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The map between Heisenberg-Weyl and Euclidean optics is comatic

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ABSTRACT The mathematics of coherent states is essentially a translation of oscillator quantum mechanics to the paraxial model of optics, and is based on the Heisenberg–Weyl algebra and group. On the other hand, "4π" optics is based on the three-dimensional Euclidean algebra and corresponding group. We show here that a global map between the two may be established. It is, in fact, third-order Seidel-Lie coma. Spherical and circular-comatic aberrations are a proper subgroup of the group of all canonical transformations of phase space, that can be subject to unique quantization and wavization.

7.1 Introduction

Our previous joint work [1] addressed the question of the behaviour of Gaussian beams, including Glauber coherent beams [2] and correlated coherent [4], [5] states, under spherical aberration. This is the only aberration present in free flight, and is increasingly important at large angles from the chosen optical axis [6].

The mathematical foundation of coherent state theory is the *Heisenberg-Weyl* Lie algebra of the quantum mechanical operators of position and momentum [7], [8], with the harmonic oscillator physical model of optical fibers of parabolic index profile [9]. This is a structure rich in results that has been profitably applied to laser optics, among many other fields. Recent work on *Euclidean* optics, *i.e.*, "4π" geometric and wave optics [10] motivated our enquiry into their exact relation. Physical three-dimensional optics requires $N = 2$ dimensional screens. We shall work in generic dimension N because, once we go beyond the simple $N = 1$ case, the mathematical formulation allows easily such generalization.

It is generally taken for granted that Heisenberg-Weyl optics is the *paraxial limit*, meaning small angles and distances from the optical axis, of 4π optics. We believe that Heisenberg-Weyl and Euclidean optics should be described as *separate* mathematical structures. We show that there exists a *semi*-global 1:1 map between the two, that assigns to each point in a $\Re^{2N}$

Heisenberg-Weyl phase space one point in the Euclidean phase space of $2N$ dimensions where the optical momentum values are bounded and rays are in the *forward* hemisphere, *i.e.*, advance in the direction of the optical axis. Another $\Re^{2N}$ space maps on the *backward* hemisphere, and continuity conditions are asked to hold between the former two 'flat' spaces.

The map between the Heisenberg-Weyl and Euclidean optics is a *point* Escher-like transformation between the N-dimensional momentum space of the former and the interior of the N-dimensional momentum sphere of the latter. In the $N = 2$ case of ordinary optics, the latter sphere interior is a *disk* of radius n, the refractive index of the medium. The canonically conjugate observables of ray *position* are then related by a map that is in general *circular comatic*, and in fact *precisely* third-order Lie-Seidel coma. This intertwining of Heisenberg-Weyl and Euclidean optics allows the linear (and nonlinear) symplectic transformations of the former to be applied to the latter. And conversely, rotations [11] and relativistic boosts [12], that act naturally on the sphere [10], are thereby applied on the $\Re^N$ space of Heisenberg-Weyl optics. The map is near to unity in the paraxial region, *i.e.*, small angles and distances from the optical axis, but extends analytically to all angles and distances. This we can do for both geometric and wave optics.

Section 2 succintly derives the Hamilton equations of motion for optics directly from Snell's law in local form. The proof of this was presented in Ref. [13] by a didactical route that is considerably shorter than that of Fermat's principle through the Lagrangian formalism [6]. The present approach is adapted from Ref. [14]. The paraxial approximation is treated in Section 3, and is extended to the full Heisenberg-Weyl phase space. We compare the (straight) trajectories of rays freely propagating in this and in Euclidean optics and introduce, in Section 4, the *opening coma* transformation between them. Section 5 then shows that in this way, indeed, we intertwine the two different régimes of propagation in homogeneous optical media.

Heisenberg-Weyl optics has a distinguished transformation group, that is *larger* than the Heisenberg-Weyl group itself [7]: *linear* symplectic transformations. Euclidean optics, on the other hand, accomodates naturally Euclidean and Lorentz transformations. In Sections 6 and 7 we apply the former group to the latter phase space, and *viceversa*. Such cross-applications constitute apparently novel nonlinear realizations of these groups. Section 8 is set towards structuring the results for general spherical plus comatic aberration maps of phase space.

The opening coma map in *wave* optics is seen from Section 9 on. From the $\mathcal{L}_2(\Re^N)$ inner product, through the map, we arrive at the Hilbert space $\mathcal{H}_k^N$ of oscillatory solutions of the Helmholtz equation with an inner product that is nonlocal. In Sections 10 and 11 we follow through the coma map the plane waves, δ's, Gaussians (including coherent and correlated coherent states), and Bessel-function nondiffracting beams. They are analogous but distinct in the paraxial (Heisenberg-Weyl) and in the global (Euclidean) regimes. Section 11 ends with the necessary context to present some open directions into fields where the formalism of Heisenberg and Weyl has been successful and where it could be extended to all ray directions.

7.2 From Snell's law to the Hamilton equations

We consider a homogeneous optical medium (generally, of $N+1$ dimensions), characterized by a refractive index n, separated by a surface σ from a second such medium of different index n'. We assume σ has a tangent plane $T_\sigma(S)$ at each point $S \in \sigma$, characterized by its normal $(N+1)$-dimensional vector $\vec{\Sigma}(S)$. Similarly, we denote by $\vec{n}$ and $\vec{n}\,'$ the ray direction vectors that range over spheres of radii given by $|\vec{n}\,| = n$ and $|\vec{n}\,'| = n'$, the two refractive indices. These are the *Descartes* spheres of rays in the two media. The 'rays' may be one-dimensional straight lines in space, or may be N dimensional *planes* with normals $\vec{n}$ and $\vec{n}\,'$ [10].

Snell's law is the statement that the change in the direction vector $\vec{n}-\vec{n}\,'$ be *along* the refracting surface normal $\vec{\Sigma}$ *at* the incidence point of the ray. When $n = n'$, then also $\vec{n} = \vec{n}\,'$.

In ordinary optics ($N+1=3$), the law may be cast in the cross-product form $\vec{\Sigma}\times\vec{n} = \vec{\Sigma}\times\vec{n}\,'$. The equality of the norm of this relation implies the familiar sine law $n\sin\theta = n'\sin\theta'$, with θ being the angle between $\vec{\Sigma}$ and $\vec{n}$, and similarly for the primed. This relation also tells us that the three vectors are in a plane. Equally important, but generally left unsaid, is that the incidence point S of the ray in medium n is the *same* as its departure point S in medium n'.

Snell's law in $N+1$ dimensions implies that the projection of $\vec{n}$ on the tangent plane $T_\sigma(S)$ is *conserved*. This is the N-dimensional vector of optical *momentum* referred to the tangent plane of the refracting surface. The N coordinates of the incidence point S on the screen are also conserved.

Let us choose a standard Cartesian coordinate system in space where points are $(N+1)$-vectors $\vec{q}$, where rays are *geometrical* lines $\vec{q}(z) = (q_1(z), q_2(z), \ldots, q_N(z), z) = (\mathbf{q}(z), z)$, $z \in \Re$ being the $(N+1)^{\text{th}}$ coordinate taken as the *optical axis*, indicating by **boldface** the first N components $\mathbf{q}(z)$, the *position* of the ray at the *standard screen:* the $z=0$ plane . This parametrization fails only for rays paralell to the screen; it is not a singularity of the space of rays but only indicates the limit of this particular coordinate chart.

The ray direction $\vec{n}$ is the $(N+1)$-dimensional vector tangent to $\vec{q}(z)$, and will depend on z when the medium is inhomogeneous through $n(\vec{q})$. The Cartesian components of $\vec{n}$ are not all independent since this vector lies on the surface of the Descartes sphere of radius n. It is convenient to single out again the $(N+1)^{\text{th}}$ component and write $\vec{n} = (p_1, p_2, \ldots p_N, h) = (\mathbf{p}, h)$, where

$$h = \tau\sqrt{n^2 - p^2}, \qquad p^2 = \mathbf{p}\cdot\mathbf{p}, \quad \tau \in \{+, 0, -\} \ (\tau = 0 \text{ when } h = 0). \tag{2.1}$$

The vector $\vec{n} = (\mathbf{p}, h)$, tangent to $\vec{q}(z)$, is thus paralell to $d\vec{q}(z) = (d\mathbf{q}, dz)$. Hence, the following proportion holds between each of their first N components,

$$\frac{d\mathbf{q}}{dz} = \frac{\mathbf{p}}{h} = -\frac{\partial h}{\partial \mathbf{p}}. \tag{2.2}$$

The second equality is a consequence of the specific function form of h on $\mathbf{p}$ given in equation (2.1).[1] The equality between the first and last members in (2.2) is the *first* Hamilton equation. The origin of this equation is thus purely geometrical. If the medium is inhomogeneous through $n(\vec{q})$, infinitesimal changes in ray direction contribute only to second differentials in $\mathbf{q}$, so they do not appear in the first-order Hamilton equation.[2]

The *second* Hamilton equations are *dynamical* and speak of the inhomogeneity of the medium $n(\mathbf{q}, z)$ through the obeyance of $\vec{n}(z)$ to Snell's law. Let the gradient of the refractive index $\vec{\nabla} n = (\partial n/\partial \mathbf{q}, \partial n/\partial z)$ take the role of the surface normal $\vec{\Sigma}$ above. This vector will be parallel to the *change* of the direction vector, $d\vec{n}/dz = (d\mathbf{p}/dz, dh/dz)$. Let the ratio be a certain $\alpha(\mathbf{p}, \mathbf{q}, z)$ that we shall find through the constraint that the direction vector remain on its Descartes sphere, $\vec{n}\cdot\vec{n} = n^2 = p^2 + h^2$.

Thus, on the one hand,

$$\frac{d}{dz} n^2 = \frac{d}{dz}\vec{n}\cdot\vec{n} = 2\vec{n}\cdot\frac{d\vec{n}}{dz} = 2\alpha\vec{n}\cdot\vec{\nabla} n, \tag{2.3a}$$

and on the other, using (2.2) and the chain rule,

$$\begin{aligned} \frac{d}{dz} n^2 &= 2n\left(\frac{d\mathbf{q}}{dz}\cdot\frac{\partial n}{\partial \mathbf{q}} + \frac{\partial n}{\partial z}\right) = 2n\left(\frac{\mathbf{p}}{h}\cdot\frac{\partial n}{\partial \mathbf{q}} + \frac{\partial n}{\partial z}\right) \\ &= 2\frac{n}{h}\left(\mathbf{p}\cdot\frac{\partial n}{\partial \mathbf{q}} + h\frac{\partial n}{\partial z}\right) = 2\frac{n}{h}\vec{n}\cdot\vec{\nabla} n. \end{aligned} \tag{2.3b}$$

From the equality of the last members we conclude that $\alpha = n/h$. Consequently,

$$\frac{d\vec{n}}{dz} = \frac{n}{h}\vec{\nabla} n. \tag{2.4}$$

Using again the function form of h, we write in components

$$\frac{d\mathbf{p}}{dz} = \frac{n}{h}\frac{\partial n}{\partial \mathbf{q}} = \frac{\partial h}{\partial \mathbf{q}}, \tag{2.5a}$$

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[1] We use the vector derivative notation $\partial/\partial\mathbf{p} = (\partial/\partial p_1, \partial/\partial p_2, \ldots, \partial/\partial p_N)$.

[2] Second differentials help to build the *ray* differential equation [15], that is of second order.

$$\frac{dh}{dz} = \frac{n}{h}\frac{\partial n}{\partial z} = \frac{\partial h}{\partial z}. \tag{2.5b}$$

The first and last members of these equalities constitute de *second* Hamilton equations of motion.

In the case of homogeneous media, $n =$ constant, the equations of motion (2.5) and (2.2) become, respectively,

$$\frac{d\mathbf{p}}{dz} = 0, \qquad \frac{d\mathbf{q}}{dz} = \frac{\mathbf{p}}{h} = \frac{\mathbf{p}}{\tau\sqrt{n^2 - p^2}}. \tag{2.6a, b}$$

The solution to these equations in terms of initial ($z = 0$) *screen* values $\mathbf{p}_0$ and $\mathbf{q}_0$, is the ray path

$$\mathbf{p}(z) = \mathbf{p}_0, \qquad \mathbf{q}(z) = \mathbf{q}_0 + z\frac{\mathbf{p}_0}{\tau\sqrt{n^2 - p_0^2}}. \tag{2.7a, b}$$

This is a z-dependent *canonical* transformation [16] that conserves the Poisson brackets

$$\{q_i(z), p_k(z)\} = \delta_{i,k}, \qquad i, k = 1, 2, \ldots, N. \tag{2.8a}$$

$$\{q_i(z), q_k(z)\} = \{p_i(z), p_k(z)\} = 0. \tag{2.8b}$$

The validity of the Hamilton equations (2.2) and (2.5) establishes $\mathbf{q}$ and $\mathbf{p}$ as *canonically conjugate* quantities of optical position and momentum. Position $\mathbf{q} \in \Re^N$ is the manifold of the standard screen plane; optical momentum $\mathbf{p}$ ranges over the interior of the N-sphere $|\mathbf{p}| < n$ (a *disk* in $N = 2$, 3-dimensional ordinary optics) once for forward rays ($h > 0$, $\tau = +$), and once for backward rays ($h < 0$, $\tau = -$). The two charts are separated by rays in the equator of the Descartes sphere ($h = 0$, $\tau = 0$), where we expect continuity conditions to hold. We shall speak for the most part of *forward* rays $\tau = +$.

Evolution along the z-axis of Euclidean geometrical optics is thus ruled by the Hamiltonian function

$$\begin{aligned} H^E &= -h = -\sqrt{n^2 - p^2} \\ &= -n + \frac{p^2}{2n} + \frac{(p^2)^2}{8n^3} + \frac{(p^2)^3}{16n^5} + \cdots, \end{aligned} \tag{2.9}$$

that is (minus) the $(N+1)^{\text{th}}$-component of the ray direction vector. The series expansion will be now subject to scrutiny.

7.3 The paraxial régime and Heisenberg–Weyl optics

We consider now the *paraxial* optical régime, *i.e.*, the approximation that is valid when the angle θ between rays $\vec{n}$ in a light beam and the chosen optical z-axis is small. In (hyper) spherical coordinates, this angle is given by the rays' momentum $\mathbf{p}$ through

$$|\mathbf{p}| = n \sin\theta. \tag{3.1a}$$

The paraxial approximation thus entails selecting a region of optical phase space restricted by

$$|\theta| \ll \pi, \quad i.e., \quad |\mathbf{p}| \ll n. \tag{3.1b}$$

For such beams the Hamiltonian (2.9), expanded in power series with respect to p^2/n^2, is approximated by the *paraxial* form

$$H^E \approx \frac{p^2}{2n} - n. \tag{3.2}$$

If the refractive index n is a constant, this is the Hamiltonian for mechanical *free motion* in N-dimensional space. The corresponding phase space is $2N$-dimensional, and its basic group of motions is the Heisenberg-Weyl group. Let us consider now *this* Hamiltonian and *its* free motion dynamics. We may omit the constant term $-n$ in (3.2) since its Lie operator is zero, and describe the free motion system by the Heisenberg-Weyl Hamiltonian

$$H^{HW} = \frac{P^2}{2n}, \qquad P^2 = \mathbf{P}\cdot\mathbf{P}. \tag{3.3}$$

Throughout this paper, we shall use capital letters $\mathbf{P}$ and $\mathbf{Q}$ to indicate that the observables belong to the Heisenberg-Weyl phase space while lower-case $\mathbf{p}$ and $\mathbf{q}$ will remain for Euclidean optical variables.

The Hamilton equations of motion in phase space (2.2)–(2.5) for the free point particle are

$$\frac{d\mathbf{P}}{dz} = -\frac{\partial H^{HW}}{\partial \mathbf{Q}} = 0, \qquad \frac{d\mathbf{Q}}{dz} = \frac{\partial H^{HW}}{\partial \mathbf{P}} = \frac{\mathbf{P}}{n}. \tag{3.4a, b}$$

The solution to these equations is

$$\mathbf{P}(z) = \mathbf{P}_0, \qquad \mathbf{Q}(z) = \mathbf{Q}_0 + z\mathbf{P}_0/n, \tag{3.5a, b}$$

where $\mathbf{P}_0$ and $\mathbf{Q}_0$ are the initial momenta and positions in phase space. Compare this with equations (2.6) and solutions (2.7) in Euclidean optics, which are also straight lines in space. These relations are here *linear*, and may be written in matrix form through $2N \times 2N$ *symplectic* matrices with

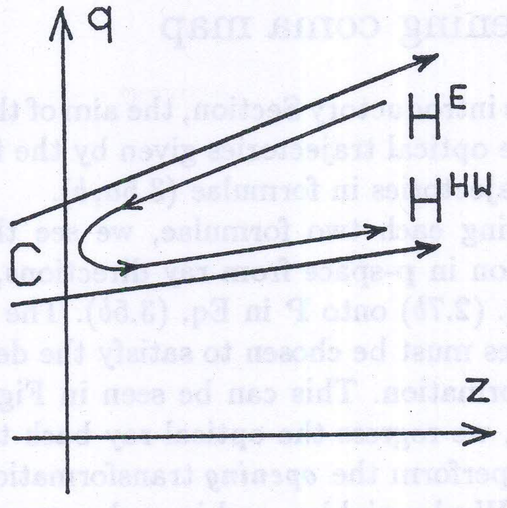

FIGURE 1. The map from Euclidean to Heisenberg-Weyl propagation: The ray coordinates are regressed back to the screen $z = 0$; next, the *opening* of Euclidean to Heisenberg-Weyl ranges; lastly, we evolve the ray from the screen to general z in the Heisenberg-Weyl regime, *i.e.*, paraxial optics.

$N \times N$ blocks that are proportional to the identity matrix. Such transformations are also canonical since they preserve the Poisson bracket as in (2.8).

What are the phase space *bounds* of this paraxial construction? In principle none that may be naturally incorporated in the Heisenberg-Weyl algebra with basic Poisson brackets (2.8) or the corresponding Lie *groups*.[3] Moreover, quantum mechanics (in the standard theory) *demands* that observables have self-adjoint operators to their name, and the familiar Schrödinger or coherent-state generators of the Heisenberg-Weyl algebra need the full real line, both for $\mathbf{Q}$ and $\mathbf{P}$. Finally, the possibility of using the linear symplectic group for paraxial optics also hinges on the assumption that the range of momentum $\mathbf{P}$ be the full plane $\Re^N$. In fact, we *know* that nonrelativistic mechanics is a consistent mathematical theory of much practical use in paraxial optics, and the basis for aberration expansions into the metaxial regime. But we stress that it is a theory globally *different* from the 4π Euclidean optics presented in last Section. We shall now proceed to find the map between the systems obeying H^E and H^{HW}.

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[3]In reference [17] we examined a 1-dimensional Heisenberg-Weyl group where the momentum parameter is cyclic —direction lies on a circle in two-dimensional optics. This entailed that the conjugate position parameter be *discrete* (*cf.* the sampling theorem) denying thus the possiblity of having an infinitesimal translation generator. The central subgroup is also forced to be cyclic —as it *should*, being a phase. The N-dimensional version of that construction yields the direction vector as ranging over an N-torus, however, instead of an N-sphere.

7.4 The opening coma map

As we stated in the introductory Section, the aim of this work is to construct the map of the free optical trajectories given by the formulae (2.7a, b) onto the free motion trajectories in formulae (3.5a, b).

In fact, comparing each two formulae, we see that we must find the point transformation in **p**-space from ray directions, that maps the factor $\mathbf{p}/\sqrt{n^2 - p^2}$ in Eq. (2.7b) onto **P** in Eq. (3.5b). The canonically conjugate position coordinates must be chosen to satisfy the demand of *canonicity* of the desired transformation. This can be seen in Figure 1, and consists of three factors: first, we regress the optical ray back to the standard screen $z = 0$; second, we perform the *opening* transformation from the Euclidean to the Heisenberg-Weyl variables, and in such a way that the two different momentum ranges are properly related; and third, we evolve the ray from the screen to general z in the Heisenberg-Weyl regime.

The first factor in the map will be thus the transformation *inverse* to (2.7), *i.e.*, *backward* free propagation of Euclidean rays; this we may write in terms of a Lie transformation [16]: [4]

$$\mathbf{p}_0 = \mathbf{p}(0) = \exp(+z\hat{H}^E)\mathbf{p}(z) = \mathbf{p}(z), \tag{4.1a}$$

$$\mathbf{q}_0 = \mathbf{q}(0) = \exp(+z\hat{H}^E)\mathbf{q}(z) = \mathbf{q}(z) - z\frac{\mathbf{p}(z)}{\sqrt{n^2 - \mathbf{p}(z)^2}}. \tag{4.1b}$$

The last transformation, Eqs. (3.5), is the *forward* evolution in Heisenberg-Weyl mechanical space. It may be similarly expressed as a Lie exponential

$$\mathbf{P}(z) = \exp(-z\hat{H}^{HW})\mathbf{P} = \mathbf{P}(0), \tag{4.2a}$$

$$\mathbf{Q}(z) = \exp(-z\hat{H}^{HW})\mathbf{Q} = \mathbf{Q}(0) + z\mathbf{P}(0)/n. \tag{4.2b}$$

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[4] We recall that, associated to every differentiable function f, we define its Lie *operator*

$$\hat{f} = \{f, \circ\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\right).$$

These operators have the important property of intertwining with commutators,

$$\{f, g\}\hat{} = [\hat{f}, \hat{g}],$$

so that we may speak of the Lie *algebra* generated by a set of functions under the Poisson bracket. The Lie *transformation* generated by f is the exponential of its Lie operator,

$$\exp\hat{f} = \sum_{n=1}^{\infty}\frac{1}{n!}(\hat{f})^n = \sum_{n=1}^{\infty}\frac{1}{n!}\{f, \{f, \cdots\{f, \circ\}\cdots\}\}.$$

These transformations are well known to be canonical.

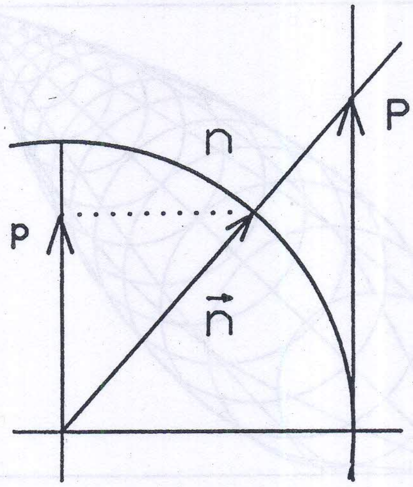

FIGURE 2. The geometric map from Euclidean to Heisenberg-Weyl momenta, **p** in the former and **P** in the latter, $\vec{n}$ is the ray direction vector.

The basic *opening* transformation at the screen is the following:

$$\mathbf{p} \mapsto \mathbf{P} \quad = \quad \mathcal{C}\mathbf{p} = \frac{\mathbf{p}}{\sqrt{1 - p^2/n^2}}, \tag{4.3c}$$

$$\mathbf{q} \mapsto \mathbf{Q} \quad = \quad \mathcal{C}\mathbf{q} = \sqrt{1 - p^2/n^2}\left(\mathbf{q} - \frac{1}{n^2}\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\right). \tag{4.3d}$$

This transformation *opens* the compact Euclidean momentum range $|\mathbf{p}| < n$ to the full Heisenberg-Weyl momentum plane $\Re^N$. We have called this map *comatic* because it is in fact a particular case of a Lie transformation [18], generated by the coma monomial ${}^2\mathcal{X}_1^2 = p^2\mathbf{p}\cdot\mathbf{q}$ [14], [19]

$$\begin{aligned} \mathbf{P}_\gamma &= \exp(\gamma p^2 \mathbf{p}\cdot\mathbf{q})\hat{}\,\mathbf{p} = \mathbf{p}/\sqrt{1 - 2\gamma p^2} && (4.4a)\\ &= \mathbf{p} + \gamma p^2 \mathbf{p} + \tfrac{3}{2}\gamma^2 (p^2)^2 \mathbf{p} + \tfrac{5}{2}\gamma^3 (p^2)^3 \mathbf{p} + \cdots. && (4.4b)\end{aligned}$$

$$\begin{aligned} \mathbf{Q}_\gamma &= \exp(\gamma p^2 \mathbf{p}\cdot\mathbf{q})\hat{}\,\mathbf{q} = \sqrt{1 - 2\gamma p^2}(\mathbf{q} - 2\gamma \mathbf{p}\cdot\mathbf{q}\,\mathbf{p}) && (4.4c)\\ &= \mathbf{q} - \gamma(p^2\mathbf{q} - 2\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}) \\ &\quad - \gamma^2 p^2 (\tfrac{1}{2}p^2\mathbf{q} - 2\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}) \\ &\quad - \gamma^3 (p^2)^2 (\tfrac{1}{2}p^2\mathbf{q} - \mathbf{p}\cdot\mathbf{q}\,\mathbf{p}) - \cdots, && (4.4d)\end{aligned}$$

for the value $\gamma = 1/2n^2$ of the parameter. To third degree in the phase space variables we have the usual third-order Seidel coma of optics; the full Lie series defines the *Seidel-Lie* global coma aberration.

In Figure 2 we show the geometric map between the Heisenberg-Weyl momentum plane and the 'forward' half-circle of ray directions and its projection on the Euclidean momentum segment $(-n, n)$. Correspondingly, in Figure 3 we show the position variables $\mathbf{q}'(\mathbf{p}, \mathbf{q})$ (in $N = 2$ dimensions)

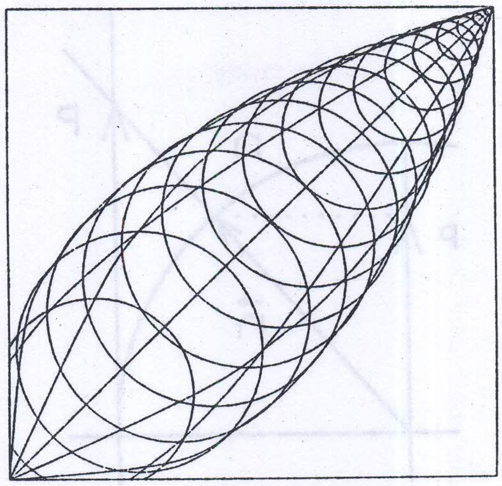

FIGURE 3. The spot diagram for the 'system' $\mathcal{C}$. A 90° pencil of rays in 5°-intervals in paralells and 16 meridians, issuing from a point at the upper right corner, upon passage through this '$\mathcal{C}$-system', maps on the indicated points on the screen.[5]

for $\mathbf{q}$ fixed and letting $\mathbf{p}$ range over a polar coordinate grid around the optical axis, *i.e.*, the *spot diagram* of the 'system' $\mathcal{C}$.

The vector function $A(p^2)p^2\mathbf{q} + B(p^2)\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}$ maps a cone of rays ($p^2 =$ constant) twice onto a *circle* on the screen, of radius $\frac{1}{2}Bp^2q$, $q = |\mathbf{q}|$, and center at $(A+\frac{1}{2}B)p^2q$. These circles are tangent to a sector with apex at $\mathbf{q}$ and half-angle of 30° at the apex. Figure 3 shows that as $|\mathbf{p}|$ increases from 0 to n, the radii grow from zero to a maximum and then decrease back to zero. The segment of circle *centers* starts at $\mathbf{q}$ and ends at the optical center $\mathbf{q} = 0$. This is the *global spot diagram* of the coma transformation.

The *inverse* comatic map $\mathcal{C}^{-1}$ may be obtained from (4.4)–(4.5) for the negative value of the parameter $\gamma = -1/2n^2$. This yields the inverse of (4.3) to be

$$\mathbf{p} = \mathcal{C}^{-1}\mathbf{P} \;=\; \frac{\mathbf{P}}{\sqrt{1+P^2/n^2}}, \tag{4.6a}$$

$$\mathbf{q} = \mathcal{C}^{-1}\mathbf{Q} \;=\; \sqrt{1+P^2/n^2}\left(\mathbf{Q} + \frac{1}{n^2}\mathbf{P}\cdot\mathbf{Q}\,\mathbf{P}\right). \tag{4.6b}$$

This map *closes* the noncompact range of Heisenberg-Weyl 'directions' into the compact region of Euclidean ray directions. Figure 4 shows the spot diagram of the 'system' $\mathcal{C}^{-1}$; the comet-like sectors in the spot diagram are now unbounded. We note the following useful identities under the

[5]Figures 3 and 4 have been prepared using the program SPOT_D developed by Guillermo Correa at IIMAS-UNAM. See: G.J. Correa-Gómez and K.B. Wolf, SPOT_D, *Programa para graficación de Diagramas de Manchas en Optica*. Comunicaciones Técnicas IIMAS, Serie Desarrollo, No. 97 (1989).

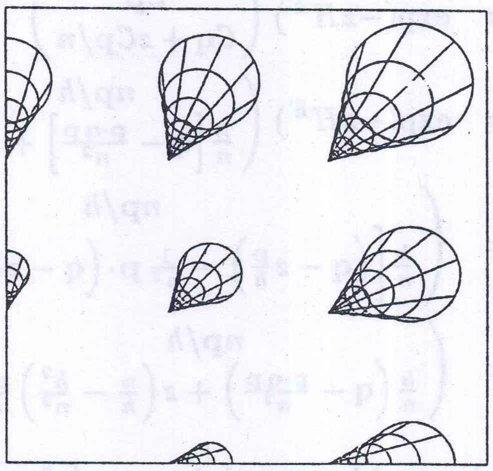

FIGURE 4. The spots diagram for the 'system' $\mathcal{C}^{-1}$. Rays with Heisenberg-Weyl $|\mathbf{P}| \le n/2$ (corresponding to $P = n \sin\Theta$, $|\Theta| \le 30°$), in six intervals of 5°, issuing from a rectangular grid of points at the apeces of the comet (*coma*) spot.

transformation:

$$1 - p^2/n^2 = \frac{1}{1+P^2/n^2}, \quad 1 + P^2/n^2 = \frac{1}{1-p^2/n^2}, \tag{4.7a}$$

$$\mathbf{p}\cdot\mathbf{q} = \mathbf{P}\cdot\mathbf{Q}(1+P^2/n^2), \quad \mathbf{P}\cdot\mathbf{Q} = \mathbf{p}\cdot\mathbf{q}(1-p^2/n^2). \tag{4.7b}$$

7.5 The map between Heisenberg-Weyl and Euclidean free rays

We have the backward transformation to the reference screen $z = 0$ for Euclidean rays in (4.1), the basic comatic map at that reference screen in (4.3), and the forward evolution from the screen to general z in (4.2). We now compose the three Lie transformations in that order:[6]

$$\begin{pmatrix} \mathbf{P}(z) \\ \mathbf{Q}(z) \end{pmatrix} = \exp(-zH^E)\,\mathcal{C}\,\exp(zH^{HW}) \begin{pmatrix} \mathbf{p}(z) \\ \mathbf{q}(z) \end{pmatrix}. \tag{5.1}$$

To compose the Lie transformations we consider the three operators acting on *dummy* variables that are called simply $\mathbf{p}$ and $\mathbf{q}$. We do this explicitly:

$$\begin{pmatrix} \mathbf{P}(z) \\ \mathbf{Q}(z) \end{pmatrix} = \exp(-zH^E)\,\mathcal{C} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} + z\mathbf{p}/n \end{pmatrix} \tag{5.2a}$$

[6] We recall that we use capital $\mathbf{P}$, $\mathbf{Q}$ for Heisenberg-Weyl phase space variables of N-dimensional mechanical systems, and lower-case $\mathbf{p}$, $\mathbf{q}$ for the N-dimensional screen in $(N+1)$-dimensional space.

$$= \exp(-zH^E)\begin{pmatrix} \mathcal{C}\mathbf{p} \\ \mathcal{C}\mathbf{q} + z\mathcal{C}\mathbf{p}/n \end{pmatrix} \tag{5.2b}$$

$$= \exp(-zH^E)\begin{pmatrix} n\mathbf{p}/h \\ \frac{h}{n}\left[\mathbf{q} - \frac{\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}}{n^2}\right] + z\frac{\mathbf{p}}{h} \end{pmatrix} \tag{5.2c}$$

$$= \begin{pmatrix} n\mathbf{p}/h \\ \frac{h}{n}\left[\left(\mathbf{q} - z\frac{\mathbf{p}}{h}\right) - \frac{1}{n^2}\,\mathbf{p}\cdot\left(\mathbf{q} - z\frac{\mathbf{p}}{h}\right)\mathbf{p}\right] + z\frac{\mathbf{p}}{h} \end{pmatrix} \tag{5.2d}$$

$$= \begin{pmatrix} n\mathbf{p}/h \\ \frac{h}{n}\left(\mathbf{q} - \frac{\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}}{n^2}\right) + z\left(\frac{n}{h} - \frac{h^2}{n^2}\right)\frac{\mathbf{p}}{n} \end{pmatrix}, \tag{5.2e}$$

where as before we have abbreviated $h = \tau\sqrt{n^2 - p^2}$, and consider the $\tau = +$ hemisphere of rays.

The z-dependent map that relates the (lower case) Euclidean and (upper-case) Heisenberg-Weyl trajectories is thus

$$\mathbf{P} = \frac{\mathbf{p}}{\sqrt{1 - p^2/n^2}}, \qquad \text{independent of } z, \tag{5.3a}$$

$$\mathbf{Q}(z) = \sqrt{1 - p^2/n^2}\left(\mathbf{q}(z) - \frac{\mathbf{p}\cdot\mathbf{q}(z)\,\mathbf{p}}{n^2}\right) + z\left(\frac{1}{\sqrt{1 - p^2/n^2}} - \left[1 - \frac{p^2}{n^2}\right]\right)\frac{\mathbf{p}}{n}. \tag{5.3b}$$

At $z = 0$ we have the simple comatic map $\mathcal{C}$ in (4.3). In the process of z-evolution, the maps start to differ from the pure comatic one by a term of *spherical aberration* that increases linearly with z. Finally, the map *inverse* to (5.3) is

$$\mathbf{p} = \frac{\mathbf{P}}{\sqrt{1 + P^2/n^2}}, \qquad \text{independent of } z, \tag{5.4a}$$

$$\mathbf{q}(z) = \sqrt{1 + P^2/n^2}\left(\mathbf{Q}(z) + \frac{\mathbf{P}\cdot\mathbf{Q}(z)\,\mathbf{P}}{n^2}\right) + z\left(\frac{1}{\sqrt{1 + P^2/n^2}} - \left[1 + \frac{P^2}{n^2}\right]\right)\frac{\mathbf{P}}{n}. \tag{5.4b}$$

We could be tempted to think that the map between trajectories may be obtained dispensing with the middle transformation $\mathcal{C}$, simply regressing the Euclidean trajectory to the screen and taking this as the initial ray parameters to advance under the Heisenberg-Weyl free-flight Hamiltonian. We must remember, however, that the two phase spaces are essentially *different* and the $\mathcal{C}$ and $\mathcal{C}^{-1}$ maps are therefore needed to intertwine the two ranges for the momentum variables, $|\mathbf{p}| < n$ and $\mathbf{P} \in \Re^N$.

7.6 The symplectic group on Euclidean phase space

The map between Heisenberg-Weyl and Euclidean phase spaces allows us to find explicitly the action of the inhomogeneous linear symplectic group ISp($2N, \Re$), well known in its Heisenberg-Weyl realization, on the N-sphere interior $|\mathbf{p}| < n$, the true phase space of optics. For the case of ordinary optics ($N = 2$ screen dimensions) the group is the four-dimensional inhomogeneous symplectic group ISp($4, \Re$).

The $2N + 1$ functions whose Lie operators generate the N-dimensional Heisenberg-Weyl group of phase space translations and its center are

$$P_i = \frac{p_i}{\sqrt{1-p^2/n^2}}, \quad Q_j = \sqrt{1-p^2/n^2}\left(q_j - \frac{\mathbf{p}\cdot\mathbf{q}\,p_j}{n^2}\right), \quad 1, \qquad (6.1a,b,c)$$

for $i, j = 1, 2, \ldots, N$. The $\frac{1}{2}N(N+1)$ generators of the linear transformations are

$$\begin{aligned} P_i P_j &= p_i p_j/(1-p^2/n^2), & (6.2a)\\ P_i Q_j &= p_i q_j + \mathbf{p}\cdot\mathbf{q}\,p_i p_j/n^2, & (6.2b)\\ Q_i Q_j &= (1-p^2/n^2) \\ &\quad \times\left(q_i q_j - \mathbf{p}\cdot\mathbf{q}(q_i p_j + q_j p_i)/n^2 + (\mathbf{p}\cdot\mathbf{q})^2 p_i p_j/n^4\right). & (6.2c) \end{aligned}$$

These functions close into the algebra of ISp($2N, \Re$) under the Poisson bracket. The maximal compact subgroup of Sp($2N, \Re$) is the rotation subgroup SO(N) within the N-dimensional screen generated by the linear combinations $R_{i,j} = P_i Q_j - P_j Q_i = p_i q_j - p_j q_i$. These generators also belong to the Euclidean algebra and will be seen below. The part of ISp($2N, \Re$) that has zero Lie brackets with the $R_{i,j}$ is the subalgebra Sp($2, \Re$) of *axis-symmetric* systems. Explicitly, the generators of this 'radial' Sp($2, \Re$) are

$$\begin{aligned} P^2 &= p^2/(1-p^2/n^2), & (6.3a)\\ \mathbf{P}\cdot\mathbf{Q} &= (1-p^2/n^2)\,\mathbf{p}\cdot\mathbf{q}, & (6.3b)\\ Q^2 &= (1-p^2/n^2)\left(q^2 - 2(\mathbf{p}\cdot\mathbf{q})^2/n^2 + p^2(\mathbf{p}\cdot\mathbf{q})^2/n^4\right). & (6.3c) \end{aligned}$$

By exponentiation, (6.1) generate the finite translations

$$\exp \mathbf{E}\cdot\hat{\mathbf{P}}\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix} = \begin{pmatrix}\mathbf{P}\\ \mathbf{Q}-\mathbf{E}\end{pmatrix}, \quad \exp \mathbf{F}\cdot\hat{\mathbf{Q}}\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix} = \begin{pmatrix}\mathbf{P}+\mathbf{F}\\ \mathbf{Q}\end{pmatrix}. \qquad (6.4a,b)$$

The quadratic functions (6.2) generate *linear* transformations that we may put in matrix form as

$$\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix} \overset{S}{\mapsto} \mathbf{S}\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}, \qquad (6.5a)$$

where $\mathbf{S}$ is a *symplectic* matrix, *i.e.*, that satisfies

$$\mathbf{S}\mathbf{M}\mathbf{S}^{\top} = \mathbf{M}, \qquad \mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.5b}$$

where $^{\top}$ indicates transposition, and $\mathbf{M}$ is the symplectic *metric* matrix. The 2×2 block realization for $\mathbf{S}$ leads to well-known relations between the $N \times N$ block matrices [6]. The axis-symmetric linear combinations (6.3) generate the subset of matrices (6.3b) where the submatrices are multiples of the $N \times N$ unit matrix.

The action of the Heisenberg-Weyl group on Euclidean phase space coordinates, for "position"-translations (6.4a), is

$$\mathbf{q} \overset{\mathbf{E}}{\mapsto} \mathbf{q} - \mathbf{e}(\mathbf{p},\mathbf{E}), \quad \mathbf{e}(\mathbf{p},\mathbf{E}) = \sqrt{1-p^2/n^2}\left(\mathbf{E} + \frac{\mathbf{p}\cdot\mathbf{E}}{1-p^2/n^2}\,\mathbf{p}\right), \tag{6.6a}$$

$$\mathbf{p} \overset{\mathbf{E}}{\mapsto} \mathbf{p}, \quad \text{and note } \mathbf{e}(\mathbf{p},\mathbf{E}_1) + \mathbf{e}(\mathbf{p},\mathbf{E}_2) = \mathbf{e}(\mathbf{p},\mathbf{E}_1+\mathbf{E}_2). \tag{6.6b}$$

For momentum translations (6.4b),

$$\begin{aligned} \mathbf{q} \overset{\mathbf{F}}{\mapsto} & \sqrt{1+2\mathbf{p}\cdot\mathbf{f}/n^2 + f^2/n^2} \\ & \times\left[\mathbf{q} + \tfrac{1}{n^2}\left\{\mathbf{f}\cdot(\mathbf{q} - \tfrac{1}{n^2}\mathbf{p}\cdot\mathbf{q}\,\mathbf{p})(\mathbf{p}+\mathbf{f}) + \mathbf{p}\cdot\mathbf{q}(1-p^2/n^2)\mathbf{f}\right\}\right], \end{aligned} \tag{6.7a}$$

$$\mathbf{p} \overset{\mathbf{F}}{\mapsto} \frac{\mathbf{p}+\mathbf{f}}{\sqrt{1+2\mathbf{p}\cdot\mathbf{f}/n^2+f^2/n^2}}, \quad \mathbf{f}(p^2,\mathbf{F}) = \sqrt{1-p^2/n^2}\,\mathbf{F}. \tag{6.7b}$$

The translation (6.6) affects only the screen coordinate; for vanishingly small $\mathbf{p}$, $\mathbf{e}(\mathbf{p},\mathbf{E}) \approx \mathbf{E}$ is the translation (6.6a) of $\mathbf{q}$; as $|\mathbf{p}|$ becomes larger, $\mathbf{E}$ starts to differ from $\mathbf{e}(\mathbf{p},\mathbf{E})$; as $|\mathbf{p}| \to n$, the translation diverges in the direction of $\mathbf{p}$. The momentum translation (6.7) is more complex since it affects both $\mathbf{q}$ and $\mathbf{p}$. Again, for vanishingly small $|\mathbf{p}|$, $\mathbf{f}(p^2,\mathbf{F}) \approx \mathbf{F}$; the size of the vector $\mathbf{f}$ decreases with increasing $p^2 < n^2$; as $|\mathbf{p}|$ approaches n, the Euclidean limit of large-angle rays, we have $\mathbf{f}(\mathbf{p},\mathbf{F}) = \mathbf{0}$. This effect may be seen in Figure 2 as we translate the Heisenberg-Weyl momentum plane $\mathbf{P}$: the $\mathbf{p}$-sphere moves as the image in Escher's reflecting balls. It is important to note that both (6.6) and (6.7) have the functional dependence of $\mathbf{p}$ and $\mathbf{q}$ given by a *point* transformation in $\mathbf{p}$, *i.e.*, $\mathbf{p}'(\mathbf{p})$ but $\mathbf{q}'(\mathbf{p},\mathbf{q})$. This is a *distortion* in momentum space that entails a *comatic* aberration in the image on the screen.

We can now write the linear transformation $\mathcal{S}$ in (6.5), generated by the quadratic functions (6.2) on the true Euclidean optical phase space variables $\mathbf{p}$, $\mathbf{q}$. For the rotation subalgebra and subgroup, the results will be given in the next Section. Here we can write the results for the $\mathrm{Sp}(2,\Re)$ action of Heisenberg-Weyl axis-symmetric optical systems generated by (6.3).

Thus:

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \mapsto \mathcal{C}^{-1}\mathcal{S}\mathcal{C}\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \tag{6.8a}$$

$$= \mathcal{C}^{-1}\mathcal{S}\begin{pmatrix} \mathbf{p}/\sqrt{1-p^2/n^2} \\ \sqrt{1-p^2/n^2}(\mathbf{q}-\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}) \end{pmatrix}$$

$$= \mathcal{C}^{-1}\begin{pmatrix} (A\mathbf{p}+B\mathbf{q})/R_{\mathbf{S}}(\mathbf{p},\mathbf{q}) \\ R_{\mathbf{S}}(\mathbf{p},\mathbf{q})[(C\mathbf{p}+D\mathbf{q})-T_{\mathbf{S}}(\mathbf{p},\mathbf{q})(A\mathbf{p}+B\mathbf{q})] \end{pmatrix}$$

$$= \begin{pmatrix} (A\mathbf{p}^c+B\mathbf{q}^c)/R^c \\ R^c([C-AT^c_{\mathbf{S}}]\mathbf{p}^c+[D-BT^c_{\mathbf{S}}]\mathbf{q}^c) \end{pmatrix} \tag{6.8b}$$

where $\mathbf{p}^c = \mathcal{C}^{-1}\mathbf{p}$ and $\mathbf{q}^c = \mathcal{C}^{-1}\mathbf{q}$ are given by (4.6), while $R^c(\mathbf{p},\mathbf{q}) = R(\mathbf{p}^c,\mathbf{q}^c)$ and $T^c_{\mathbf{S}}(\mathbf{p},\mathbf{q}) = T_{\mathbf{S}}(\mathbf{p}^c,\mathbf{q}^c)$ are to be replaced from

$$R_{\mathbf{S}}(\mathbf{p},\mathbf{q}) = \sqrt{1-(A^2p^2+2AB\mathbf{p}\cdot\mathbf{q}+B^2q^2)/n^2}, \tag{6.9a}$$

$$T_{\mathbf{S}}(\mathbf{p},\mathbf{q}) = (2ACp^2+[AD+BC]\mathbf{p}\cdot\mathbf{q}+2BDq^2)/n^2. \tag{6.9b}$$

This is a case of a point-linear-point transformation (in N dimensions) of the kind studied by Leyvraz and Seligman [20], exchanging $\mathbf{p}$ and $\mathbf{q}$.

We can explicitly find the linear transformation subgroup generated by Heisenberg-Weyl free flight,

$$\mathcal{A}_\alpha\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} = \exp\alpha\widehat{P^2}\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ \mathbf{Q}-2\alpha\mathbf{P} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2\alpha & 1 \end{pmatrix}\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}, \tag{6.10a}$$

on the Euclidean phase space coordinates. It is

$$\mathcal{A}_\alpha\,\mathbf{p} = \mathbf{p}, \tag{6.10a}$$

$$\mathcal{A}_\alpha\,\mathbf{q} = \mathbf{q}-2\alpha\frac{\mathbf{p}}{(1-p^2/n^2)^2}. \tag{6.10b}$$

For small $|\mathbf{p}|$, the relation of $\mathcal{A}_\alpha\,\mathbf{q}$ with $\mathbf{q}$ and $\mathbf{p}$ appears linear. For $|\mathbf{p}|$ approaching n, the screen transformation (6.10c) of $\mathbf{q}$ diverges in the direction of $-\mathbf{p}$ when $\alpha > 0$. This is the inverse of *spherical aberration*, *i.e.*, the error we incurr when we propagate Euclidean rays by Heisenberg-Weyl free flight.

Pure magnification,

$$\mathcal{B}_\beta\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} = \exp\beta\widehat{\mathbf{P}\cdot\mathbf{Q}}\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} e^\beta\mathbf{P} \\ e^{-\beta}\mathbf{Q} \end{pmatrix} = \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix}\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}, \tag{6.11a}$$

on Euclidean phase space is

$$\mathcal{B}_\beta\,\mathbf{p} = \frac{e^\beta\mathbf{p}}{\sqrt{1+(e^{2\beta}-1)p^2/n^2}}, \tag{6.11a}$$

$$\mathcal{B}_\beta\,\mathbf{q} = \sqrt{1+(e^{2\beta}-1)p^2/n^2}\left[e^{-\beta}\mathbf{q}+2\frac{1}{n^2}\sinh\beta\,\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\right]. \tag{6.11b}$$

Again we check that for $|\mathbf{p}| \ll n$ the map resembles a linear one, while for $|\mathbf{p}| \to n$ the action becomes the identity, keeping the boundary of the $|\mathbf{p}| = n$ sphere in its place. Again, a glance at Figure 2 confirms intuition.

Finally, a "Gaussian thin lens" transformation,

$$\mathcal{C}_\gamma \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} = \exp \gamma \widehat{Q^2} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} \mathbf{P} + 2\gamma \mathbf{Q} \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} 1 & 2\gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}, \qquad (6.12a)$$

on the full direction hemisphere appears as

$$\mathcal{C}_\gamma \, \mathbf{p} = \frac{\mathbf{p}_\gamma}{\sqrt{1 - p^2/n^2 + p_\gamma^2/n^2}}, \qquad (6.12b)$$

where

$$\mathbf{p}_\gamma(\mathbf{p}, \mathbf{q}, \gamma) = \mathbf{p} + 2\gamma(1 - p^2/n^2)(\mathbf{q} - \frac{1}{n^2}\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}), \qquad (6.12c)$$

and

$$\mathcal{C}_\gamma \, \mathbf{q} = \sqrt{1 - p^2/n^2 + p_\gamma^2/n^2}\left(1 + \frac{\mathbf{p}_\gamma \mathbf{p}_\gamma \cdot}{n^2 - p^2}\right)\left(\mathbf{q} - \frac{1}{n^2}\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\right). \qquad (6.12d)$$

The function $\mathbf{p}_\gamma(\mathbf{p}, \mathbf{q}, \gamma)$ in (6.12*c*) becomes linear for small $|\mathbf{p}|$ and approaches the identity at n. The phase space transformations in (6.12*b*) and (6.12*d*) have the same behaviour; the boundary $|\mathbf{p}| = n$ stays put.

Much, if not the whole Lie theory of aberrations, has thus far been based on transformations generated by polynomial functions of the Heisenberg-Weyl phase space variables [1], [6]. We may now let these act on the phase space of Euclidean optics. Thus far we have involved only the *forward* hemisphere of ray directions, $h > 0$. If we take the negative sign of the square roots in (4.3) we open the Euclidean momentum sphere onto a *second* Heisenberg-Weyl momentum plane that describes *backward*-moving rays.

7.7 The Euclidean and Lorentz groups on Heisenberg-Weyl phase space

The natural group of motions of Euclidean optical phase space is precisely the Euclidean group. For the general case of $(N+1)$-dimensional optics with N-dimensonal screens this is the group ISO$(N+1)$. We shall now discuss how this group acts on the $\Re^{2N}$ phase space of Heisenberg-Weyl optics, why *two* copies of the momentum space $\Re^N$ space are required, and a certain similarity group that seems to be operating.

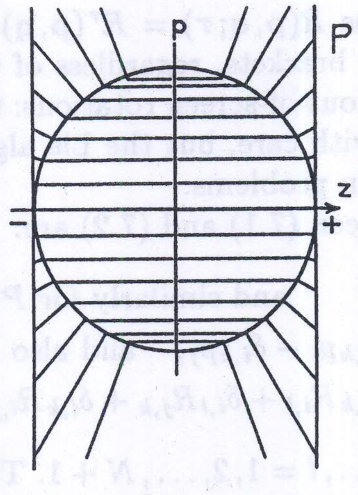

FIGURE 5. The map of the sphere of ray directions between two Euclidean disks and two Heisenberg-Weyl planes.

The N functions that generate translations in the Euclidean screen and the $(N+1)^{\text{th}}$ generator of evolution *normal* to the screen are

$$p_i = \frac{P_i}{\sqrt{1+P^2/n^2}}, \qquad i = 1,2,\ldots,N, \tag{7.1a}$$

$$h^\tau = \tau\sqrt{n^2 - p^2} = \frac{\tau n}{\sqrt{1+P^2/n^2}}, \qquad \tau = -1, 0, +1. \tag{7.1b}$$

The last generator is (minus) the Hamiltonian; its two signs give the two forms of the generator on each of the two Euclidean momentum spheres (*disks* in ordinary $N=2$ optics), corresponding to forward- and backward-directed rays.[7] The latter are the mirror images of the first reflected on the reference plane, exhibiting the same value of the momentum $\mathbf{p}$, but differing in behaviour under z-translations due to the sign τ of h, the label of the *chart*. See Figure 5.

The generators of rotations *within* the Euclidean screen submanifold are

$$R_{i,j} = p_i q_j - p_j q_i = P_i Q_j - P_j Q_i, \qquad i \neq j = 1,2,\ldots N. \tag{7.2a}$$

The forms are the same in both spaces because they depend only on the *angular* transformation properties of $\{\mathbf{p},\mathbf{q}\}$ and $\{\mathbf{P},\mathbf{Q}\}$, and not of their sizes, nor τ; they close into the Lie algebra of SO(N).

Rotations *out* of the plane of the screen are generated by

$$R^\tau_{i,N+1} = q_i h^\tau = \tau(nQ_i + \tfrac{1}{n}\mathbf{P}\cdot\mathbf{Q}\,P_i), \qquad i = 1,2,\ldots N. \tag{7.2b}$$

[7] Strictly, we should speak of forward and backward hemispheres, joined by an *equatorial circle*, corresponding to $\tau = +1, -1$ and 0. The equatorial rays are disregarded here; they should serve as limit points for sequences of rays in either hemisphere [10].

These are *two-chart* functions $R(\mathbf{p},\mathbf{q};\tau) = R^{\tau}(\mathbf{p},\mathbf{q}) = \tau R^{+1}(\mathbf{p},\mathbf{q})$. They formally close under Poisson brackets, regardless of the fact that the two charts are mixed under finite out-of-screen rotations: the corresponding Lie exponentials must be dealt with care, but the Lie algebraic properties are less sensitive to global domain problems.

The Poisson brackets between (7.1) and (7.2) are:

$$\{p_i, p_j\} = 0, \quad \text{and similarly for } P\text{'s}, \tag{7.3a}$$
$$\{R_{i,j}, p_k\} = \delta_{j,k} p_i - \delta_{i,k} p_j, \quad \text{and also } P\text{'s}, \tag{7.3b}$$
$$\{R_{i,j}, R_{k,l}\} = \delta_{j,k} R_{i,l} + \delta_{i,l} R_{j,k} + \delta_{i,k} R_{l,j} + \delta_{j,l} R_{k,i}. \tag{7.3c}$$

The Lie brackets hold for $i,\ldots,l = 1,2,\ldots,N+1$. The first two also hold when we replace p_i by q_i for $i = 1,2,\ldots,N$. The functions q_i, however, are not among the *generators* of the Euclidean group of motions; the handling of q_i's by themselves follows from the canonical formalism. In Ref. [10] we presented arguments to distrust the Euclidean wavization of the q_i into a self-adjoint operator. In the Heisenberg-Weyl setting, on the other hand, the natural and consistent quantization of Q_i is into the familiar self-adjoint Schrödinger operator.

The N-sphere of ray directions of Euclidean optics lends itself for the action of the Lorentz group SO(N+1,1). This group of relativity is a *deformation* of the basic Euclidean group ISO(N+1) [12]. The boost generators in N dimensions are given by the $(N+1)$-vector

$$B_i = nq_i - \tfrac{1}{n}\mathbf{p}\cdot\mathbf{q}\, p_i = \sqrt{n^2 + P^2}\, Q_i, \tag{7.4a}$$
$$B^{\tau}_{N+1} = -\tfrac{1}{n}\mathbf{p}\cdot\mathbf{q}\, h^{\tau} = \tau\sqrt{1 + P^2/n^2}\,\mathbf{P}\cdot\mathbf{Q}. \tag{7.4b}$$

The brackets between (7.4) and (7.2) close into the Lorentz algebra:

$$\{R_{i,j}, B_k\} = \delta_{j,k} B_i - \delta_{i,k} B_j, \tag{7.5a}$$
$$\{B_i, B_j\} = -R_{i,j}. \tag{7.5b}$$

The last function (7.4b) generates boosts along the optical axis; it is also a two-chart function due to the factor h^{τ} containing τ; we shall omit the index τ on h since it is given by the latter's sign. The Lie transformation $\exp \alpha \hat{B}_{N+1}$ distorts the angle θ (between a ray and the optical axis) through [12] $\tan \frac{1}{2}\theta \mapsto \tan \frac{1}{2}\theta' = e^{-\alpha} \tan \frac{1}{2}\theta$, mapping thus the backward direction region $\frac{1}{2}\pi \le \theta < 2 \arctan e^{\alpha}$ into the forward hemisphere.

We should point out the striking similarity between the expressions for out-of-screen rotations (7.2b) in terms of $\mathbf{p}$ and $\mathbf{q}$, and that of boosts (7.4a) in $\mathbf{P}$ and $\mathbf{Q}$. Indeed, $R_{i,N+1}(q_i, \sqrt{1-p^2/n^2}) = B_i(Q_i, \sqrt{1+P^2/n^2})$. The R's close under Poisson bracket to a rotation algebra and the B's to a Lorentz algebra because of the sign of p^2 in the square root is negative while that of P^2 is positive.

The Euclidean translations are generated by (7.1). The Lie transformation $\overset{\mathbf{e}}{\mapsto}$ of translation by $\mathbf{e} = (e_1, e_2, \ldots, e_N)$ in the screen plane is $\exp \sum e_i \hat{p}_i$. Its action on Heisenberg-Weyl phase space is the converse of (6.6), namely

$$\mathbf{Q} \overset{\mathbf{e}}{\mapsto} \mathbf{Q} - \mathbf{E}(\mathbf{P}, \mathbf{e}), \quad \mathbf{E}(\mathbf{P}, \mathbf{e}) = \frac{1}{\sqrt{1 + P^2/n^2}} \left(\mathbf{e} + \frac{\mathbf{P}\cdot\mathbf{e}}{n^2 + P^2} \mathbf{P} \right) \quad (7.6a)$$

$$\mathbf{P} \overset{\mathbf{e}}{\mapsto} \mathbf{P}, \qquad \text{and note } \mathbf{E}(\mathbf{P}, \mathbf{e}_1) + \mathbf{E}(\mathbf{P}, \mathbf{e}_2) = \mathbf{E}(\mathbf{P}, \mathbf{e}_1 + \mathbf{e}_2). \quad (7.6b)$$

Euclidean rotations *in* the N-screen, generated by $R_{i,j}$ in (7.2*a*), commute with the opening coma map, and provide the Petzval invariant as $\frac{1}{2}\sum_{i,j} R_{i,j} R_{i,j}$, the SO($N$) second-order Casimir operator. These rotations coincide with the maximal compact subgroup SO(N) $\subset$ Sp($2N, \Re$). On the other hand, SO($N+1$) rotations (7.2*b*) *out* of the screen plane, are generated by $\sum \alpha_i R^T_{i,N+1}$. For any single i, the effect of the Lie transformation $\mathcal{R}_i(\alpha) = \exp \alpha \hat{R}^T_{i,N+1} = \exp \alpha \widehat{q_i h}$ on Euclidean ray direction is

$$\mathcal{R}_i(\alpha) : p_j = \begin{cases} p_i \cos\alpha + h \sin\alpha, & i = j, \\ p_j, & i \neq j, \end{cases} \quad (7.7a)$$

$$\mathcal{R}_i(\alpha) : h = -p_i \sin\alpha + h \cos\alpha. \quad (7.7b)$$

In writing h explicitly in the transformation we can distinguish between the forward and backward hemispheres through its sign.

To obtain the action on Euclidean ray position q_i, we note that the transformation properties (7.7) of the p_j are the same as those of $R^T_{j,N+1} = q_i h$, hence

$$\mathcal{R}_i(\alpha) : R_{i,N+1} = \begin{cases} R_{i,N+1}, & i = j \\ R_{i,N+1} \cos\alpha + R_{i,j} \sin\alpha, & i \neq j. \end{cases} \quad (7.8)$$

Thus, writing $R_{i,N+1} = q_i h \mapsto R'_{i,N+1} = q'_i h'$, we find the effect of the Lie transformation on Euclidean position as

$$\mathcal{R}_i(\alpha) : q_j = \begin{cases} \dfrac{q_i}{\cos\alpha - p/h \sin\alpha}, & i = j, \\[2ex] \dfrac{q_j \cos\alpha - (p_i q_j - p_j q_i)/h \sin\alpha}{\cos\alpha - p_i/h \sin\alpha} & i \neq j. \end{cases} \quad (7.9)$$

The denominator may vanish for $\tan\alpha = h/p_i$: rays that become paralell to the screen. The singularity only indicates that a change of chart is required to go beyond, because then the ray will come from the opposite side of the screen.

Let us write explicitly the Euclidean action on the Heisenberg-Weyl variables $\{\mathbf{P}, \mathbf{Q}\} \in \Re^{2N}$, obtained through the opening coma map. Translations generated by $p_i(\mathbf{P})$ in (7.1) were shown to have the action (7.6). The SO(N)

rotations generated by (7.2a) have the same action on $\{\mathbf{P},\mathbf{Q}\}\in\Re^{2N}$ as on $\{\mathbf{p},\mathbf{q}\}\in\Re^{N}_{p\le n}\otimes\Re^{N}$. Out-of-screen rotations in are SO$(N+1)$ generated by (7.2b) and their action on p-space was seen in (7.7a) and on q-space in (7.9). Aiding ourselves with the transformation (7.7b) of h as component $N+1$ in Euclidean optics, we find for the Heisenberg-Weyl momentum space

$$\mathcal{R}_i(\alpha): P_j \;=\; \begin{cases} \dfrac{P_i\cos\alpha+n\sin\alpha}{\cos\alpha-P_i/n\,\sin\alpha}, & i=j,\\[2mm] \dfrac{P_j}{\cos\alpha-P_i/n\,\sin\alpha} & i\neq j,\end{cases} \tag{7.10a}$$

$$\mathcal{R}_i(\alpha):\sqrt{1+P^2/n^2} \;=\; \frac{\sqrt{1+P^2/n^2}}{\cos\alpha-P_i/n\,\sin\alpha}. \tag{7.10b}$$

To obtain the action on position Q_i through the opening coma map (4.3b), the rotation on q_j from (7.9) and the inverse coma map (4.6b), it is useful to know the transformation properties of $\mathbf{p}\cdot\mathbf{q}$ under such rotations. From (7.7) and (7.9), and in view of (4.8) and (7.10b),

$$\mathcal{R}_i(\alpha):\mathbf{p}\cdot\mathbf{q}=\frac{\mathbf{p}\cdot\mathbf{q}\cos\alpha+n^2/h\,(q_i-\frac{1}{n^2}\mathbf{p}\cdot\mathbf{q}\,p_i)\sin\alpha}{\cos\alpha-p_i/h\sin\alpha}. \tag{7.11}$$

The expression in parentheses above is $q_i-\frac{1}{n^2}\mathbf{p}\cdot\mathbf{q}\,p_i=Q_i\sqrt{1+P^2/n^2}$. Through this we obtain

$$\mathcal{R}_i(\alpha):Q_j \;=\; \begin{cases}(\cos\alpha-P_i/n\,\sin\alpha)(Q_i\cos\alpha-\frac{1}{n}\mathbf{P}\cdot\mathbf{Q}\,\sin\alpha), & i=j,\\ Q_j\,(\cos\alpha-P_i/n\,\sin\alpha), & i\neq j,\end{cases} \tag{7.12a}$$

$$\mathcal{R}_i(\alpha):\mathbf{P}\cdot\mathbf{Q} \;=\; (\cos\alpha-P_i/n\,\sin\alpha)(-nQ_i\sin\alpha+\mathbf{P}\cdot\mathbf{Q}\cos\alpha). \tag{7.12b}$$

We write the last expression because the out-of-plane rotations $\mathcal{R}_i(\alpha)$ will transform the components of B_i, $i=1,2,\ldots,N,N+1$ amongst themselves;[8] modulo the factor $\sqrt{n^2+P^2}$, $\mathbf{P}\cdot\mathbf{Q}/n$ suggests itself as the $(N+1)^{\text{th}}$ component of the N-vector $\mathbf{Q}$. The $(N+1)^{\text{th}}$ component of the Euclidean N-vector $\mathbf{q}$ is zero, as can be seen in the same equations (7.3) because we are in the standard Euclidean screen. In a similar way, the factor $H=\sqrt{n^2+P^2}$ transforms as the $(N+1)^{\text{th}}$ component of the N-vector $\mathbf{P}$, as was given in Eq. (7.10b); this fittingly corresponds with h, *cf.* Eqs. (4.7), as the $(N+1)^{\text{th}}$ component of the $(N+1)$-vector of ray direction on the Descartes sphere $p^2+h^2=n^2$. We are now on a *hyperboloid* $H^2-P^2=n^2$, however.

[8]Indeed, the B_i may be written as $R_{i,N+2}$ and appended to the list (7.2) with commutation relations (7.3c), except that the Kronecker δ's that appear should be replaced by $g_{i,j}$ that is 1 when $i=j\le N$ and -1 when $i=j\ge N+1$, and zero otherwise.

To obtain the action of the Lorentz boost operators (7.4) on Euclidean and Heisenberg-Weyl rays, we use the curious similarity noted above. Indeed, transformations generated by $B_i = nq_i - \frac{1}{n}\mathbf{p}\cdot\mathbf{q}\,p_i$ on $\{\mathbf{p},\mathbf{q}\}$ are 'similar' to those generated by $R_{i,N+1} = nQ_i + \frac{1}{n}\mathbf{P}\cdot\mathbf{Q}\,P_i$ on $\{\mathbf{P},\mathbf{Q}\}$, the latter given by (7.10) and (7.12). Also, transformations generated by $B_i = HQ_i$ on $\{\mathbf{P},\mathbf{Q}\}$ will be similar to $R_{i,N+1} = hq_i$ acting on $\{\mathbf{p},\mathbf{q}\}$, in (7.7) and (7.9). Replacing $\{q_i, h\} = -p_i/h$ by $\{Q_i, H\} = +P_i/H$ seems to have the effect of changing the series $\sin\alpha \mapsto -\sinh\alpha$ and $\cos\alpha \mapsto \cosh\alpha$. As a check, we may compare $\mathcal{R}_i(\alpha) : Q_j$ in (7.12) through the upper-by-lower-case and series replacements with equations (13.6) in reference [10], found in [12].

We are left with the computation of the transformation generated by the boost normal to the screen, $B_{N+1} = -\frac{1}{n}\mathbf{p}\cdot\mathbf{q}\,h = -\mathbf{P}\cdot\mathbf{Q}\,H$. This is self-similar in the same sense as the $R_{i,j}$ in (7.2a). The boost formulas appear in the companion article [10], Eqs. (13.1) and (13.3), and were also presented in [12]. For completeness, we reproduce the formulas here for the Euclidean phase space variables:

$$\mathcal{B}_z : \mathbf{p} \;=\; \frac{\mathbf{p}}{\cosh\alpha + h/n\,\sinh\alpha}, \tag{7.13a}$$

$$\mathcal{B}_z : \mathbf{q} \;=\; (\cosh\alpha + h/n\,\sinh\alpha)\left(\mathbf{q} - \frac{\sinh\alpha}{n\sinh\alpha + h\cosh\alpha}\,\frac{\mathbf{p}\cdot\mathbf{q}}{n}\,\mathbf{p}\right) \tag{7.13b}$$

The same transformation holds for the Heisenberg-Weyl variables upon replacement of $\mathbf{q}$, $\mathbf{p}$,and h by $\mathbf{Q}$, $\mathbf{P}$ and $H = \sqrt{n^2 + P^2}$.

7.8 Spherical aberration, coma, and point transformations in phase space

Let us remark a general feature of the maps seen here and, indeed, of the generator functions themselves. They have the structure $P \mapsto f(P)$, $Q \mapsto g(P)Q + h(P)$ and similarly for the lower-case Euclidean variables, and in many dimensions. Such are *point* transformations in P-space, a proper subset of the group of *all* canonical transformations in phase space. The generators of $(2m-1)^{\text{th}}$-order spherical aberration [6] are $(P^2)^m$, $m = 1, 2, \ldots$, while those of $(2m-1)^{\text{th}}$-order circular coma[9] are $(P^2)^{m-1}\mathbf{P}\cdot\mathbf{Q}$.

The Lie transformations generated by (7.1b), or by any functions $f(p^2) = F(P^2)$, will map $\{\mathbf{P},\mathbf{Q}\} \in \Re^{2N}$ in the following way:

$$\exp A\hat{F}(P^2)\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix} = \begin{pmatrix}\mathbf{P}\\ \mathbf{Q} - 2A\,F'(P^2)\mathbf{P}\end{pmatrix}. \tag{8.1a}$$

[9] For $m = 1$ we have first-order 'aberrations'. The linear symplectic 'free flight' P^2 is first-order spherical aberration; magnification $\mathbf{P}\cdot\mathbf{Q}$ is first-order coma and, as its own Fourier transform, first-order distortion.

In Optics this is termed *spherical aberration* because a pencil of rays $\mathbf{P} \in \Re^2$ issuing from an object at $\mathbf{Q}_0$, produces an image $\mathbf{Q}(\mathbf{P}, \mathbf{Q}_0) = \mathbf{Q}_0 + \alpha\Phi(\mathbf{P})$ containing a spread Φ paralell to $\mathbf{P}$, unbounded, and independent of the location of the source $\mathbf{Q}_0$ in the object screen.

The generators (7.1*a*) of translations *in* the screen are of the form $F(P^2)\mathbf{P}$. The Lie transformation generated by a coefficient vector $\mathbf{B}$ is

$$\exp[F(P^2)\,\mathbf{B}\cdot\mathbf{P}]\hat{}\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix} = \begin{pmatrix}\mathbf{P}\\ \mathbf{Q} - [F + 2F'\,\mathbf{P}\mathbf{P}\cdot]\mathbf{B}\end{pmatrix}. \tag{8.1b}$$

Rotations generated by (7.2*a*) will transform transformations (8.1*b*) among themselves.

In-screen rotations (7.2*a*) are standard, but (7.2*b*) and (7.4) are of the generic form $g(\mathbf{p}, \mathbf{q}) = g_i^I(p^2)q_i + g_i^{II}(p^2)\mathbf{p}\cdot\mathbf{q}\,p_i = G(\mathbf{P}, \mathbf{Q}) = G_i^I(P^2)Q_i + G_i^{II}(P^2)\mathbf{P}\cdot\mathbf{Q}\,P_i$, while the SO($N$) scalar (7.4*b*) has the form $h(p^2)\mathbf{p}\cdot\mathbf{q} = H(P^2)\mathbf{P}\cdot\mathbf{Q}$. They are *linear* in the components of position: $g(\mathbf{p}, x\mathbf{q}) = xg(\mathbf{p}, \mathbf{q})$ and $G(\mathbf{P}, X\mathbf{Q}) = XG(\mathbf{P}, \mathbf{Q})$. The Lie transformations they generate are of the form

$$\exp \mathbf{C}\cdot\hat{\mathbf{G}}\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix} = \begin{pmatrix}\mathbf{P}'(\mathbf{P})\\ \mathbf{Q}'(\mathbf{P}, \mathbf{Q}; \mathbf{C})\end{pmatrix}, \tag{8.2}$$

where $\mathbf{Q}'(\mathbf{P}, \mathbf{Q}; \mathbf{C})$ is a function of the *same* kind as G described above with vector indices 'balanced' to one. Similarly for the lower-cased letters. These are *point* transformations of momentum space (*i.e.*, *distortions* of it). As far as (7.4*b*) is concerned, and as detailed in Refs. [10] and [12], the Fourier transform of distortion is circular comatic aberration of the image position space. The spot diagrams of Lorentz boosts (7.4*a*) in the screen plane (Ref.[10], Fig. 5*b*), are SO($N+1$)*–rotated* versions of the basic comatic aberration (in Ref.[10], Figs. 5*a* and 5*c*).

For $N = 1$ dimension, it is well known that functions $\Phi(P, Q) = \Phi^I(P)Q + \Phi^{II}(P)$ close under Poisson brackets and thus generate an infinite-dimensional subalgebra in the enveloping algebra of Heisenberg and Weyl. They thus serve also as function space for the Lie transformations. In addition, such functions are uniquely quantized [7]. [10] For $N = 1$ dimension, we have explicit formulas (with an indefinite integral) for $Q'(P, Q; C)$; singularities may occur, having to do with the Φ's range. General formulas for the N-dimensional case are not available, but discussions with F. Leyvraz [20], IF–UNAM, suggest that they can be found explicitly.

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[10] For such functions, Poisson brackets and commutators of their Schrödinger operators follow each other. All quantization rules leading to self-adjoint operators give the same result on functions that are linear in one of the conjugate observables.

Spherical aberration and circular coma (of all orders), together with their rotated aberrations and the opening coma introduced here, constitute a subring of Heisenberg-Weyl whose study should be interesting. The Euclidean and Lorentz algebra presented above are only (the only?) finite-dimensional subalgebras.

7.9 The Hilbert spaces for Heisenberg-Weyl and Euclidean optics

Quantization in the Heisenberg-Weyl model of optical phase space is essentially $\mathcal{L}_2(\Re^N)$ quantization à la Schrödinger. The Fourier transform plays a prominent role intertwining the configuration and momentum representations [8], so the subject is also called Fourier optics. Wavization of Euclidean optics, on the other hand, involves solutions of the wave equation; Fourier frequency analysis of a signal decomposes the space of waves into subsets that are solutions of the Helmholtz equation for each wavenumber [10]. There is a 'Helmholtz-Hilbert space' that uses for function range and inner product integration the two momentum spheres (*disks* in $N = 2$ dimensions [10]), joined at their surfaces (perimeter in the $N = 2$ case). Here we shall show how the opening coma transformation intertwines between functions in Heisenberg-Weyl and Euclidean-Helmholtz wave optics.

The opening coma map (4.3a)–(4.6a) in radial and angular variables, is

$$P = \frac{p}{\sqrt{1 - p^2/n^2}}, \qquad \Omega_\mathbf{P} = \omega_\mathbf{p} \in \mathcal{S}_{N-1}, \tag{9.1a}$$

and we also need the radial differential

$$dP = (1 - p^2/n^2)^{-3/2} dp. \tag{9.1b}$$

We express the inner product of the $\mathcal{L}_2(\Re^N)$ Hilbert space of square-integrable Heisenberg-Weyl wavefunctions in the momentum representation $\mathbf{P} \in \Re^N$, in terms of an integral of the Euclidean momentum $\mathbf{p}$ over *its* proper range $|\mathbf{p}| < n$. Thus,

$$(\Phi, \Psi)_{\mathcal{L}_2(\Re^N)} = \int_{\Re^N} d^N\mathbf{P}\, \Phi(\mathbf{P})^* \Psi(\mathbf{P}) \tag{9.2a}$$

$$= \int_0^\infty P^{N-1} dP \times \int_{\mathcal{S}_{N-1}} d^{N-1}\Omega_\mathbf{P}\, \Phi(P, \Omega_\mathbf{P})^* \Psi(P, \Omega_\mathbf{P}) \tag{9.2b}$$

$$= \int_0^n \frac{p^{N-1}\, dp}{(1 - p^2/n^2)^{N/2+1}} \times \int_{\mathcal{S}_{N-1}} d^{N-1}\omega_\mathbf{p}\, \Phi(\mathbf{P}(p, \omega_\mathbf{p}))^* \Psi(\mathbf{P}(p, \omega_\mathbf{p})) \tag{9.2c}$$

$$= \int_{\substack{\mathbf{p}\in\Re^N \\ |\mathbf{p}|<n}} \frac{d^N\mathbf{p}}{(1-p^2/n^2)^{N/2+1}} \Phi(\mathbf{P}(\mathbf{p}))^* \Psi(\mathbf{P}(\mathbf{p})) \qquad (9.2d)$$

$$= \int_{\substack{\mathbf{p}\in\Re^N \\ |\mathbf{p}|<n}} \frac{d^N\mathbf{p}}{\sqrt{1-p^2/n^2}} \phi(\mathbf{p})^* \psi(\mathbf{p}) \qquad (9.2e)$$

$$= \int_{\mathcal{S}_N^{(+)}} d^N S(\vec{p})\, \phi(\vec{p})^*\, \psi(\vec{p}) = (\phi, \psi)_{\mathcal{L}_2(\mathcal{S}_N^{(+)})}. \qquad (9.2f)$$

The $\mathcal{L}_2(\Re^N)$ inner product $(9.2a)$ is expressed in the radial and angular variables of $\mathbf{P}$ in $(9.2b)$ and of $\mathbf{p}$ in $(9.2c)$. The integral $(9.2d)$ shows that the inner product in Euclidean momentum variables has a weight factor $(1-p^2/n^2)^{-(N/2+1)}$. The integrand in $(9.2e)$ defines ϕ and ψ such that the last integral, $(9.2f)$, be over the *forward hemisphere* $\mathcal{S}_N^{(+)}$ of the N-dimensional surface of the *Descartes* sphere of ray directions. The sphere $\mathcal{S}_N$ is inmersed in an $(N+1)$-dimensional space whose *ray* vectors are $\vec{p} = \{\mathbf{p} = n\sin\theta\,\omega_{\mathbf{p}},\ h = n\cos\theta > 0\}$, whose radii are $\vec{p}\cdot\vec{p} = p^2 + h^2 = n^2$, and we may write $p = |\mathbf{p}| = n\sin\theta$. The surface element of the Descartes sphere is then

$$d^N S(\vec{p}) = n^N \sin^{N-1}\theta\, d\theta\, d^{N-1}\omega_{\mathbf{p}} = \frac{p^{N-1}\, dp\, d^{N-1}\omega_{\mathbf{p}}}{\sqrt{1-p^2/n^2}} = \frac{d^N\mathbf{p}}{\sqrt{1-p^2/n^2}}. \qquad (9.3)$$

In $\mathcal{S}_N^{(+)}$ the lower-case Greek wavefunctions of $\mathbf{p}$ are related to their upper-case counterparts through a measure normalization factor,

$$\begin{aligned} \psi(\vec{p}) &= \psi_+(\mathbf{p}) = |1-p^2/n^2|^{-(N+1)/4}\Psi(\mathbf{P}(\mathbf{p})) \\ &= |1+P^2/n^2|^{(N+1)/4}\Psi(\mathbf{P}(\mathbf{p})), \quad h = \sqrt{1-p^2/n^2} > 0. \qquad (9.4a) \\ \Psi(\mathbf{P}) &= |1+P^2/n^2|^{-(N+1)/4}\psi_+(\mathbf{p}(\mathbf{P})) \\ &= |1-p^2/n^2|^{(N+1)/4}\psi_+(\mathbf{p}(\mathbf{P})). \qquad (9.4b) \end{aligned}$$

In reference [10] we built the Hilbert space $\mathcal{L}_2(\mathcal{S}_N)$ on the *whole* Descartes sphere, natural subject to Euclidean transformations, and then projected it flat on *two* $|\mathbf{p}| < n$ disks sewn at the boundary. Equation $(9.4a)$ defines the function $\psi(\vec{p})$ only on the forward $(h > 0)$ hemisphere. A second Heisenberg-Weyl $\Re^N$ space of fuctions $\Psi_b(\mathbf{P})$ is needed to map onto the *backward* $(h < 0)$ hemisphere by

$$\begin{aligned} \psi(\vec{p}) &= \psi_-(\mathbf{p}) = |1-p^2/n^2|^{(N+1)/4}\Psi_b(\mathbf{P}(\mathbf{p})) \\ &= |1+P^2/n^2|^{(N+1)/4}\Psi_b(\mathbf{P}(\mathbf{p})), \quad h = \sqrt{1-p^2/n^2} < 0. \qquad (9.5a) \\ \Psi_b(\mathbf{P}) &= |1+P^2/n^2|^{-(N+1)/4}\psi_-(\mathbf{p}(\mathbf{P})) \\ &= |1-p^2/n^2|^{(N+1)/4}\psi_-(\mathbf{p}(\mathbf{P})). \qquad (9.5b) \end{aligned}$$

7.10 Plane waves and the coma kernel

In Fourier-Heisenberg-Weyl optics, the image wave function on the screen is given by the *Fourier* transform of $\Psi(\mathbf{P})$,

$$\widetilde{\Psi}(\mathbf{Q}) = \left(\frac{k}{2\pi n}\right)^{N/2} \int_{\Re^N} d^N\mathbf{P}\,\Psi(\mathbf{P})\exp(ik\mathbf{P}\cdot\mathbf{Q}/n), \quad (10.1a)$$

$$\Psi(\mathbf{P}) = \left(\frac{k}{2\pi n}\right)^{N/2} \int_{\Re^N} d^N\mathbf{Q}\,\widetilde{\Psi}(\mathbf{Q})\exp(-ik\mathbf{P}\cdot\mathbf{Q}/n). \quad (10.1b)$$

The scale is given by the quantity k/n, of units of $[PQ]^{-1}$. The reduced wavelength $\lambda/2\pi = n/k$ is equivalent to $\hbar = h/2\pi$ in quantum mechanics. The inner product on the Fourier screen is, through the Parseval identity,

$$\begin{aligned}(\widetilde{\Phi},\widetilde{\Psi})_{\mathcal{L}_2(\Re^N)} &= \int_{\Re^N} d^N\mathbf{Q}\,\widetilde{\Phi}(\mathbf{Q})^*\,\widetilde{\Psi}(\mathbf{Q})\\ &= \int_{\Re^N} d^N\mathbf{P}\,\Phi(\mathbf{P})^*\,\Psi(\mathbf{P}) = (\Phi,\Psi)_{\mathcal{L}_2(\Re^N)}. \quad (10.2)\end{aligned}$$

In Euclidean optics [10] we combine plane waves coming from all directions in the Descartes sphere. The basic linear combination function is $\psi(\vec{p})$ in (9.4a) and (9.5a), covering both backward and forward ray hemispheres. The wavefunction at the screen is then

$$\widetilde{\phi}(\vec{q})|_{q_N=0} = \left(\frac{k}{2\pi n}\right)^{N/2} \int_{S_N} d^N S(\vec{p})\,\phi(\vec{p})\exp(ik\vec{p}\cdot\vec{q}/n)\Big|_{q_N=0} \quad (10.3a)$$

$$= \widetilde{\phi}(\mathbf{q}) = \left(\frac{k}{2\pi n}\right)^{N/2} \int_{\substack{\mathbf{p}\in\Re^N\\|\mathbf{p}|<n}} \frac{d^N\mathbf{p}}{\sqrt{1-p^2/n^2}}[\phi_+(\mathbf{p})+\phi_-(\mathbf{p})] \times \exp(ik\mathbf{p}\cdot\mathbf{q}/n), \quad (10.3b)$$

and the *normal derivative* at the screen is

$$\left.\frac{\partial\widetilde{\phi}(\vec{q})}{\partial q_N}\right|_{q_N=0} = \frac{ik}{n}\left(\frac{k}{2\pi n}\right)^{N/2} \times \int_{S_N} d^N S(\vec{p})\,p_N\,\phi(\vec{p})\exp(ik\vec{p}\cdot\vec{q}/n)\Big|_{q_N=0} \quad (10.3c)$$

$$= \widetilde{\phi}'(\mathbf{q}) = ik\left(\frac{k}{2\pi n}\right)^{N/2} \times \int_{\substack{\mathbf{p}\in\Re^N\\|\mathbf{p}|<n}} d^N\mathbf{p}\,[\phi_+(\mathbf{p})-\phi_-(\mathbf{p})]\exp(ik\mathbf{p}\cdot\mathbf{q}/n). \quad (10.3d)$$

The wavefunction $\widetilde{\phi}(\vec{q})$ is a solution of the Helmholtz equation $\sum_{i=1}^{N+1} \partial^2/\partial q_i^2 \ \phi(\vec{q}) = -k^2\phi(\vec{q})$ [10]. The inverse spectrum analysis of Helmholtz solutions (10.3*b*) requires both its value and its normal derivative at the screen, given by

$$\phi_\pm(\mathbf{p}) = \frac{1}{2}\left(\frac{k}{2\pi n}\right)^{N/2} \int_{\Re^N} d^N\mathbf{q} \left[\sqrt{1-p^2/n^2}\,\widetilde{\phi}(\mathbf{q}) \pm \frac{1}{ik}\widetilde{\phi}'(\mathbf{q})\right] \times \exp(-ik\mathbf{p}\cdot\mathbf{q}/n). \tag{10.4}$$

The Parseval relation between the $\mathcal{L}_2(\mathcal{S}_N)$ inner product (9.2*f*) over the full sphere defines a *non-local* inner product over the screen involving again both the wavefunction and its normal derivative thus [10]:

$$(\phi,\psi)_{\mathcal{L}_2(\mathcal{S}_N)} = \int_{\mathcal{S}_N} d^N S(\vec{p})\,\phi(\vec{p})^*\psi(\vec{p}) \tag{10.5a}$$

$$\begin{aligned} &= \left(\frac{k}{2\pi n}\right)^N \int_{\substack{\mathbf{p}\in\Re^N\\|\mathbf{p}|<n}} \frac{d^N\mathbf{p}}{\sqrt{1-p^2/n^2}} \\ &\quad\times \int_{\Re^N} d^N\mathbf{q} \int_{\Re^N} d^N\mathbf{q}' \exp(-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n) \\ &\quad\times \left[(1-p^2/n^2)\widetilde{\phi}(\mathbf{q})^*\widetilde{\psi}(\mathbf{q}') + \frac{1}{k^2}\widetilde{\phi}'(\mathbf{q})^*\widetilde{\psi}'(\mathbf{q}')\right] \end{aligned}$$

$$\begin{aligned} &= \left(\frac{k}{2\pi n}\right)^2 \int_{\Re^N} d^N\mathbf{q} \int_{\Re^N} d^N\mathbf{q}' \\ &\quad\times \Big[\omega(|\mathbf{q}-\mathbf{q}'|)\widetilde{\phi}(\mathbf{q})^*\widetilde{\psi}(\mathbf{q}') \\ &\quad\quad + \varpi(|\mathbf{q}-\mathbf{q}'|)\widetilde{\phi}'(\mathbf{q})^*\widetilde{\psi}'(\mathbf{q}')\Big] \end{aligned} \tag{10.5b}$$

$$= (\widetilde{\phi},\widetilde{\psi})_{\mathcal{H}_k^N}. \tag{10.5c}$$

The integral over the sphere $\mathcal{S}_N$ has been performed to yield two nonlocal weight functions, ω and ϖ, that are functions of the norm of the coordinate vector difference $|\mathbf{q}-\mathbf{q}'|$

$$\begin{aligned} \omega(|\mathbf{q}-\mathbf{q}'|) &= \frac{1}{2}\int_{\mathcal{S}_N} d^N S(\vec{p})\,(1-p^2/n^2)\,\exp(-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n) \\ &= \frac{1}{2}\int_0^n p^{N-1}\,dp\,\sqrt{1-p^2/n^2} \\ &\quad\times \int_{\mathcal{S}_{N-1}} d\omega_{\mathbf{p}} \exp(-ikp\,|\mathbf{q}-\mathbf{q}'|\cos[\angle(\mathbf{p},\mathbf{q}-\mathbf{q}')/n]) \\ &= \sqrt{\frac{\pi}{2}}\left(\frac{k\,|\mathbf{q}-\mathbf{q}'|}{2\pi n}\right) \\ &\quad \int_0^n p^{N/2}dp\,\sqrt{1-p^2/n^2}\,J_{N/2-1}(kp\,|\mathbf{q}-\mathbf{q}'|/n) \end{aligned}$$

$$= \tfrac{1}{4}(2\pi)^{(N+1)/2} n^N \frac{J_{(N+1)/2}(k\,|\mathbf{q}-\mathbf{q}'|)}{(k\,|\mathbf{q}-\mathbf{q}'|)^{(N+1)/2}}. \quad (10.6)$$

The integral ([21], Eq. 6.567.1) leads to a Bessel function of the first kind $J_m(x)$. Similarly, we find

$$\varpi(|\mathbf{q}-\mathbf{q}'|) = \frac{1}{2k^2}\int_{S_N} d^N S(\vec{p})\, \exp(-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n)$$

$$= \tfrac{1}{4}(2\pi)^{(N+1)/2} n^N \frac{1}{k^2}\frac{J_{(N-1)/2}(k\,|\mathbf{q}-\mathbf{q}'|)}{(k\,|\mathbf{q}-\mathbf{q}'|)^{(N-1)/2}}. \quad (10.7)$$

These equations may be checked with the results for $N = 1$ in Ref. [22], and for the case of ordinary optics $N = 2$ with Eqs. (10.10)–(10.13) in Ref. [10]. They involve the pleasant function $J_m(x)/x^m$ whose value at $x = 0$ is a maximum, $[2^m\Gamma(m+1)]^{-1}$; it decreases to its first zero[11] at $2.405, \pi, 3.832, 4.493$ for $m = 0, \frac{1}{2}, 1, \frac{3}{2}$ (for $N = 1$ dimension we need $m = 0$ and 1, for the case of ordinary optics $N = 2$, $m = \frac{1}{2}$ and $\frac{3}{2}$); thereafter it oscillates and decreases as $\sqrt{2/\pi}\, x^{-m-1}$ asymptotically. This is the picture of the nonlocality of the Helmholtz-Hilbert space $\mathcal{H}_k^N$ on the N-dimensional screen. The $\mathcal{H}_k^N$ space of functions does not support Dirac δ's as valid objects: images cannot be perfectly pointlike since such need $\mathbf{p} \in \Re^N$, and momentum ranges only on the subset $p < n$. The position coordinate is *not* a well defined concept in Euclidean optics as it is in Heisenberg-Weyl quantum mechanics.

The face of contact of Euclidean and Heisenberg-Weyl phase spaces is in the momentum subspace. *Plane waves* are well defined objects in both. With 'forward Heisenberg-Weyl direction' $\mathbf{P}_0 \in \Re^N$, the wave will appear in the $\mathbf{P}$-representation as

$$\Phi_{\mathbf{P}_0}(\mathbf{P}) = \delta^N(\mathbf{P}-\mathbf{P}_0), \quad (10.8a)$$

and on the $\mathbf{Q}$-screen as the complex amplitude (10.1a),

$$\widetilde{\Phi}_{\mathbf{P}_0}(\mathbf{Q}) = \left(\frac{k}{2\pi n}\right)^{N/2} \exp(i\,\mathbf{P}_0\cdot\mathbf{Q}). \quad (10.8b)$$

This is a periodic function with wavelength $\lambda = 2\pi n/kP_0$ between crests. The wavelength may range from infinity ($\mathbf{P}_0 = 0$, a ray paralell to the optical axis, its wavefronts paralell to the screen), through decreasing real values for increasing ray angles, down to zero for $P_0 \to \infty$ (this limit is nonrelativistic-mechanical, not optical).

[11] Except for π, the numbers are truncated to three decimals.

The (inverse) opening coma transformation asigns to $\Phi_{\mathbf{P}_0}$ through (9.4a) a Euclidean plane wave in momentum representation

$$\begin{aligned} \phi_{\mathbf{p}_0}(\mathbf{p}) &= |1-p^2/n^2|^{-(N+1)/4}\Phi_{\mathbf{P}_0(\mathbf{p}_0)}(\mathbf{P}(\mathbf{p})) \\ &= |1-p^2/n^2|^{-(N+1)/4}\delta^N\left(\frac{\mathbf{p}}{\sqrt{1-p^2/n^2}}-\frac{\mathbf{p}_0}{\sqrt{1-p_0^2/n^2}}\right) \\ &= |1-p^2/n^2|^{(N+3)/4}\delta^N(\mathbf{p}-\mathbf{p}_0). \end{aligned} \tag{10.9a}$$

This yields through (10.3b) that the Helmholtz solution on the screen is a plane wave in the Euclidean direction $\mathbf{p}_0 = \mathbf{P}_0/\sqrt{1+P_0^2/n^2}$,

$$\widetilde{\phi}_{\mathbf{p}_0}(\mathbf{q}) = \left(\frac{k}{2\pi n}\right)^{N/2}(1-p_0^2/n^2)^{(N+1)/4}\exp(ik\mathbf{p}_0\cdot\mathbf{q}/n). \tag{10.10a}$$

With the assumption that our Heisenberg-Weyl plane wave is in the *forward* hemisphere of ray directions, the normal derivative is given to us by (10.3d), and is

$$\begin{aligned} \widetilde{\phi}'_{\mathbf{p}_0}(\mathbf{q}) &= \left(\frac{k}{2\pi n}\right)^{N/2}(1-p_0^2/n^2)^{(N+3)/4}\exp(ik\mathbf{p}_0\cdot\mathbf{q}/n) \\ &= ik\sqrt{1-p_0^2/n^2}\,\widetilde{\phi}_{\mathbf{p}_0}(\mathbf{q}). \end{aligned} \tag{10.10b}$$

A backward-directed wave in Helmholtz optics has the opposite sign of the normal derivative. Linear combinations of forward and backward rays allows for *real* (or purely imaginary) standing waves throughout space, and permits the decoupling of the function from its normal derivative. This freedom can not be obtained in Heisenberg-Weyl optics for a single $\Re^N$ screen; there, we can at most linearly combine waves $\mathbf{P}_0$ and $-\mathbf{P}_0$ with real results on the screen, but complex oscillation elsewhere.

In the last formula there is a decrease factor of $(1-p_0^2/n^2)^{(N+1)/4}$ in the amplitude of the wave, which drops to zero for $p_0 = n$. A grid of Heisenberg-Weyl plane wave vectors in $\Re^N$ will map on a 'noncartesian' grid in the Descartes sphere in Escher-like distortion; as we draw near the equator the intensities will drop to zero.

Euclidean plane waves that are Dirac-normalized on $\mathcal{S}_N$ may be constructed from (10.3). These are [10]:

$$\widetilde{w}_{\mathbf{p}_0,\tau}(\mathbf{q}) = \left(\frac{k}{2\pi n}\right)^{N/2}\exp(ik\mathbf{p}_0\cdot\mathbf{q}), \tag{10.11a}$$

$$\widetilde{w}'_{\mathbf{p}_0,\tau}(\mathbf{q}) = ik\tau\left(\frac{k}{2\pi n}\right)^{N/2}\sqrt{1-p_0^2/n^2}\exp(ik\mathbf{p}_0\cdot\mathbf{q}). \tag{10.11b}$$

Their normalization is evident on momentum space

$$w_{\mathbf{p}_0,\tau}(\mathbf{p}) = \sqrt{1-p_0^2/n^2}\,\delta^N(\mathbf{p}-\mathbf{p}_0)\,\delta_{\tau,\mathrm{sign}\sqrt{}} = \delta^N_{\mathcal{S}_N}(\vec{p}-\vec{p}_0). \tag{10.11c}$$

In the corresponding Heisenberg-Weyl direction $\mathbf{P}_0 = \mathbf{p}_0/\sqrt{1-p_0^2/n^2}$, for the corresponding value of τ, the waves are

$$W_{\mathbf{P}_0}(\mathbf{P}) = (1+P_0^2/n^2)^{(N+1)/4}\,\delta^N(\mathbf{P}-\mathbf{P}_0), \tag{10.12a}$$

$$\widetilde{W}_{\mathbf{P}_0}(\mathbf{Q}) = \left(\frac{k}{2\pi n}\right)^{N/2}(1+P_0^2/n^2)^{(N+1)/4}\exp(ik\mathbf{P}_0\cdot\mathbf{Q}). \tag{10.12b}$$

Let us now consider Heisenberg-Weyl Dirac δ's on the screen, at points $\mathbf{Q}_0$, in their position and momentum representations,

$$\widetilde{D}_{\mathbf{Q}_0}(\mathbf{Q}) = \delta^N(\mathbf{Q}-\mathbf{Q}_0), \tag{10.13a}$$

$$D_{\mathbf{Q}_0}(\mathbf{P}) = \left(\frac{k}{2\pi n}\right)^{N/2}\exp(-ik\mathbf{P}\cdot\mathbf{Q}_0/n). \tag{10.13b}$$

The coma map (10.9) leads to the Euclidean wavefunction on the right momentum range:

$$d_{\mathbf{Q}_0}(\mathbf{p}) = \left(\frac{k}{2\pi n}\right)^{N/2}\frac{1}{(1-p^2/n^2)^{(N+1)/4}}\exp -i\frac{k}{n}\left(\frac{\mathbf{p}\cdot\mathbf{Q}_0}{\sqrt{1-p^2/n^2}}\right). \tag{10.14}$$

Finally, the image on the Euclidean screen is found through the integral in (10.3) to be

$$\widetilde{d}_{\mathbf{Q}_0}(\mathbf{q}) = \left(\frac{k}{2\pi n}\right)^{N}\int_{\Re^N}\frac{d^N\mathbf{p}}{(1-p^2/n^2)^{(N+3)/4}}\exp i\frac{k}{n}\left(\mathbf{p}\cdot\mathbf{q}-\frac{\mathbf{p}\cdot\mathbf{Q}_0}{\sqrt{1-p^2/n^2}}\right). \tag{10.15}$$

This integral, when and if evaluated, will be the kernel of the unitary operator of symplectic coma transformation between $\mathcal{L}_2(\Re^N)$ and $\mathcal{H}_k^N$. If $\widetilde{F}(\mathbf{Q})$ is a wavefunction calculated to carry an image in Heisenberg-Weyl optics, its actual image on a Euclidean screen will be

$$\widetilde{f}(\mathbf{q}) = \int_{\Re^N} d^N\mathbf{Q}\,\widetilde{d}_{\mathbf{Q}}(\mathbf{q})\,\widetilde{F}(\mathbf{Q}). \tag{10.16}$$

This integral is *not quite* a convolution because the kernel $\widetilde{d}_{\mathbf{Q}}(\mathbf{q})$ is *not quite* a function of $\mathbf{q}-\mathbf{Q}$.

When only rays at small angles to the optical axis are involved, then both momentum representation wavefunctions $F(\mathbf{P})$ and $f(\mathbf{p})$ are significantly different from zero only at a small neighborhood of $\mathbf{p}=\mathbf{0}$. The integral in (10.15) is then approximated with the integrand $\exp ik\,\mathbf{p}\cdot(\mathbf{q}-\mathbf{Q}_0)$, both in company with this $f(\mathbf{p})$. In the small wavelength limit ($k\to\infty$), the exponential oscillates everywhere except at $\mathbf{q}=\mathbf{Q}_0$ where it diverges as a Dirac δ in $\Re^N$. For small ray angles thus, Euclidean images are sharp, with no magnification or distortion. For realistic ranges of $\mathbf{P}$, the integral in (10.15) should be interesting to study as it approximates also the *Airy* function when the next higher term in the exponent is computed.

7.11 Gaussians, non-diffracting beams, and concluding remarks

Coherent states, correlated (*squeezed*) and discrete [4], and generally Gaussian beams [1], are prime fruit of the Heisenberg-Weyl model of optics. This is so because we may *complexify* the inhomogeneous linear symplectic transformations. Imaginary-z free propagation, generated by P^2, produces Gaussians out of the Dirac δ in $\Re^N$, just as heat diffusion does. The $\Re^N$-Fourier transform of a Gaussian is generically a Gaussian. As operators, neither $\mathbf{Q}$ nor $\mathbf{P}$ have sharp eigenvalues in this set of functions, but their dispersions satisfy the minimum of the Heisenberg uncertainty relation.

Through the coma map (9.4), the Heisenberg-Weyl Gaussians with exponents $-AP^2$ $(A > 0)$ and/or $\mathbf{B}\cdot\mathbf{P}$, will become exponentials of $-Ap^2/(1-p^2/n^2)$ and $\mathbf{B}\cdot\mathbf{p}/\sqrt{1-p^2/n^2}$, times the measure factor $(1-p^2/n^2)^{(N+1)/4}$. These functions, although not *quite* Gaussians (because their exponent is not simply $-ap^2$ and $\mathbf{B}\cdot\mathbf{p}$), do adequately go to zero at $|\mathbf{p}| \to n$; *i.e.*, approaching the *surface* of the solid N-sphere of Euclidean momenta. In the ordinary case $N = 2$, this is the 1-dimensional *rim* of the forward momentum disk, the equatorial circle. Since only forward (or backward) rays are involved, the Helmholtz solution and its normal derivative at the screen are related by a factor of $\pm ik\sqrt{1-p^2/n^2}$, as was the case with plane waves in (10.10b). The $\mathcal{H}_k^N$ and $\mathcal{L}_2(\mathcal{S}_N)$ inner product properties of these projected Heisenberg-Weyl coherent states will be the same as their well-known counterparts in $\mathcal{L}_2(\Re^N)$ because of the Parseval relation (9.2).

What is the natural counterpart of a Gaussian distribution on the surface of the Descartes sphere of ray directions $\mathcal{S}_N$? Take a Dirac δ in the context of $\mathcal{L}_2(\mathcal{S}_N)$ and let it *diffuse*, as if it were heat, with $\exp -w\Delta_{\mathcal{S}_N}$, where $\Delta_{\mathcal{S}_N}$ is the Laplace-Beltrami operator on $\mathcal{S}_N$. For small width w the Descartes $\mathcal{S}_N$-Gaussian looks like a Heisenberg-Weyl $\Re^N$-Gaussian centered on the original δ, but as w increases it asymptotically tends to a nonzero constant over the finite volume of $\mathcal{S}_N$ and both momentum charts are necessarily involved. For the case of $N = 1$ dimensional screens, we can point to Ref. [17] to show that this Gaussian is the heat kernel on a conducting ring $\mathcal{S}_1$ —one of the Jacobi ϑ-functions. The Fourier *series* decomposition of $\mathcal{L}_2(\mathcal{S}_1)$ yields naturally a description of the images on the screen in terms of equally-spaced *sampling points*; that is unitarily equivalent to the Hilbert space $\mathcal{H}_k^1$. The Euclidean Gaussian in this discrete image space is a *modified* Bessel function $I_j(kw)$ that, on the *integers* j, is symmetric about a smooth peak and falls off asymptotically $(|j| \to \infty)$ as an ordinary Gaussian of width w does. This takes us to the theory of coherent states on group manifolds and their sampling theorem, that we hope to develop elsewhere.

Among the subgroup-adapted wavefunction bases we should draw also special attention to the *diffraction-free* beams [23]. We may define these to be the Euclidean wavefunction, p–$\omega_{\mathbf{p}}$ separated solutions of the Helmholtz equation on the screen, that are eigenfunctions of the generator of translations along the $(N+1)^{\text{th}}$ axis (the optical z-axis), namely h^{τ} in (7.1b). We can treat the two charts separately and work only with *forward* beams; this fixes the normal derivative of the solution as for plane waves in (10.10b). The solution of $h\phi_a = \lambda(a)\phi_a$ in $\mathbf{p}$-space is a Dirac δ with support on a single value of the radius $|\mathbf{p}| = p = a$, times a function $\Upsilon(\omega_{\mathbf{p}})$ of the S_{N-1} angular variables in $\mathbf{p} = p\omega_{\mathbf{p}}$, times a normalization constant $\nu_E(a)$. In the ordinary $N = 2$ case the Dirac δ has support on a circle inside the momentum disk. In Refs. [23] a mask with an annular slit provided the source for a ray distribution that, after a Fourier transformer lens, provided a *diffraction-free* or J_0 beam, with a Bessel function amplitude on successive z-translated screens.[12]

In the N-dimensional case, a momentum wavefunction $\phi_a(p, \omega_{\mathbf{p}}, \tau = +)$ that contains $\delta(p-a)$ will cast an image $\widetilde{\phi}_a(\mathbf{q})$ given by (10.3a,b). The steps needed to decompose the integration into radius p and S_{N-1}-angles $\omega_{\mathbf{p}}$ follow closely the integration in (10.6), except that the integrand factor $(1-p^2/n^2)$, is replaced by $\delta(p-a)$, and we have plain $\mathbf{q}$ here instead of $(\mathbf{q}-\mathbf{q}')$. The angular integral over S_{N-1} now contains the angular function $\Upsilon(\omega_{\mathbf{p}})$. If this is taken to be a constant, the diffraction-free beam $\widetilde{\phi}_a(\mathbf{q})$ will have the form of a Bessel function $J_{N/2-1}(ka|\mathbf{q}|/n)$. In $N = 2$ dimensions this is the J_0 beam of reference [23].

If the angular function $\Upsilon(\omega_{\mathbf{p}})$ is *not* a constant, the diffraction-free beam $\widetilde{\phi}_a(\mathbf{q})$ will also exhibit angular dependence. In the ordinary case $N = 2$ this leads to nondiffracting solutions $e^{im\omega} J_{|m|}(kaq/n)$, where $m = 0, \pm 1, \pm 2, \ldots$ is the in-screen rotations' SO(2) irreducible representation label, and ω the single polar angle. For general N we may use SO(N) spherical harmonics as a basis where the vectors are classified by a complete set of labels following $\mathrm{SO}(N) \supset \mathrm{SO}(N-1) \supset \cdots \supset \mathrm{SO}(2)$. This set of labels will replace the single m. The SO(N) Casimir is found as $\frac{1}{2}\sum_{i,j}(R_{i,j})^2$ from (7.2a), corresponding to the well-known Petzval invariant $(\mathbf{p} \times \mathbf{q})^2$ in the $N = 2$ case. While $\mathbf{p} \times \mathbf{q}$ by itself yields m, it is the Casimir eigenvalue $\kappa^{\mathrm{SO}(N)} = \ell(\ell + N - 2)$, $\ell = 0, 1, 2, \ldots$, that will appear in the radial part as ℓ in $J_{N/2+\ell-1}(ka|\mathbf{q}|/n)$ for the general-N case.

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[12]The very interesting experimental property of such beams is that, due to the finiteness of the wavefront, the beam mantains its peaked J_0 shape up to a certain distance only; thereafter, the peak vanishes rather abruptly. Very much as Luke Skywalker's 3-foot-long red laser sword, and Darth Vader's blue one. (This would *still* not explain why such light swords would clash among themselves, unless a nonlinear régime is encountered.)

The Euclidean $q_{N+1} = z$-evolution is produced by $\exp(ikz\hat{h}/n)$. On its eigenfunction $\widetilde{\phi}_a(\mathrm{q})$ this yields a phase of $\exp(ikz\sqrt{1-a^2/n^2})$ where a is, as before, the direction $p = n\sin\theta_a$, $\theta_a = \arcsin(a/n) \leq \frac{1}{2}\pi$, of the constituent forward plane waves that form the nondiffracting beam. Finally, the normal derivative $\widetilde{\phi}'_a(\mathrm{q}, z)$ is $ik\sqrt{1-a^2/n^2}$ times $\widetilde{\phi}_a(\mathrm{q}, z)$.

Bessel functions of the first kind *also* appear in Heisenberg-Weyl optics, as was pointed out in Ref. [24] for the $N = 2$ case of 'physical' optics. Indeed, the subgroup reduction $\mathrm{Sp}(2N, \Re) \supset \mathrm{Sp}(2, \Re) \otimes \mathrm{SO}(N)$ in the oscillator realization has the same $\mathrm{SO}(N)$ generators $R_{i,j}$ of the Euclidean group (7.2a), the $\mathrm{Sp}(2, \Re)$ generators in (6.3), and the subgroups are *conjugate*, *i.e.*, their Casimir operators are related by $\kappa^{\mathrm{Sp}(2)} = -\frac{1}{4}\kappa^{\mathrm{SO}(N)} + \frac{1}{16}N(4-N)$. The value of the Bargmann label of $\mathrm{Sp}(2, \Re)$ is $\frac{1}{2}(\ell + \frac{1}{2}N)$. We look at the *eigenbasis* of $\frac{1}{2}P^2$, the generator of a *parabolic* orbit in $\mathrm{Sp}(2, \Re)$ [25] in the Q-realization. The eigenfunction corresponding to eigenvalue $\frac{1}{2}A^2 \in \Re^+$ contains a Bessel factor of $J_{N/2+\ell-1}(kA|\mathrm{Q}|/n)$, an $\mathcal{L}_2(\Re^N)$ normalization factor $\nu_R(A)$, and an $\mathrm{SO}(N) \subset \mathrm{SO}(N-1) \subset \ldots \subset \mathrm{SO}(2)$ set of labels for the angular part, just like their Euclidean counterparts, but for $A = a/\sqrt{1-a^2/n^2}$ now unbounded. The forward z-evolution is in this context the *paraxial* one, *i.e.*, generated by $H = P^2/2n$. This will produce on the wavefunction a phase of $\exp(-ikzA^2/2n^2)$. Comparing this with the phase $\exp(ikz\sqrt{1-a^2/n^2}) = \exp ikz\ \exp(-ikza^2/2n^2) \times \exp(-ikza^4/8n^4)\cdots$ obtained for Euclidean propagation, we see that the Heisenberg-Weyl wavefunction still needs the well-known (central) phase $\exp ikz$, and approximates well the phase anomaly $-ika^2/2n^2$ for $A \approx a \ll n$. It was the point of Ref. [25] to show that the Heisenberg-Weyl nondiffracting beams, subject to arbitrary paraxial transformations, generically mantain their dependence on a Bessel function (scaled by z and/or multiplied by an oscillating Gaussian), *i.e.*, the beams *self-reproduce* under the group of paraxial transformations.

The above two classes of beams have been given as illustrations of the opening coma map between Euclidean and Heisenberg-Weyl wave optics. The latter has served for many developments that still await to be defined on the former. Signal theory, in particular its formulation through the Wigner phase space distribution function and the Gabor expansion [26], do not seem applicable as such to wide-angle wave optics. For example, we may inquire about the optimal sampling-point set for Helmholtz solutions for 4π or stopped beams, with or without axial symmetry, or about the natural spread of coherent states and the behaviour of their correlation (*squeezing*). The purpose of illustration is served by leaving these subjects for further development elsewhere.

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7.13 REFERENCES

[1] V.I. Man'ko and K.B. Wolf, The influence of spherical aberration on gaussian beam propagation. In *Lie Methods in Optics*, ed. by J. Sánchez-Mondragón and K.B. Wolf. Lecture Notes in Physics, Vol. 250 (Springer Verlag, Heidelberg, 1986); K.B. Wolf and V.I. Man'ko, *ibid.* [in Russian]. Trudy Fiz. Inst. P.N. Lebedev, Vol. **176** (Nauka, Moscow, 1986); translated in *Classical and Quantum Effects in Electrodynamics*, ed. by A.A. Komar (Nova Science Publ., Commack N.Y., 1988), pp. 169–200.

[2] R. Glauber, Coherent states of the quantum oscillator, *Phys. Rev. Lett.* **10**, 84 (1963).

[3] *See for example*, M.M. Nieto, What are squeezed states really like? In *Frontiers of Nonequilibrium Statistical Physics*, ed. by G.T. Moore and M.O. Scully (Plenum Publ. Corp, New York, 1986), pp. 287–307.

[4] V.V. Dodonov, E.V. Kurmyshev, and V.I. Man'ko Exact bounds for the uncertainty relation in correlated coherent states [in Russian]. In *Group-theoretical Methods in Physics*, Proceedings of the Zvenigorod Seminar, November 1979. (Nauka, Moscow, 1980).

[5] V.V. Dodonov, E.V. Kurmyshev, and V.I. Man'ko Generalized uncertainty relation and correlated coherent states. *Phys. Lett.* **A79**, 150–152 (1980).

[6] A.J. Dragt, E. Forest, and K.B. Wolf, Foundations of a Lie algebraic theory of geometrical optics. In *Lie Methods in Optics*, *op. cit.*

[7] K.B. Wolf, The Heisenberg–Weyl ring in quantum mechanics. In *Group Theory and its Applications*, Vol. III, ed. by E.M. Loebl (Academic Press, New York, 1975), pp. 189–247.

[8] H. Raszillier and W. Schempp, Fourier optics from the perspective of the Heisenberg group. In *Lie Methods in Optics, op. cit.*

[9] J.A. Arnaud, *Beam and Fiber Optics* (Academic Press, New York, 1976).

[10] K.B. Wolf, Elements of Euclidean Optics. In this volume.

[11] K.B. Wolf, Symmetry in Lie optics, *Ann. Phys.* **172**, 1–25 (1986).

[12] N.M. Atakishiyev, W. Lassner, and K.B. Wolf, The relativistic coma aberration. I. Geometrical optics. *Comunicaciones Técnicas IIMAS* No. 509 (1988); *ib.* II. Helmholtz wave optics. No. 517 (1988), to appear in *Journal of Mathematical Physics.*

[13] T. Sekiguchi and K.B. Wolf, The Hamiltonian formulation of optics, *Am. J. Phys.* **55**, 830–835 (1987).

[14] K.B. Wolf, Nonlinearity in aberration optics. In *Symmetries and Nonlinear Phenomena*, Proceedings of the International School on Applied Mathematics, Centro Internacional de Física, Paipa, Colombia, 22–26 Feb. 1988, ed. by D. Levi and P. Winternitz, CIF Series Vol. 9 (World Scientific, Singapore, 1989).

[15] O.N. Stavroudis, *The Optics of Rays, Wavefronts, and Caustics* (Academic Press, New York, 1972); Eq. (II-19) on p. 26.

[16] S. Steinberg, Lie series, Lie transformations, and their applications. In *Lie Methods in Optics, op. cit.*; pp. 45–103.

[17] K.B. Wolf, A Euclidean algebra of Hamiltonian observables in Lie optics, *Kinam* **6**, 141–156 (1985).

[18] A.J. Dragt, A Lie algebraic theory of geometrical optics and optical aberrations. *J. Opt. Soc. Am.* **72**, 372–379 (1982).

[19] J.R. Klauder, Wave theory of imaging systems. In *Lie methods in optics, op. cit.* pp. 183–191; Eqs. (11) on p. 186.

[20] F. Leyvraz and T.H. Seligman, Sequences of point transformations and linear canonical transformations in classical and quantum mechanics. Preprint IFUNAM (1988), to appear in *J. Phys.*

[21] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, sums, series and products* (Academic Press, New York, 1975).

[22] S. Steinberg and K.B. Wolf, Invariant inner products on spaces of solutions of the Klein-Gordon and Helmholtz equations, *J. Math. Phys.* **22**, 1660–1663 (1981).

[23] J. Durnin, J.J. Miceli, and J.H. Eberly, Diffraction-free beams. *Phys. Rev. Lett.* **58**, 1499–1501 (1987); J. Durnin, Exact Solutions for non-diffracting beams. I. The scalar theory. *J. Opt. Soc. Am. A* **4**, 651–654 (1987).

[24] K.B. Wolf, Diffraction-free beams remain diffraction-free under all paraxial optical transformations. *Phys. Rev. Lett.* **60**, 757–759 (1988).

[25] D. Basu and K.B. Wolf, The unitary irreducible representations of *SL(2,R)* in all subgroup reductions. *J. Math. Phys.* **23**, 189–205 (1982).

[26] M.J. Bastiaans, *Local-Frequency Description of Optical Signals and Systems*, Eindhoven University of Technology report 88-E-191 (April, 1988); lectures delivered at the First International School and Workshop on Photonics (Oaxtepec, June 28 – July 8, 1988).