-\boldsymbol{\mu}) \end{pmatrix}. \tag{27}$$

This generalizes (11).

A.4.6 $\overline{Sp(2N,\Re)}$ and *Mp(2N,ℜ)*

For the matrix realizations of *Sp(2N,ℜ)* in (24) and (27), only $\omega \equiv \omega(\mathrm{mod}\ 2\pi)$ makes sense. As in the $(N = 1)$-dimensional case, however, the composition law (25), taken form $\omega \in \Re$, suffices to define the universal covering group $\overline{Sp(2N,\Re)}$ of $Sp(2N,\Re)$. We shall denote its elements by $\overline{u}\{\omega,\boldsymbol{\lambda},\boldsymbol{\mu}\}$. The group unit is given by $\overline{u}\{0,\mathbf{1},\mathbf{0}\}$, and $\overline{u}\{\omega,\boldsymbol{\lambda},\boldsymbol{\mu}\}^{-1}=\overline{u}\{-\omega,\boldsymbol{\lambda}^{\dagger},-e^{2i\omega}\boldsymbol{\mu}^{\top}\}$. The center of $\overline{Sp(2N,\Re)}$ is the set $\overline{u}\{n\pi,\mathbf{1},\mathbf{0}\}$, $n\in\mathfrak{Z}$, and the symplectic group is given by $Sp(2N,\Re)=\overline{Sp(2N,\Re)}/\{\overline{u}\{2n\pi,\mathbf{1},\mathbf{0}\},\ n\in\mathfrak{Z}\}$.

The N-dimensional *metaplectic* group, defined by

$$Mp(2N,\Re):=\overline{Sp(2N,\Re)}/\{\overline{u}\{4\pi n,\mathbf{1},\mathbf{0}\},\ n\in\mathfrak{Z}\}, \tag{28}$$

is the two-fold cover of $Sp(2N,\Re)$, and its elements may be denoted $\overline{g}(\omega,\boldsymbol{\lambda},\boldsymbol{\mu})$, with $\omega\equiv\omega(\text{mod }4\pi)$. Again, we have no representation through finite matrices.

References

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