6

Normal Mode Expansion and Bessel Series

The eigenfunctions of the Laplacian operator in function spaces with certain sets of boundary conditions constitute orthogonal sets of functions on the region enclosed by the boundaries. This is developed in Section 6.1 for rectangular boundaries and in Sections 6.2 and 6.3 for circular, sectorial, and annular boundaries in the plane. These are a few of the systems which appear in physics and engineering, where a great variety of operators and boundaries occur. The Laplacian applies mainly to wave and diffusion phenomena, which makes it specially relevant. As for boundary value problems, the above have been chosen for simplicity and because Fourier and *Bessel* series appear. Bessel series are a family of expansions in terms of orthonormal sets of functions which include those of Fourier as a particular case. In Section 6.4 we give a broad survey of the variants of eigenfunction expansions and some references.

6.1. Eigenfunctions of the Laplacian on Finite Regions: The Rectangular Membrane

Chapter 5 dealt with one-dimensional problems of diffusion and vibration where the key element was the expansion of the solution in series of eigenfunctions of the Laplacian with boundary conditions which restricted the "physical" space to a region of finite (*compact*) extent: 2π for the conducting ring and L for the fixed-end vibrating string. Here we shall see some general features of these expansions in more than one dimension. If the boundary conditions are given on certain coordinate lines or surfaces, the solutions can be obtained exactly in terms of known functions. In this section we shall refer mostly to Cartesian coordinates, while in the rest of this chapter polar coordinates in the plane will be used.

6.1.1. Vector Spaces of Functions on $\mathscr{R}^N$

In working with the space of functions of N variables

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathscr{R}^N$$

we can endow it with a sesquilinear inner product which is the natural extension of the one-variable function space inner product (4.7), namely

$$(\mathbf{f}, \mathbf{g})_R := \int_R d^N\mathbf{x} f(\mathbf{x})^* g(\mathbf{x}), \tag{6.1}$$

where R is a region in N-dimensional space which for simplicity we consider to be a connected subset of Euclidean space with finite volume. The set of functions with finite norm [i.e., $\|\mathbf{f}\| := (\mathbf{f}, \mathbf{f})_R^{1/2} < \infty$ with (6.1) being a *Lebesgue* integral] which *vanish* on the boundary B of R can be shown to be in a Hilbert space. We denote it by $\mathscr{L}_0{}^2(R)$.

6.1.2. The N-Dimensional Laplacian

In $\mathscr{L}_0{}^2(R)$, the N-dimensional Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \cdots + \frac{\partial^2}{\partial x_N{}^2} \tag{6.2}$$

is self-adjoint. The weaker condition of hermiticity is easy to prove—without reference to Cartesian coordinates—by integration by parts using the Gauss theorem for $\mathbf{f}, \mathbf{g} \in \mathscr{C}^{(2)}$,

$$\begin{aligned}(\mathbf{f}, \nabla^2\mathbf{g})_R &= \int_R d^N\mathbf{x} f(\mathbf{x})^* \nabla \cdot \nabla g(\mathbf{x}) \\ &= \oint_B d^{N-1}\mathbf{s} \cdot f(\mathbf{x})^* \nabla g(\mathbf{x}) - \int_R d^N\mathbf{x} [\nabla f(\mathbf{x})]^* \cdot \nabla g(\mathbf{x}) \\ &= \oint_B d^{N-1}\mathbf{s} \cdot \{f(\mathbf{x})^* \nabla g(\mathbf{x}) - [\nabla f(\mathbf{x})]^* g(\mathbf{x})\} + \int_R d^N x [\nabla^2 f(\mathbf{x})]^* g(\mathbf{x}) \\ &= (\nabla^2\mathbf{f}, \mathbf{g})_R,\end{aligned} \tag{6.3}$$

where as usual $d^{N-1}\mathbf{s}$ is the directed surface element of B. The vanishing of the boundary term is due to the restriction $f(\mathbf{x}), g(\mathbf{x}) = 0$ at $\mathbf{x} \in B$ which has been assumed in defining $\mathscr{L}_0{}^2(R)$.

6.1.3. The Laplacian Eigenbasis

The hermiticity property (6.3) is sufficient to guarantee that if we find the eigenvectors of ∇^2 in $\mathscr{L}_0{}^2(R)$,

$$\nabla^2 \varphi_{\mathbf{n}}(\mathbf{x}) = \lambda_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x}), \tag{6.4}$$

their eigenvalues $\lambda_{\mathbf{n}}$ will be *real*, and any two eigenvectors corresponding to different eigenvalues will be *orthogonal*. The proof of these facts follows (1.106). Pending its exact specification, the label $\mathbf{n}$ attached to the eigenvectors and eigenvalues in (6.4) will be assumed to belong to a denumerable set $\mathscr{N}$. Actually, one can in all cases establish a natural correspondence between $\mathscr{N}$ and $\mathscr{Z}^N$, N-dimensional vectors $\mathbf{n}$ of integer components. The set $\{\boldsymbol{\varphi}_{\mathbf{n}}\}_{\mathbf{n}\in\mathscr{N}}$ can then be chosen *orthonormal* by appropriate normalization. The fact that ∇^2 has the stronger property of being self-adjoint has the consequence of allowing the statement that $\{\boldsymbol{\varphi}_{\mathbf{n}}\}_{\mathbf{n}\in\mathscr{N}}$ is not only an orthogonal set but a *complete basis* for $\mathscr{L}_0{}^2(R)$, i.e., any function $f(\mathbf{x})$ in this space can be expanded as

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathscr{N}} f_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x}), \tag{6.5a}$$

with *generalized Fourier coefficients*

$$f_{\mathbf{n}} := (\boldsymbol{\varphi}_{\mathbf{n}}, \mathbf{f})_R = \int_R d^N\mathbf{x}\, \varphi_{\mathbf{n}}(\mathbf{x})^* f(\mathbf{x}), \tag{6.5b}$$

and the Parseval identity holds in the form

$$(\mathbf{f}, \mathbf{g})_R = \int_R d^N\mathbf{x}\, f(\mathbf{x})^* g(\mathbf{x}) = \sum_{\mathbf{n}\in\mathscr{N}} f_n^* g_n. \tag{6.5c}$$

Exercise 6.1. Note that the vanishing of the functions on the boundary B of the region R is not necessary to guarantee the *hermiticity* of the Laplacian in (6.3). It is only necessary that the surface integral over B vanish. This can be brought about if the *directional derivatives* of the functions involved along the normal to B are proportional to the functions themselves, i.e., $d^{N-1}\mathbf{s}\cdot\nabla f(\mathbf{x}) = \sigma f(\mathbf{x})\, d^{N-1}\mathbf{s}$, where σ can depend on the points of B where it is taken. We have treated the case $\sigma = \infty$. The case $\sigma = 0$ corresponds to functions whose normal derivative vanishes at B.

The reader can see that for the one-dimensional case the ordinary sine Fourier series (4.134) is described by (6.5) with $R = (0, L)$, $B = \{0, L\}$, $\varphi_n(x) = (2/L)^{1/2} \sin(n\pi x/L)$ and $n \in \mathscr{Z}^+$.

6.1.4. Boundary Conditions along Cartesian Coordinates

Our next example concerns N-dimensional space when the region R is a hyperprism $R_\square$ extending along Cartesian axes x_j from 0 to L_j, $j = 1, 2, \ldots, N$.

In Cartesian coordinates ∇^2 has the form (6.2). It is simplest to solve the eigenfunction equation (6.4) by proposing *separable* solutions of the form $\varphi(\mathbf{x}) = X_1(x_1)X_2(x_2)\cdots X_N(x_N)$, substituting them into (6.2)–(6.4), applying the Leibnitz rule, and dividing by $\varphi(\mathbf{x})$. We obtain

$$X_1^{-1}X_1'' + X_2^{-1}X_2'' + \cdots + X_N^{-1}X_N'' = \lambda, \tag{6.6}$$

where primes indicate differentiation with respect to the function's argument. Every summand $X_j^{-1}X_j''$ can depend only on x_j so its transfer to the right-hand side would leave an equality between a function of x_j and a sum of functions of all x's *but* x_j. Hence every summand $X_j^{-1}X_j''$ can only be a constant λ_j and $\lambda_\mathbf{n} = \sum_{j=1}^{N} \lambda_j$. The independent eigenfunction equations we are left with are $X_j''(x_j) = \lambda_j X_j(x_j)$ with the boundary conditions $X_j(0) = 0 = X_j(L_j)$, $j = 1, 2, \ldots, N$. Their solution has been given in Section 5.2, so we can write the $\varphi_\mathbf{n}(\mathbf{x})$ in (6.4) as

$$\varphi_\mathbf{n}(\mathbf{x}) = (2^N/L_1L_2\cdots L_N)^{1/2}\sin(n_1\pi x_1/L_1)\sin(n_2\pi x_2/L_2)\cdots\sin(n_N\pi x_N/L_N) \tag{6.7a}$$

and label the function by the N-tuple

$$\mathbf{n} := (n_1, n_2, \ldots, n_N), \qquad n_j \in \mathscr{Z}^+, j = 1, 2, \ldots, N. \tag{6.7b}$$

In the solution process we have found $\lambda_j = -(n_j\pi/L_j)^2$, so that the eigenvalue corresponding to (6.7) is

$$\lambda_\mathbf{n} = -\pi^2 \sum_{j=1}^{N} (n_j/L_j)^2. \tag{6.8}$$

The spectrum of ∇^2 in $\mathscr{L}_0{}^2(R_\square)$ with $R_\square$ as described here is then the set of all $\lambda_\mathbf{n}$ for $n_j \in \mathscr{Z}^+$. We note that all values in the spectrum in (6.4) are *negative*.

6.1.5. Mode Labeling Degeneracy

It should also be noted, however, that the numerical value of $\lambda_\mathbf{n}$ may not label the eigenfunction uniquely. Assume all L_j's are equal so $R_\square$ is a hypercube. Then clearly any permutation of n_j's will yield the same value of $\lambda_\mathbf{n}$. This situation is referred to as *degeneracy* and has been mentioned before in Section 1.7, where we pointed out that in order to *resolve* the degeneracy and provide a unique numerical labeling for the eigenfunctions, hermitian operators commuting with the first one had to be found. In the process of deriving (6.7) we have used $\partial^2/\partial x_j{}^2$, $j = 1, 2, \ldots, N$, as the N labeling operators. They are all self-adjoint and obviously commute with each other. Any $N - 1$

linear combinations of these and ∇^2—itself such a linear combination—thus provide a commuting set whose common eigenfunctions (6.7) are a complete and orthonormal basis for $\mathscr{L}_0{}^2(R_\square)$. It is the set of eigenvalues $\{\lambda_j\}_{j=1}^N$ which labels the eigenbases uniquely. This is equivalent to their specification by the N-tuple $\mathbf{n} = \{n_1, n_2, \ldots, n_N\}$, $n_j \in \mathscr{Z}^+$.

6.1.6. The Two-Dimensional Case

The use which can be made of the eigenfunctions and values of ∇^2 in $\mathscr{L}^2(R)$ has been shown in Sections 5.1 and 5.2 for the heat and wave equations. Let us now proceed along the same lines briefly to analyze the vibrations of a two-dimensional rectangular elastic membrane. The extension to a prismoidal three-dimensional cavity or higher-dimensional such systems will then be evident. Any function $f(x_1, x_2, t)$ on $R_\square$ can be expanded in the functions (6.7) with coefficients (6.5b) which are time dependent:

$$f(x_1, x_2, t) = (4/L_1 L_2)^{1/2} \sum_{n_1 n_2 \in \mathscr{Z}^+} f_{n_1 n_2}(t) \sin(n_1 \pi x_1/L_1) \sin(n_2 \pi x_2/L_2). \tag{6.9}$$

For the function (6.9) to be a solution of the two-dimensional wave equation, its Fourier coefficients must satisfy [as for (5.22)]

$$c^{-2} \frac{\partial^2}{\partial t^2} f_{n_1 n_2}(t) = \lambda_{n_1 n_2} f_{n_1 n_2}(t), \tag{6.10}$$

i.e., they are oscillatory functions of time,

$$f_{n_1 n_2}(t) = b_{n_1 n_2} \sin[\omega_{n_1 n_2}(t - t_0)] + c_{n_1 n_2} \cos[\omega_{n_1 n_2}(t - t_0)] \tag{6.11a}$$

with angular frequency

$$\omega_{n_1 n_2} := c(-\lambda_{n_1 n_2})^{1/2} = \pi c[(n_1/L_1)^2 + (n_2/L_2)^2]^{1/2} \tag{6.11b}$$

and constants $b_{n_1 n_2}$, $c_{n_1 n_2}$ which can be fixed by the initial conditions at time t_0.

6.1.7. Nodal Lines, Frequency Lattice, and Accidental Degeneracy in the Two-Dimensional Case

Rather than analyze the Green's function (which will be discussed in Chapter 8), we shall point out some features of the normal modes

$$\dot{\varphi}_{n_1 n_2}(x_1, x_2, t) := (4/L_1 L_2)^{1/2} \sin(n_1 \pi x_1/L_1) \sin(n_2 \pi x_2/L_2) \cos \omega_{n_1 n_2} t \tag{6.12}$$

and their time antiderivatives $\varphi_{n_1 n_2}(x_1, x_2, t)$: (a) They are waveforms which start from rest and maximum elongation and from equilibrium and maximum

separable sol's in x & t

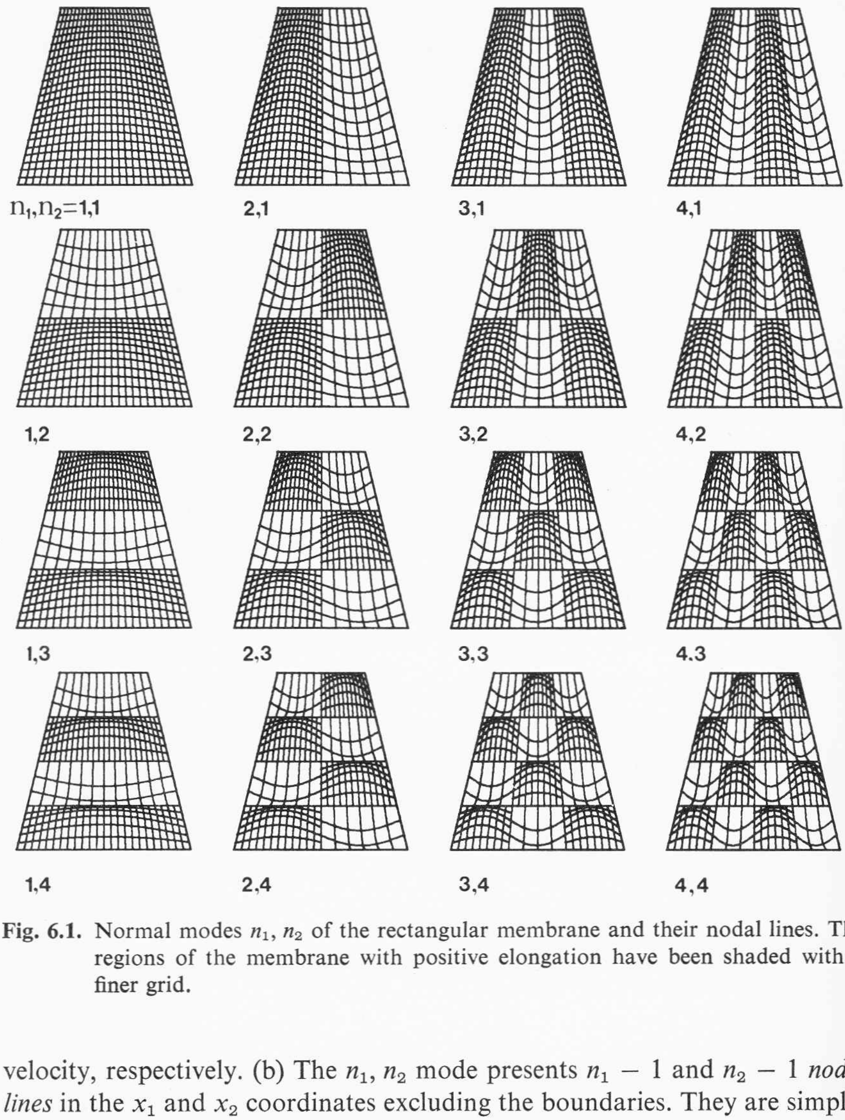


Fig. 6.1. Normal modes n_1, n_2 of the rectangular membrane and their nodal lines. The regions of the membrane with positive elongation have been shaded with a finer grid.

velocity, respectively. (b) The n_1, n_2 mode presents $n_1 - 1$ and $n_2 - 1$ *nodal lines* in the x_1 and x_2 coordinates excluding the boundaries. They are simple-zero lines within the membrane walls, across which the functions change sign (Fig. 6.1). The nodal lines are fixed in time. (c) The normal modes oscillate with angular frequencies $\omega_{n_1 n_2}$ as given by (6.11b) which are discrete and whose allowed values form a two-dimensional *lattice*. See Fig. 6.2. (d) The corresponding periods are $T_{n_1 n_2} = 2\pi/\omega_{n_1 n_2} = mT_{mn_1, mn_2}$. Thus in Fig. 6.2 the periods of modes lying on straight lines passing through the origin are multiples of a fundamental period $T_{p_1 p_2}$ with p_1 and p_2 relatively prime. (e) "Accidental" degeneracies can occur whenever L_1 and L_2 are commensurable. In Fig. 6.3 are some low-lying degeneracies for a 1:2 ratio of the rectangle sides. As the corresponding modes have the same angular velocity, so will any linear combination of them. These give rise to *degenerate subspaces* of modes, the elements of which have fixed nodes which are not straight lines. In Fig. 6.4 is a sequence of such linear combinations. Note that the *total number of nodal lines is conserved.* (f) The **x**-dependent factors of the normal modes (6.12) are orthonormal under the inner product

$$(\boldsymbol{\varphi}_{n_1n_2}, \boldsymbol{\varphi}_{n_1'n_2'})_\square := \int_{R_\square} d^2\mathbf{x}\varphi_{n_1n_2}(\mathbf{x})^*\varphi_{n_1'n_2'}(\mathbf{x}) = \delta_{n_1n_1'}\delta_{n_2n_2'}. \tag{6.13}$$

Exercise 6.2. Consider a membrane in the form of a narrow annulus of radius ρ and width ω. A fair description of the vibration characteristics is to assume that the radial functions are those of a string of length ω with fixed ends and the angular functions are *periodic* with period $2\pi\rho$. What is the relevant inner product? Is ∇^2 hermitian in such a region? Show that the mode labels would be (n, m), $n \in \mathscr{Z}^+$ labeling the radial functions and $m \in \mathscr{Z}$ the angular ones.

Exercise 6.3. Analyze the modes and oscillation frequency degeneracy of a vibrating cubic cavity. Note that there is degeneracy between $\omega_{n_1n_2n_3}$ and the ω's with the same permuted indices. Relate this to the fact that the system—differential equation and boundary conditions—is *invariant* under symmetry transformations of the cube.

Exercise 6.4. Show that the degeneracies of the oscillation frequency fix uniquely the ratios of the sides of the vibrating membrane or cavity.

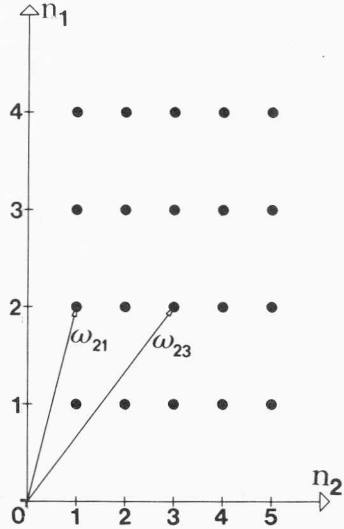


Fig. 6.2. "Reciprocal" lattice of allowed frequency values $\omega_{n_1n_2}$ for a rectangular membrane with a length ratio $L_1:L_2 :: 1:2$. The distance from the origin to each of the points gives the magnitude of $\omega_{n_1n_2}$.

The method of separation of variables in finding the eigenfunctions of the Laplacian operator in more than one variable will be applied in Section 6.2 to polar coordinates in the plane. At the end of this chapter we shall add some remarks on other coordinate systems in the plane where this is possible. In each case, when the region R is finite, the spectrum of ∇^2 is negative and the eigenfunctions are orthogonal, giving rise to a corresponding generalized Fourier series.

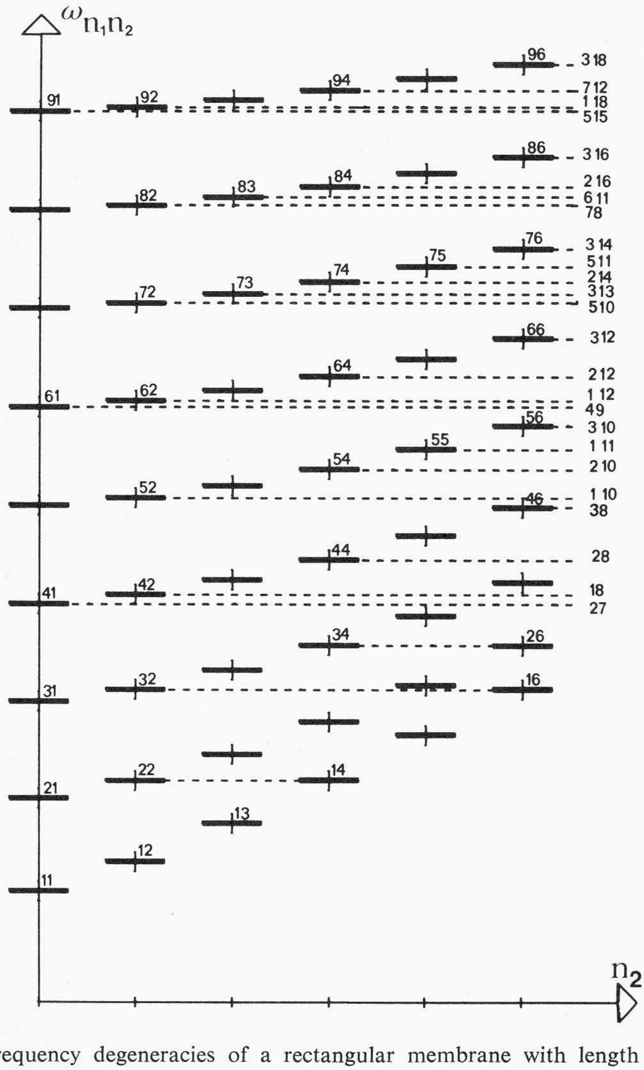


Fig. 6.3. Frequency degeneracies of a rectangular membrane with length ratio 1:2. Dashed lines join the degenerate pairs of $\omega_{n_1 n_2}$'s. If a partner lies beyond $n_2 = 6$, only the n_1, n_2 values are indicated.

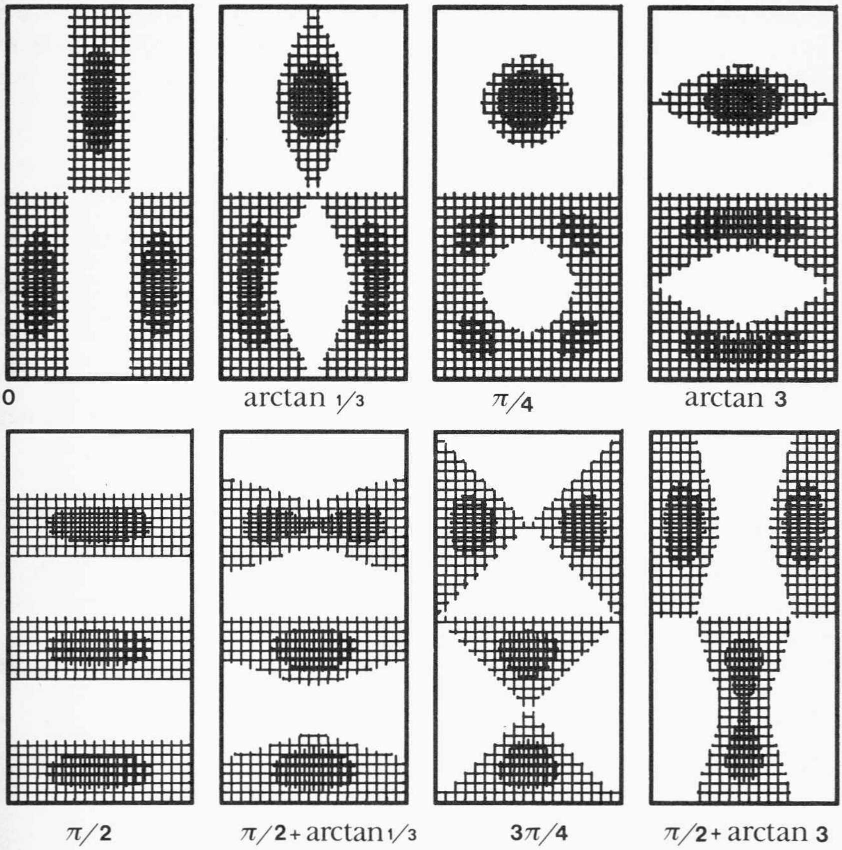


Fig. 6.4. Degenerate subspaces of normal modes. The modes $\boldsymbol{\varphi}_{3,1}$ and $\boldsymbol{\varphi}_{1,6}$ of a 1:2 rectangular membrane have the same angular frequency $(10)^{1/2}$. We plot here the linear combinations $\boldsymbol{\varphi}_\theta = \cos\theta\boldsymbol{\varphi}_{31} + \sin\theta\boldsymbol{\varphi}_{16}$ for various selected values of θ indicated below each figure. Blank and lightly shaded regions indicate negative and positive values of $\boldsymbol{\varphi}_\theta$, the fixed nodal lines being the boundaries. Heavily shaded areas indicate values of $\boldsymbol{\varphi}_\theta$ larger than 0.6.

6.2. Laplacian on the Unit Disk: The Circular Membrane

The eigenfunctions of the Laplacian operator will be found now when the domain is the space of square-integrable functions on the unit disk, which vanish on its boundary circle. In polar coordinates, we shall see that these consist of circular functions for the angular variable times *Bessel* functions for the radial part. The former have been treated extensively in Chapter 4 as the basis functions for the Fourier series expansion, while the latter (a summary of whose properties can be found in Appendix B) are a basis for one of the Bessel series expansions. The product of the two functions provides the spatial part of normal modes for the description of a vibrating circular membrane.

6.2.1. Polar Coordinates

When the region R in (6.1) is the unit disk $R_\bigcirc$, it is convenient to parametrize the plane in *polar* coordinates,

$$x_1 = r\cos\phi, \qquad x_2 = r\sin\phi, \qquad r\in[0,\infty),\ \phi\in(-\pi,\pi], \tag{6.14a}$$

$$d^2\mathbf{x} = dx_1\,dx_2 = r\,dr\,d\phi, \tag{6.14b}$$

so that the inner product between two functions on this region can be written as

$$(\mathbf{f},\mathbf{g})_\bigcirc = \int_0^1 r\,dr\int_{-\pi}^{\pi} d\phi f(r,\phi)^* g(r,\phi). \tag{6.15}$$

The space of functions with finite norm [induced by (6.15)] which vanish for $r = 1$ will be denoted again as $\mathscr{L}_0{}^2(R_\bigcirc)$. It is a Hilbert space. The functions in this space are of course periodic in ϕ with period 2π.

The expression for the Laplacian operator in polar coordinates is well known to be

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}. \tag{6.16}$$

Exercise 6.5. Verify directly that (6.16) is hermitian. This is just (6.3) in coordinate form using (6.15) and $\mathbf{f},\mathbf{g}\in\mathscr{C}^{(2)}$.

6.2.2. Separation of Variables

To solve the eigenfunction equation for (6.16) on $\mathscr{L}_0{}^2(R_\bigcirc)$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right) f_{\mathbf{n}}(r,\phi) = -\lambda_{\mathbf{n}} f_{\mathbf{n}}(r,\phi), \tag{6.17}$$

we propose separable solutions $f(r,\phi) = R(r)\Phi(\phi)$. We have put a minus sign in front of the $\lambda_{\mathbf{n}}$ on the basis of the observation in (6.8) that the spectrum of ∇^2 there was negative. By substituting the proposed solution form in (6.17), applying the Leibnitz rule, and dividing by $r^{-2}f(r,\phi)$, the equation is transformed into

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} - \lambda r^2 = c = -\frac{\Phi''(\phi)}{\Phi(\phi)}. \tag{6.18}$$

As before, the purpose of the separation of variables method is to be able to write Eq. (6.17) in a form (6.18) in which one side depends only on one variable and the other side only on the other, independent, variable. Both sides can only be equal to the same constant c, and we are left with two ordinary differential equations coupled by the separation constant. As the functions in $\mathscr{L}_0{}^2(R_\bigcirc)$ are to be periodic in ϕ, the right-hand side yields the well-known circular functions

$$\Phi_m{}^\bigcirc(\varphi) = (2\pi)^{-1/2} \exp(im\phi), \qquad m \in \mathscr{Z}, \tag{6.19}$$

fixing the separation constant as $c = m^2$ and providing one label for the Laplacian eigenfunctions. The left-hand side of (6.18) then becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right) R_m(r) = -\lambda R_m(r). \tag{6.20}$$

6.2.3. General Solution of the Radial Part

Upon the simple change of scale $\lambda^{1/2} r \to r$, Eq. (6.20) is *Bessel's* differential equation. (See Appendix B.) The general solution of (6.20) is then

$$R_m(r) = a_m J_m(\lambda^{1/2} r) + b_m N_m(\lambda^{1/2} r), \tag{6.21}$$

where a_m and b_m are as yet arbitrary constants, m is an integer, and J_m and N_m are the *Bessel* and *Neumann* functions of order m. [These are also called Bessel functions of the *first* and *second* kind; see the National Bureau of Standards tables edited by Abramowitz and Stegun (1964). There, the symbol Y_m is employed for the latter, for which the name *Weber* function is also occasionally used, as in Watson's classic treatise (1922). In mathematical physics, however, Neumann's name seems to be more popular. See Morse and Feshbach (1953)].

6.2.4. Boundary Conditions and Frequency Quantization

We now require the space and boundary conditions to hold for Eq. (6.21). A first observation (Figs. B.1 and B.2) is that the Neumann function becomes infinity at $r = 0$ and is in fact not square-integrable, so it cannot belong to $\mathscr{L}_0{}^2(R_\bigcirc)$ and therefore, *unless* $b_m = 0$, neither will $R_m(r)$. A second remark is that (6.20) is the same equation for $+m$ and $-m$. No essential features in (6.21) distinguish between the two since, for integer m, $J_{-m} = (-1)^m J_m$. The third argument, we note, is that the boundary condition of the function's vanishing at the membrane edge, $R_m(1) = 0$, fixes the allowed values of λ and thereby the spectrum of the Laplacian. Indeed, this condition implies $J_m(\lambda^{1/2}) = 0$. Now, this is clearly valid only if $\lambda^{1/2}$ is a *zero* of the Bessel

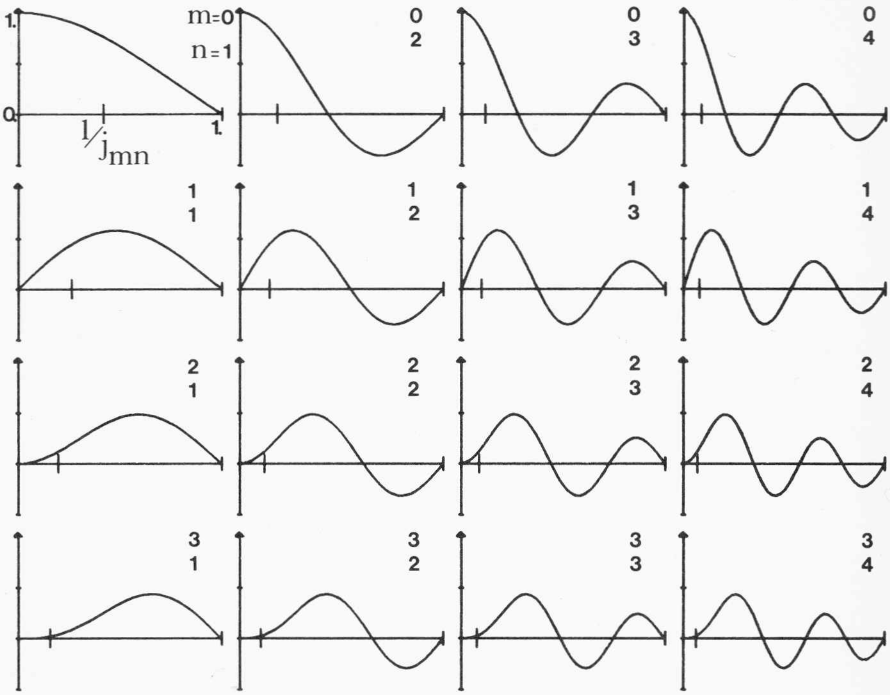


Fig. 6.5. Radial functions in the circular membrane normal modes φ°_{mn}. The left edge is the membrane center, while the right edge is the fixed boundary. All $m \neq 0$ modes are zero at the former, and all vanish at the latter, coinciding with the nth zero of the Bessel function.

function. The Bessel function of any order m has a denumerable infinity of simple zeros. [A small table of the first few is given in Appendix B. A more complete list can be found in Abramowitz and Stegun (1964, Table 9.5).] The effect of this condition is then to fix

$$\lambda_{mn} = (j_{m,n})^2, \qquad m \in \mathscr{Z}, n \in \mathscr{Z}^+, \tag{6.22}$$

where j_{mn} is the nth *zero of the Bessel function of order m* (*not* including $r = 0$). The effect of λ_{mn} when placed in the Bessel function in (6.21) is to change the scale in the argument of $R_m(r)$ so that for $n = 1, 2, \ldots$ the nth zero of the Bessel function coincides with the region's edge at $r = 1$. This is shown in Fig. 6.5. Finally the restriction (6.22) also provides a second label, n, to mark uniquely the eigenvalues and functions. The latter are thus recombined from (6.19) and (6.21) as

$$\varphi^{\bigcirc}_{mn}(r, \phi) := c^{\bigcirc}_{mn} J_m(j_{mn} r) \exp(im\phi), \qquad m \in \mathscr{Z}, n \in \mathscr{Z}^+. \tag{6.23a}$$

The constant $c^{\bigcirc}_{mn}$ in (6.23a) is introduced in order to *normalize* the functions with respect to the inner product (6.14). It can be shown to be

$$c^{\bigcirc}_{mn} = (2\pi)^{-1/2} g_{mn}, \qquad g_{mn} := 2^{1/2}\left[\left|\frac{dJ_m(s)}{ds}\right|_{s=j_{mn}}\right]^{-1} \tag{6.23b}$$

[see Tolstov (1962, Section 8-13)].

6.2.5. Normal Modes on the Disk

Having found the expressions (6.22) and (6.23), we have completed our task of finding and classifying the Laplacian eigenvalues and functions in $\mathscr{L}_0{}^2(R_{\bigcirc})$. They are a complete and orthonormal set of basis functions in this space. Equations (6.5), the generalized Fourier series for $\mathscr{L}_0{}^2(R_{\bigcirc})$ functions on the unit disk, can thus be written using this set. We can now use this in order to expand the time-dependent function,

$$f(r, \phi, t) = \sum_{m\in\mathscr{Z}, n\in\mathscr{Z}^+} f_{mn}(t)\varphi^{\bigcirc}_{mn}(r, \phi), \tag{6.24a}$$

$$f_{mn}(t) = \int_0^1 r\,dr \int_{-\pi}^{\pi} d\phi f(r, \phi, t)\varphi^{\bigcirc}_{mn}(r, \phi)^* = (\boldsymbol{\varphi}^{\bigcirc}_{mn}, \mathbf{f})_{\bigcirc}, \tag{6.24b}$$

which will be required to be a solution of the wave equation describing the vibrations of a circular membrane of unit radius fixed along its perimeter. As in (6.9)–(6.10), for $f(r, \phi, t)$ to be a solution of the wave equation, the Fourier coefficients must satisfy

$$c^{-2}\frac{\partial^2}{\partial t^2} f_{mn}(t) = -\lambda_{mn} f_{mn}(t), \tag{6.25}$$

which are two independent sinusoidal, oscillatory functions of time with angular frequency

$$\omega^{\bigcirc}_{mn} = c\lambda^{1/2}_{mn} = cj_{mn}, \tag{6.26}$$

exactly as in (6.11a).

The *normal modes* of the circular membrane will thus be the solutions for which the Fourier coefficients (6.24b) are different from zero one at a time,

$$\dot{\varphi}^{\bigcirc}_{mn}(r, \phi, t) = c^{\bigcirc}_{mn} J_m(j_{mn} r) \exp(im\phi) \cos \omega^{\bigcirc}_{mn} t \tag{6.27}$$

and their time antiderivatives which involve sine functions of $\omega^{\bigcirc}_{mn} t$.

6.2.6. Properties of the Disk Normal Modes

The properties we noted for the rectangular membrane normal nodes have their counterparts here: (a) The modes (6.27) start from rest at $t = 0$,

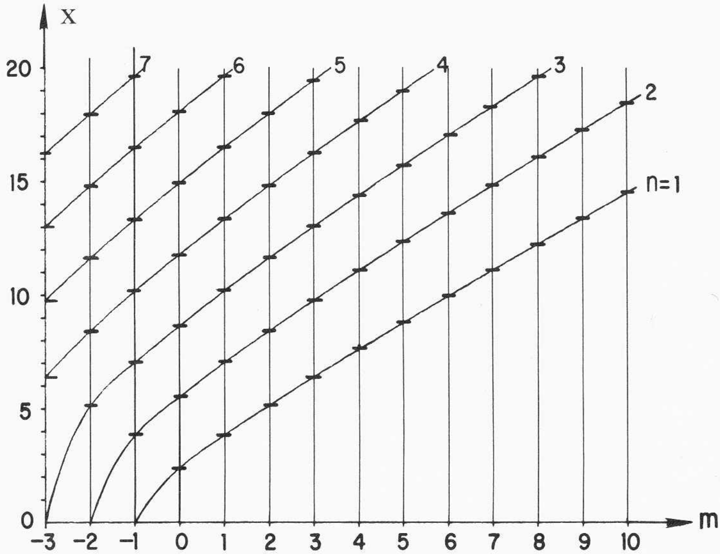


Fig. 6.6. The allowed angular frequencies of the circular membrane normal modes lie on the zeros of the Bessel function $J_m(x)$. The latter are indicated by the sloping lines corresponding to the first, second, etc., zero. For m integer (vertical lines), the position of the zero (heavier bars) gives the value of ω_{mn}.

while their antiderivatives start with maximum velocity. (b) If we consider the real or the imaginary part of (6.27), the m, n mode presents m nodal diameter lines and n nodal circles, including the boundary. They are simple-zero lines, fixed in time. (c) The angular frequencies (6.26) can be arranged in an m-n diagram as in Fig. 6.6, which is the counterpart of Fig. 6.3. They appear as points—for integer m—on the zero lines in the (m, x)-plane of the Bessel function $J_m(x)$. We can see that, quite naturally, the ω_{mn} fall into *trajectories* characterized by n. (d) The oscillation periods $T_{mn} = 2\pi/\omega_{mn}$ are all mutually incommensurable, except for $T_{mn} = T_{-mn}$. The lack of harmonic frequencies accounts for the "nonmusical" sound of a drum as compared with a guitar string, where all frequencies are multiples of a basic one. (e) The *twofold* degeneracy of all $m \neq 0$ modes is a consequence of the invariance of the system—differential equation and boundary conditions—under the group $O(2)$ of rotations and reflections across any line which passes through the origin. Clearly, as one reflection (across a line by 0°) replaces ϕ by $-\phi$, the $\varphi^{\bigcirc}_{mn}(r, \phi)$ modes are transformed into the $\varphi^{\bigcirc}_{-mn}(r, \phi)$ ones. Linear combination in these equal-frequency subspaces only rotates the position of the angular nodes.

In Fig. 6.7 we show some of the lower-lying vibrational modes of the circular membrane.

Exercise 6.6. Assume the initial conditions are invariant under rotations [i.e., $f(r, \phi, t_0) = f(r, \phi_{+}\alpha, t_0)$ and similarly for the time derivatives, for arbitrary α]. Show that the only normal modes present in such a vibration are the $m = 0$ modes and that this invariance will hold for all time.

Exercise 6.7. Assume the initial conditions of the membrane are eigenfunctions of an element of the group $O(2)$ of rotations and reflections. Find the normal modes present in this state. Show that this symmetry will be preserved forever.

Exercise 6.8. Find the Parseval identity for (6.23).

Exercise 6.9. Consider the region R to be a *circular cylinder cavity* of unit radius and length L. The Laplacian eigenfunctions will then be

$$(2/L)^{1/2}\varphi^{\bigcirc}_{mn}(r, \phi)\sin(k\pi z/L)$$

for $m \in \mathscr{Z}$ and $n, k \in \mathscr{Z}^+$. The system is the "direct product" of a circular membrane times a string. The normal mode oscillation angular frequencies will be the "Pythagorean sum" of those of the constituent systems, that is, $\omega_{mnk} = (\omega^2_{mn} + \omega_k{}^2)^{1/2}$ in terms of (6.26) and (5.23b). Note that one needs one label for each dimension of the space.

6.2.7. Bessel Series of Integral Order

The orthogonality and completeness of the normal mode expansion (6.24) on the disk, for any fixed time, has one rather immediate consequence for functions $f(r, \phi)$ which are of the form $f(r, \phi) = f(r)\Phi^{\bigcirc}_{m_0}(\phi)$ for a fixed m_0 [Eq. (6.19)]. Equation (6.24b), when integration over φ is performed, will yield a factor $2\pi\delta_{m,m_0}$. The generalized Fourier synthesis in (6.24a) will contain only an $m = m_0$ term so the $\Phi^{\bigcirc}_{m_0}$'s can be canceled on both sides, turning the pair of equations (6.24) into

$$f(r) = \sum_{n\in\mathscr{Z}^+} f_n^{J_m} g_{mn} J_m(j_{mn}r), \tag{6.28a}$$

$$f_n^{J_m} = g_{mn}\int_0^1 r\,dr f(r) J_m(j_{mn}r). \tag{6.28b}$$

Equations (6.28) are the *order-m Bessel series* and *Bessel partial waves*: the expansion of an arbitrary $\mathscr{L}_0{}^2(0, 1)$ function $f(r)$ into a series of Bessel functions of order m and its corresponding order-m Bessel coefficient $f_n^{J_m} = (2\pi)^{-1/2} f_{mn}$.

The Bessel series (6.28) is one example, in addition to the Fourier sine series, of expansion of an arbitrary function in $\mathscr{L}_0{}^2(a, b)$ in terms of a complete and orthogonal set of functions with respect to a given inner product. Note that the relevant inner product here is

$$(\mathbf{f}, \mathbf{g})_J := \int_0^1 r\,dr f(r)^* g(r) = \sum_{n\in\mathscr{Z}^+} f_n^{J_m *} g_n^{J_m}, \tag{6.28c}$$

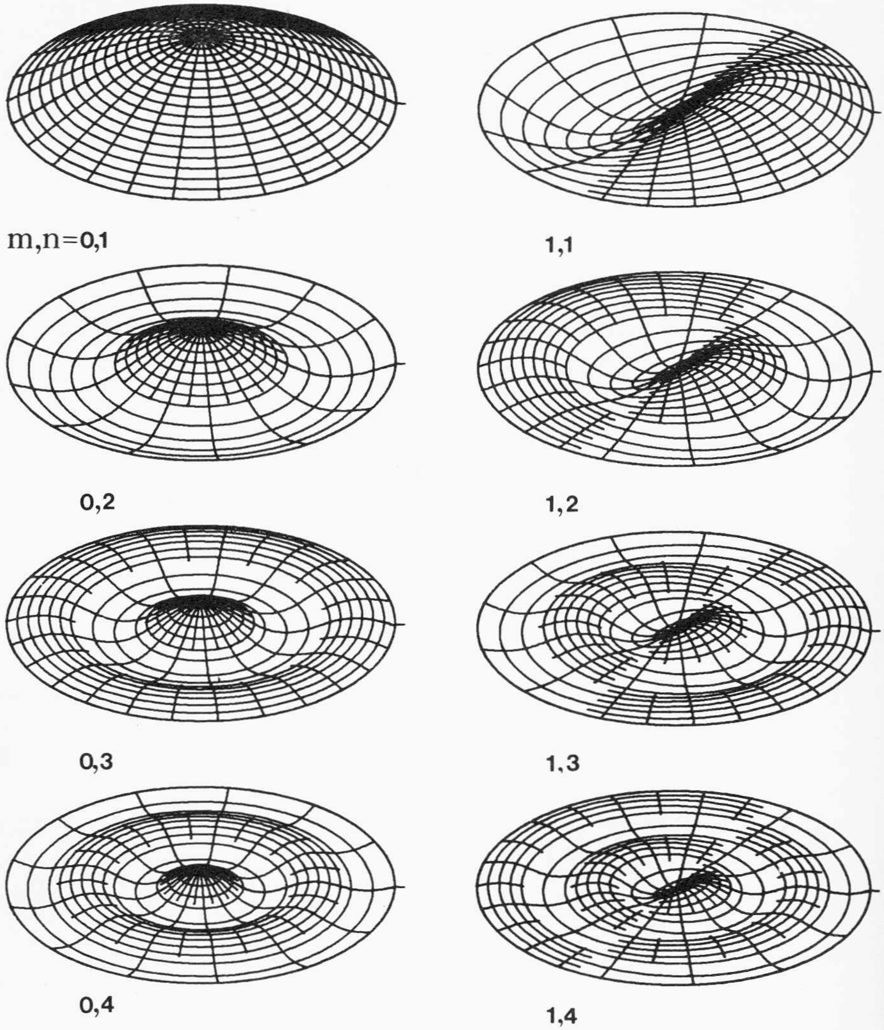
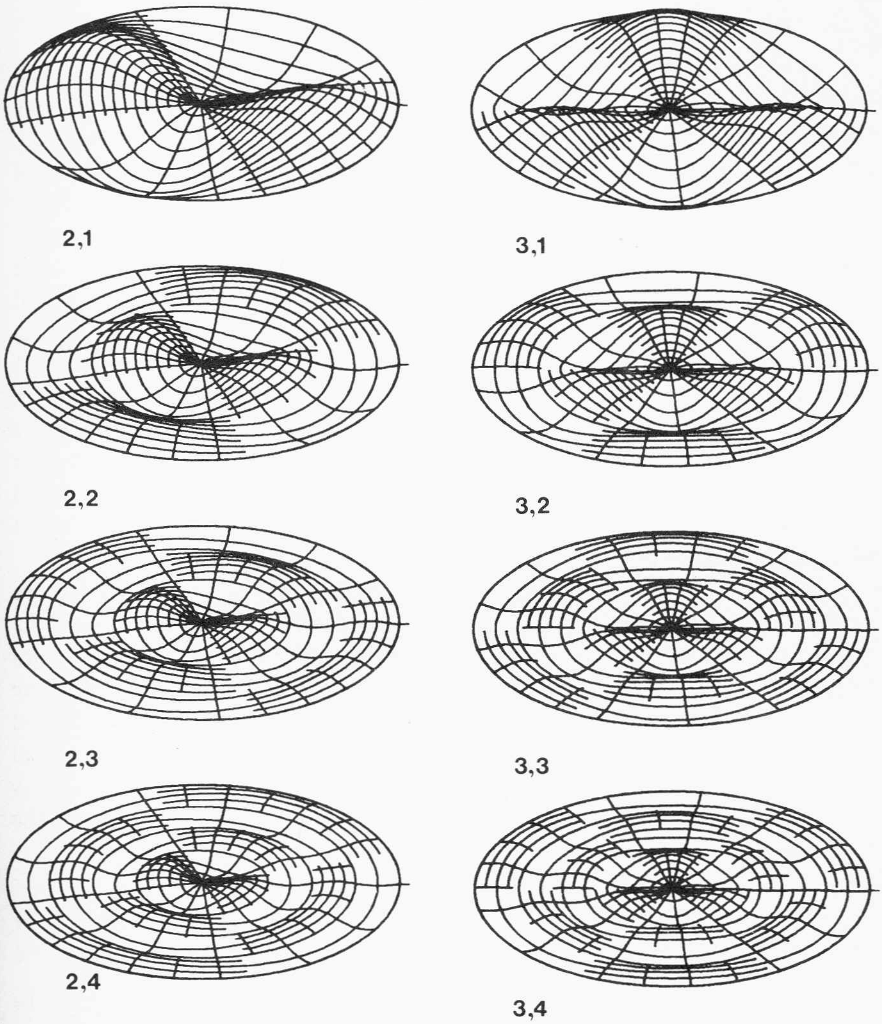


Fig. 6.7. The normal modes φ°_{mn} of the circular membrane. For positive values of the function [Eq. (6.27), $t = 0$] the grid is finer. The fixed radial and angular nodes

this being the form of the Parseval identity. A trivial change of function $f(r) \to r^{-1/2}f(r)$ transforms (6.28) into an expansion of the new $f(r)$ in terms of $r^{1/2}J_m(j_{mn}r)$ having the advantage that the integral in (6.28b) and (6.28c) contains dr rather than $r\,dr$ as its differential.

Exercise 6.10. Verify the orthogonality of the Bessel functions $J_m(j_{mn}r)$ with respect to the index n under the integral (6.28). This can be done by verifying



of the membrane are the boundaries between the single- and double-gridded regions.

first that the differential operator on the left-hand side of (6.17), with $\partial^2/\partial\phi^2$ replaced by $-m^2$, is hermitian with respect to the inner product. Then apply the argument (1.106).

Exercise 6.11. Consider the $f(r)$ to be expanded in a Bessel series to be the Dirac δ, $a^{-1}\delta(r - a)$, $a \in (0, 1)$. This acts as the *reproducing kernel* in the inner product (6.28c). Find its Bessel series coefficients from (6.28b); (6.28a) then gives another divergent series representation for the Dirac δ and its derivatives.

6.2.8. Other Boundary Conditions

The series (6.28) is called, to be precise, the *Fourier–Bessel* series. The roots j_{mn} of the order-m Bessel functions appear in it due to the boundary condition that the solution to (6.20) at the membrane edge $r = 1$ be zero. This condition can be replaced by any other condition which ensures that the boundary terms in (6.3), namely $[f(r)g'(r) - f'(r)g(r)]|_{r=1}$, vanish. This is achieved if $f(1) = 0$, as we demanded from (6.21), or by the more general condition that the ratio of $f(r)$ to $f'(r)$ at the boundary be constant. In particular, if one asks for $rf'(r) + kf(r) = 0$ at $r = 1$ to be satisfied by all solutions, one finds a normal mode basis of the type (6.23a), where the roots $\{j_{mn}\}_{n=1}^{\infty}$ are replaced by the roots $\{\lambda_{mn}\}_{n=1}^{\infty}$ of $rJ_m'(r) + kJ_m(r)$, and the series analogous to (6.28) will contain these roots. The resulting series has been called the *Dini* series and includes the Fourier–Bessel series as a particular case. Watson (1922, Section 18.3) discusses this series in some detail. For further examples and physical problems, the reader can refer to Churchill (1941), Relton (1946), Courant and Hilbert (1953, Section V-5), Morse and Feshbach (1953, Section 11.2), and Tolstov (1962, Chapter 9).

Tables of Bessel functions and their derivatives, products, and *roots* are necessary for any actual calculation. These tables have proliferated with the advent of electronic computation. See, for instance, the Bessel function tables of the British Association for the Advancement of Science (1950, 1952), the Royal Society Mathematical Tables (1960) (this includes zeros and associated values in Part III), and the National Physical Laboratory Mathematical Tables (1962). Tables of Bessel functions of large orders have been edited by the USSR Academy of Sciences (Fadeeva and Gavurin, 1950) and by the Harvard Computation Laboratory (1947–1951).

6.2.9. The Limit of Infinite Radius: Hankel Transforms

The expansion of functions $f(r)$ in terms of Bessel series need not be constrained to the interval $r \in (0, 1)$. A change of variables will allow for any interval $(0, R)$, as was done for simple Fourier series in Section 4.7. Let $q := rR$, and introduce the discrete variable $p_n := j_{mn}/R$ for fixed m, so that p_n takes on a discrete set of values proportional to the roots of the Bessel function, which can be numbered by the natural numbers n. If we further introduce $f_1(q) := (q/R)^{1/2}f(q/R)$, $f_1{}^H(p_n) := \pi^{-1/2}Rf_n$, and $h_{mn} := (\pi j_{mn})^{-1/2}g_{mn}$, Eqs. (6.28) become

$$f_1(q) = \sum_{n\in\mathscr{Z}^+} h_{mn}(\pi/R)f_1{}^H(p_n)(p_nq)^{1/2}J_m(p_nq), \tag{6.29a}$$

$$f_1{}^H(p_n) = h_{mn}\int_0^R dq f_1(q)(p_nq)^{1/2}J_m(p_nq), \tag{6.29b}$$

$$\int_0^R dqf_1(q)^* g_1(q) = \sum_{n\in\mathscr{Z}^+} (\pi/R) f_1{}^H(p_n)^* g_1{}^H(p_n). \tag{6.29c}$$

The limit $R \to \infty$ of Eqs. (6.29) can be found. We have to watch the discrete variable $p_n := j_{mn}/R$. As $R \to \infty$, higher and higher roots of the Bessel function will correspond to finite values of p_n. For large values of the argument, the Bessel function $J_m(z)$ behaves (see Appendix B) like $(2/\pi z)^{1/2} \cos[z - \pi(m + \frac{1}{2})/2]$, i.e., the roots approach asymptotically the equally spaced sequence $\pi(n + m/2 + \frac{3}{4})$. The values of p_n hence also approach equal spacing $\Delta p := \pi/R$, which vanishes as $R \to \infty$. Finally, the values of h_{mn} approach unity. This is seen from (6.23b) and the asymptotic behavior of the Bessel function derivative evaluated at the roots. The factor $(2/\pi z)^{1/2}$ yields $g_{mn} \sim (\pi j_{mn})^{1/2}$ as $n \to \infty$ and thus $h_{mn} \to 1$. Introducing these considerations into (6.29), we observe that sums $\sum_p \Delta p \cdots$ of functions of p appear. As $R \to \infty$ these will become Riemann integrals $\int_0^\infty dp \cdots$, giving, finally,

$$f(q) = \int_0^\infty dpf^H(p)(pq)^{1/2} J_m(pq), \tag{6.30a}$$

$$f^H(p) = \int_0^\infty dqf(q)(pq)^{1/2} J_m(pq), \tag{6.30b}$$

$$\int_0^\infty dqf(q)^* g(q) = \int_0^\infty dpf^H(p)^* g^H(p), \tag{6.30c}$$

where we have dropped the subscripts. Equation (6.30b) defines the *Hankel integral transform* of $f(q)$ and (6.30a) its inverse, while (6.30c) is the corresponding Parseval identity. Although the derivation (6.28)–(6.30) has not been rigorous for basically the same reasons as in Section 4.7, the results (6.30) and their range of validity will be established independently in Section 8.4.

Exercise 6.12. Consider a similar limit for the Dini series.

The Bessel–Fourier series in the form (6.28) is but one of a family of similar series in Bessel and Neumann functions. These will be discussed in Section 6.3.

6.3. Sectorial and Annular Membranes

We continue our consideration of the Laplacian operator when the functions in its domain vanish on boundaries which follow polar coordinates in the plane. These are used for the description of sectorial and annular membranes. The results are nontrivial extensions of the results on the

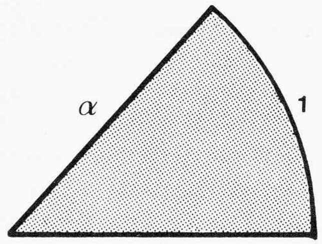


Fig. 6.8. Circular sector membrane.

circular membrane normal modes in Section 6.3 and are meant to illustrate the use of the normal mode method in finding explicit results for a variety of systems.

6.3.1. Inner Product on a Sector

Let the region $R_{\triangleleft}$ be a *sector* of the unit circle extending between the lines $\phi = 0$ and $\phi = \alpha$ (Fig. 6.8). The inner product on this region, in polar coordinates, will be

$$(\mathbf{f}, \mathbf{g})_{\triangleleft} = \int_0^1 r\,dr \int_0^\alpha d\phi f(r, \phi)^* g(r, \phi). \tag{6.31}$$

The space of functions we want to consider is $\mathscr{L}_0{}^2(R_{\triangleleft})$: square-integrable functions under (6.31) which vanish on the boundary of $R_{\triangleleft}$, i.e., $f(r, 0) = 0 = f(r, \alpha)$ and $f(1, \phi) = 0$.

The expression for the Laplacian in polar coordinates is again (6.16), and the eigenfunction problem with the same coordinate separation is also (6.18). The solutions of the right-hand side of this equation, however, because of our new boundary conditions in ϕ, will be akin to the functions for a fixed-end string of length α, namely,

$$\Phi_m{}^{\triangleleft}(\phi) = (2/\alpha)^{1/2} \sin \mu\phi, \qquad \mu := m\pi/\alpha, \qquad m \in \mathscr{Z}^+, \tag{6.32}$$

and the separation constant in (6.18) will be $c = \mu^2$.

6.3.2. Solution to the Problem

The Bessel differential equation, which is the left-hand side of Eq. (6.18), is identical to (6.20), except that $m \in \mathscr{Z}$ is replaced by $\mu = m\pi/\alpha$, $m \in \mathscr{Z}^+$. This replacement applies also to the solutions, Eq. (6.23), which for $R_{\triangleleft}$ now read

$$\varphi_{mn}^{\triangleleft}(r, \phi) := (2/\alpha)^{1/2} g_{\mu n} J_\mu(j_{\mu n} r) \sin \mu\phi, \qquad m, n \in \mathscr{Z}^+, \tag{6.33}$$

where all quantities have the same meaning as before, $j_{\mu n}$ being the nth zero of the Bessel function of order μ. The main *difference* between (6.33) and (6.23) is that the "angular" label μ takes on equally spaced but in general noninteger values. The set of Laplacian eigenfunctions (6.33) will be orthogonal with respect to the inner product (6.31) and complete for $\mathscr{L}_0^2(R_\triangleleft)$. A generalized Fourier series can be written for (6.33) identical to (6.24) except for the ranges of summation over m and integration over ϕ.

6.3.3. Normal Modes and Frequencies for a Sector Membrane

An elastic membrane governed by the wave equation over the region $R_<$ and fixed at its boundary will exhibit normal modes

$$\dot{\varphi}^{\triangleleft}_{mn}(r, \phi, t) = (2/\alpha)^{1/2} g_{\mu n} J_\mu(j_{\mu n} r) \sin \mu\phi \cos \omega_{\mu n} t, \tag{6.34}$$

and their time antiderivatives, the oscillation frequencies $\omega_{\mu n}$ being c times the nth zero of the Bessel function of order μ. In Fig. 6.9 are these allowed angular frequencies for the normal modes of a sectorial membrane of angle $\alpha = \pi/3$: the allowed μ's are positive integer multiples of 3. As the figure suggests, the opening of the angle α produces a "sliding down" of the allowed frequencies along their *trajectories*. In particular when α reaches π we have a half-circular membrane. The allowed values of μ are the positive

Fig. 6.9. Allowed angular frequencies of the normal modes of a sectorial membrane of angle $\alpha = \pi/3$. These are given by the zeros of $J_m(x)$ for m a nonzero multiple of 3.

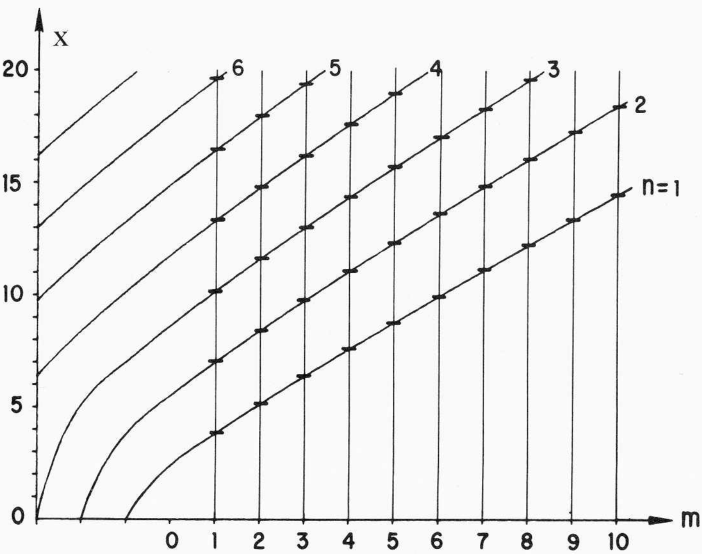


Fig. 6.10. Allowed angular frequencies of the normal modes of an $\alpha = \pi$ sectorial (semicircular) membrane.

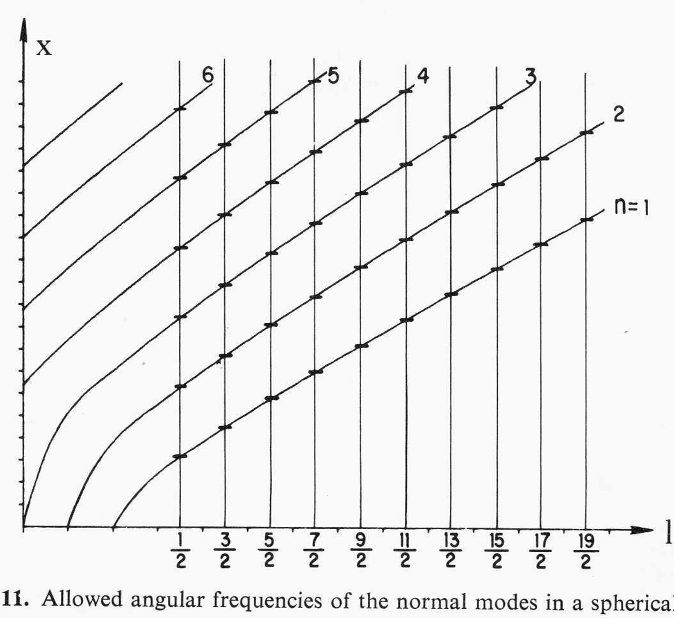


Fig. 6.11. Allowed angular frequencies of the normal modes in a spherical cavity.

integers (Fig. 6.10). Except for the $m = 0$ modes, therefore, the sounds one can produce on a circular drum are identical to those one can get from a half-drum. The mechanics of actual drumming, however, tend to generate mostly $m = 0$ normal modes. These are the only circular membrane modes where the center is in motion.

Exercise 6.13. Extend the sector angle α to 2π. You have then a circular membrane with a fixed strut extending to the center. The allowed angular frequencies will include $\omega_{1/2,n} = cn\pi$, capable of producing harmonic sounds. (See the particular function $J_{1/2}$ in Appendix B.) Describe the corresponding normal modes.

Exercise 6.14. Provided you are familiar with spherical harmonics, solve the wave equation for a resonating spherical cavity. Show that the allowed angular frequencies are only half-integers, as given by Fig. 6.11. These are the allowed ω's for Exercise 6.13, minus integers.

6.3.4. Bessel Series for Real Order

As in Section 6.2, in considering the generalized Fourier expansion of functions in $\mathscr{L}_0^2(R_\triangleleft)$ in series of $\phi_{mn}^\triangleleft(r, \phi)$, we can consider those which have the form $f(r)\Phi_m^\triangleleft(\phi)$ so that ϕ integration and cancellation can be made. This gives rise to the pair of Bessel series equations (6.28) for general *real* order m.

Exercise 6.15. Prove this in detail.

Exercise 6.16. Show that for $m = \frac{1}{2}$ the Bessel series (6.28) becomes the Fourier sine series.

6.3.5. Annular Boundary Conditions and Solutions

Consider now a region $R_\odot$ which is an *annulus* of interior and exterior radii ρ_1 and ρ_2 (Fig. 6.12). The relevant inner product is then

$$(\mathbf{f}, \mathbf{g})_\odot = \int_{\rho_1}^{\rho_2} r\, dr \int_{-\pi}^{\pi} d\varphi f(r, \varphi)^* g(r, \varphi), \tag{6.35}$$

defining a space $\mathscr{L}_0^2(R_\odot)$ in analogy with the former cases. The search for the eigenfunctions of ∇^2 in this space follows that of Section 6.2; the angular functions are here periodic and identical to (6.19), while the radial functions have the form (6.21). The boundary conditions on the latter are different, however:

$$R_m(\rho_1) = a_m J_m(\lambda^{1/2}\rho_1) + b_m N_m(\lambda^{1/2}\rho_1) = 0, \tag{6.36a}$$

$$R_m(\rho_2) = a_m J_m(\lambda^{1/2}\rho_2) + b_m N_m(\lambda^{1/2}\rho_2) = 0. \tag{6.36b}$$

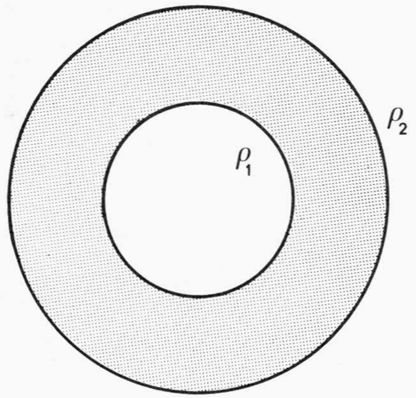


Fig. 6.12. Annular membrane.

This set of homogeneous equations will have a solution for the ratio b_m/a_m only if their determinant vanishes, i.e., for those values of $k := \lambda^{1/2}$ for which

$$D_m^{\rho_1\rho_2}(k) := J_m(\rho_1 k)N_m(\rho_2 k) - J_m(\rho_2 k)N_m(\rho_1 k) = 0. \tag{6.37}$$

6.3.6. Frequencies and Normal Modes for an Annular Membrane

The problem, then, is to find the zeros of the function $D_m^{\rho_1\rho_2}(k)$, which can be shown to be *simple*. This is not too difficult with standard numerical computer methods. See Fig. 6.13. Once these are found as $k_{m1}, k_{m2}, \ldots$, $k_{mn}, \ldots$, they can be introduced in (6.36) and the ratios $\rho_{mn} := b_m/a_m$ thereby determined for the nth zero of (6.37). The radial functions will then be

$$B_{mn}(r) := c_{mn}[J_m(k_{mn}r) + \rho_{mn}N_m(k_{mn}r)], \qquad m \in \mathscr{Z}, n \in \mathscr{Z}^+. \tag{6.38}$$

The normalization coefficient c_{mn} is chosen so that

$$\int_{\rho_1}^{\rho_2} r\, dr B_{mn}(r) B_{mn'}(r) = \delta_{n'n}. \tag{6.39}$$

Bessel and Neumann functions are real, so no complex conjugation is necessary.

Exercise 6.17. Verify the orthogonality (6.39) of the $B_{mn}(r)$ with respect to the index n. This can be done as in Exercise 6.10.

Once the radial functions (6.38) have been found, the rest of the program follows as before: The eigenfunctions of ∇^2 on the annulus $R_\odot$ are $\varphi_{mn}^\odot(r, \phi) := B_{mn}(r)\Phi_m{}^\circ(\phi)$ and constitute a complete and orthonormal set of functions on $\mathscr{L}_0{}^2(R_\odot)$ [compare with (6.23) and (6.33)], giving rise to a generalized Fourier series on $R_\odot$. Normal modes for the annular membrane can be built as the $\varphi^{\odot}_{mn}(r, \phi)$ times oscillating functions of time, of angular frequency $\omega_{mn} = ck_{mn}$ determined by the roots of (6.37). These can be plotted as in Fig. 6.6. In fact, Fig. 6.13, seen sidewise, is just such a diagram.

Exercise 6.18. Verify and explain the twofold degeneracy of ω_{mn} and ω_{-mn}.

Exercise 6.19. Investigate the case when the interior radius of the annulus, ρ_1, becomes zero. Show that, as $N_m(\rho) \to \pm\infty$ for $\rho \to 0$, $\rho_{mn} \to 0$ in (6.38). The annular normal modes thus become the circular ones, except for the $m = 0$ ones. Why are these absent?

Exercise 6.20. Consider an annular–sectorial membrane bounded between $r = \rho_1$ and ρ_2, $\phi = 0$ and α.

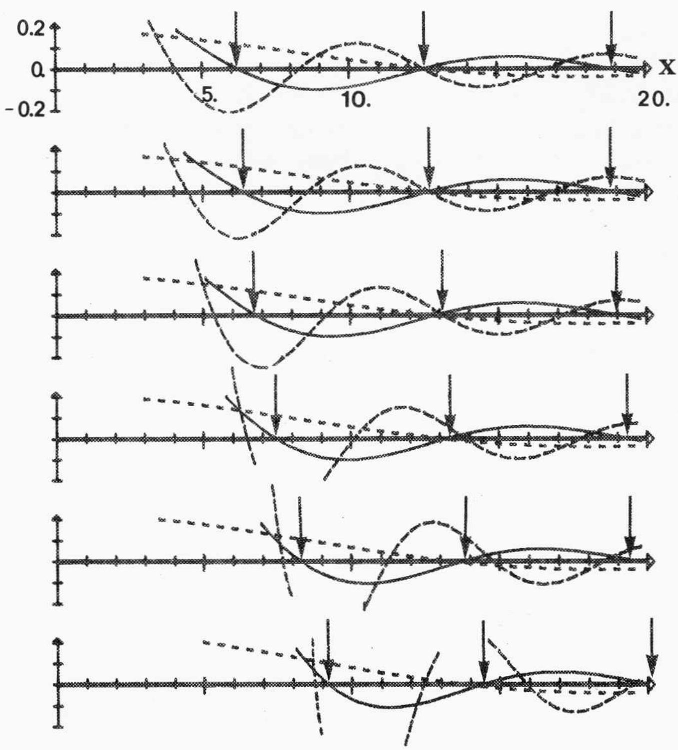


Fig. 6.13. The function $D_m^{\rho_1\rho_2}(k)$ in Eq. (6.37) for $m = 0, 1, \ldots, 5$. We draw three curves: long dashes for $\rho_1 = 0.25$, $\rho_2 = 1$; continuous for $\rho_1 = 0.5$, $\rho_2 = 1$, indicating the zeros by arrows; and short dashes for $\rho_1 = 0.75$, $\rho_2 = 1$. The zeros of the last function can be seen to lie at higher values of the argument and to tend toward equal spacing.

6.3.7. Bessel Series with the Annular Functions

The radial functions $B_{mn}(r)$ we have found for the annulus also provide an orthonormal (and *complete*) basis for the space of square-integrable functions on the interval (ρ_1, ρ_2). In fact, by arguments parallel to those which lead from the generalized Fourier expansions on $R_\bigcirc$ and $R_\triangleleft$ to the Bessel series in Eq. (6.28), we are led to the general Bessel series of order m,

$$f(r) = \sum_{n \in \mathscr{Z}^+} f_n^{B_m} B_{mn}(r), \tag{6.40a}$$

$$f_n^{B_m} = \int_{\rho_1}^{\rho_2} r\, dr f(r) B_{mn}(r), \tag{6.40b}$$

$$(\mathbf{f}, \mathbf{g})_B = \int_{\rho_1}^{\rho_2} r\, dr f(r)^* g(r) = \sum_{n \in \mathscr{Z}^+} f_n^{B_m *} g_n^{B_m}, \tag{6.40c}$$

which, as suggested by Exercise 6.20, is valid for real m.

The use of cylindrical functions for the series expansion of functions is not restricted to those types seen here, which arose out of the normal mode expansion in regions of the plane bounded by polar-coordinate boundaries. Among these "other" series expansions we should mention the *Neumann series*, which are of the form $\sum_{n=0}^{\infty} a_n J_{\nu+n}(r)$; the *Kapteyn series*, of the form $\sum_{n=0}^{\infty} b_n J_{\nu+n}[(\nu + n)r]$; and the *Schlömlich series*, of the form $\sum_{n=0}^{\infty} c_n J_\nu(nr)$. The region of convergence of the two first series is determined by the analytic properties of the functions to be expanded. In this sense, they are similar to the ordinary Taylor expansions. We shall not elaborate on these but refer the interested reader to Watson's treatise (1922, Chapters XVI, XVII, and XIX) for further details and applications. Neumann series also appear in the series on transcendental functions by Erdelyi *et al.* (1953–1955, Vol. 2, Chapter 7).

Tables of the roots of Eq. (6.37) are needed for any practical calculation. Two articles dealing with problems of this kind which offer reasonably extensive tables are those by Dwight (1948) and Bridge and Angrist (1962).

6.4. Other Series of Orthonormal Functions

In this chapter we have seen the Fourier and Bessel series—and many of their variants—arise in the description of the normal modes of an elastic medium enclosed by rectangular and polar-coordinate boundaries. There are at least two directions in which this approach can be generalized: first, by consideration of more general boundary conditions and surfaces in higher dimensions and, second, as "normal mode" solutions of other types of equations. In both cases, though, finding orthonormal and complete sets of functions is a *Sturm–Liouville* problem, which can be posed as follows.

6.4.1. The Sturm–Liouville Problem

Define an inner product over the interval (a, b) given by

$$(\mathbf{f}, \mathbf{g})_\omega = \int_a^b \omega(x)\, dx f(x)^* g(x), \qquad \omega(x) > 0,\, x \in (a, b), \tag{6.41}$$

with a positive *weight function* $\omega(x)$. Such an inner product defines the (*Hilbert*) space $\mathscr{L}_\omega^2(a, b)$ of functions $\mathbf{f}$ on (a, b) such that $(\mathbf{f}, \mathbf{f})_\omega < \infty$. Consider now a second-order differential operator with $p(x), q(x), r(x)$ real.

$$\mathbb{H} = p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x). \tag{6.42}$$

We want to examine the conditions under which $\mathbb{H}$ is *hermitian*, i.e., $(\mathbb{H}\mathbf{f}, \mathbf{g})_\omega = (\mathbf{f}, \mathbb{H}\mathbf{g})_\omega$. Performing the necessary integrations by parts, denoting d/dx by ∇ for the sake of brevity, and suppressing arguments, we find

$$\begin{aligned}(\mathbb{H}\mathbf{f}, \mathbf{g})_\omega &= \int_a^b \omega\, dx[(p\nabla^2 + q\nabla + r)f^*]g \\ &= \{\omega p(g\nabla f^* - f^*\nabla g) + [\omega q - \nabla(\omega p)]f^* g\}|_a^b + \int_a^b \omega\, dx f^* \\ &\quad \times \{p\nabla^2 + \omega^{-1}[2\nabla(\omega p) - \omega q]\nabla + \omega^{-1}[\nabla^2(\omega p) - \nabla(\omega q)] + r\}g.\end{aligned} \tag{6.43}$$

So that (6.43) will equal $(\mathbf{f}, \mathbb{H}\mathbf{g})_\omega$ for arbitrary $\mathbf{f}$ and $\mathbf{g}$ it is sufficient that (a) $\nabla(\omega p) = \omega q$, which turns the operator in curly brackets into $\mathbb{H}$; the boundary term disappears if either (b) $\nabla h/h|_{x=a} = \nabla h/h|_{x=b}$ for $h = f$ and g or (b′) $\nabla h/h|_{x=a} = k_a$ and $\nabla h/h|_{x=b} = k_b$, k_a and k_b constants. These conditions direct us to consider operators (6.42) of the form

$$\mathbb{H} = [\omega(x)]^{-1}\frac{d}{dx}\omega(x)p(x)\frac{d}{dx} + r(x), \tag{6.44}$$

which, we are assured, are *hermitian* with respect to the inner product (6.41) in spaces of functions which satisfy boundary conditions which are either periodic or fix the logarithmic derivative at the interval ends.

We now pose ourselves the task of finding the solutions $\varphi_\lambda(x)$ to the eigenvalue equation

$$\mathbb{H}\varphi_\lambda(x) = \lambda\varphi_\lambda(x) \tag{6.45}$$

which are in $\mathscr{L}_\omega^2(a, b)$ and which satisfy the vanishing of the boundary term by (b) or (b′). We assume here for mathematical tractability that the set of values over which λ can range—the spectrum of $\mathbb{H}$—is an infinite, discrete set.

The solutions to (6.45), once they are explicitly found, will provide us with an *orthogonal* set of functions which can be normalized so that

$(\boldsymbol{\varphi}_\lambda, \boldsymbol{\varphi}_\mu)_\omega = \delta_{\lambda\mu}$. The proof of this fact is completely analogous to the proof in (1.106) for finite-dimensional vector spaces and was briefly commented upon following Eq. (6.4) in discussing eigenfunctions of the Laplacian which vanish on finite, closed boundaries.

Exercise 6.21. Verify the validity of the above construction for the operator d^2/dx^2, Eq. (6.44) with $\omega(x) = 1 = p(x)$, $r(x) = 0$, for the interval $(-\pi, \pi]$ with periodic boundary conditions (b) leading to the functions $(2\pi)^{-1/2}\exp(imx)$, $m \in \mathscr{Z}$, $\lambda = -m^2$. Note the slightly disturbing feature that the nonzero eigenvalues of d^2/dx^2 are doubly degenerate.

Exercise 6.22. Verify the validity for d^2/dx^2 under conditions (b′) in a general interval (a, b). The solutions to (6.45) have the general form $f_\mu(x) = c_\mu \sin(\mu x) + d_\mu \cos \mu x$, $\lambda = -\mu^2$. Assuming that $\nabla f_\mu / f_\mu|_{x=a} = k_a$ and $\nabla f_\mu / f_\mu|_{x=b} = k_b$, find the allowed values of μ and the corresponding ratio of c_μ/d_μ. Normalize. [See Titchmarsh (1946, Section 4.1).]

Exercise 6.23. Study the Bessel series of annular functions (6.38) as stemming from the eigenvalue equation (6.20), which has the form (6.44)–(6.45) with $\omega(x) = x$, $p(x) = 1$, $r(x) = -m^2/x^2$, $m > 0$. The solutions (6.21) are further curtailed by the boundary conditions on (ρ_1, ρ_2): $k_{\rho_1} = \infty = k_{\rho_2}$. Other boundary conditions will give versions of the Dini series. [See Titchmarsh (1946, Section 4.7 *et seq.*).]

Exercise 6.24. Consider Bessel's differential equation (B.12) written as an eigenfunction equation with eigenvalues m^2 of the form (6.44)–(6.45) with $\omega(x) = 1/x$, $p(x) = x^2 = r(x)$. What boundary conditions give the orthogonality relations for the expanding functions employed in the Neumann series?

6.4.2. On Eigenvalues, Orthogonality, and Completeness

It should be observed that only orthogonality of the eigenfunction set is guaranteed by the hermiticity of the operator $\mathbb{H}$. *Completeness* is a more difficult property to prove or verify. When the operator is *self-adjoint* (see the discussion in Section 4.6) and the spectrum as assumed above, the eigenfunction set *is* complete. An arbitrary function $f(x) \in \mathscr{L}_\omega^2(a, b)$ can then be written (approximated *in the norm*) as a normalized eigenfunction series

$$f(x) = \sum_{n=0}^{\infty} f_n \varphi_n(x), \qquad x \in (a, b), \tag{6.46a}$$

where the generalized Fourier coefficients f_n are

$$f_n = (\boldsymbol{\varphi}_n, \mathbf{f})_\omega = \int_a^b \omega(x)\, dx \varphi_n(x)^* f(x). \tag{6.46b}$$

The generalized Parseval identity reads

$$(\mathbf{f}, \mathbf{g})_\omega = \int_a^b \omega(x)\, dx f(x)^* g(x) = \sum_{n=0}^{\infty} f_n^* g_n. \tag{6.46c}$$

Quantum mechanics, in its Schrödinger formulation, is mathematically a Sturm–Liouville theory, the operator in question being typically the system's Hamiltonian $-\frac{1}{2}\nabla^2 + V(\mathbf{x})$, where $V(\mathbf{x})$ is the potential function. The region R where $\mathbf{x}$ is allowed to range is usually the whole three-dimensional space, so, in a sense, the wave-function expansion lies outside the class considered in this part. Yet if the potential is such that is classically constrains a particle with finite energy to a bounded region in space, or if we are using coordinate systems such as cylindrical or spherical where one or more of the coordinates range, due to geometry, over a bounded interval, the result is a wave-function *series*. Of particular importance are the *bound-state Coulomb* and *harmonic oscillator* systems. The eigenfunction expansions associated with the latter will be detailed in Section 7.5; those of the former can be seen in most quantum mechanics texts, such as Messiah (1964, Chapter 11).

6.4.3. Orthogonal Polynomial Series

Due to their ubiquity, a class of eigenfunction expansions which we can hardly escape mentioning is that of the classical *orthogonal polynomials*. There are three families of these, according to whether the interval (a, b) in (6.41) is finite, half-infinite [i.e., (a, ∞)], or infinite. The first family is that of *Jacobi* polynomials, $P_n^{(\alpha,\beta)}(x)$, orthogonal under (6.41) with $(a, b) = (-1, 1)$ and $\omega(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, $\alpha, \beta > -1$. When $\alpha = \beta =: \gamma - \frac{1}{2}$, i.e., $\omega(x) = (1 - x^2)^{\gamma - 1/2}$, these become the *Gegenbauer* polynomials $C_n^{(\gamma)}(x)$ which appear in connection with hyperspherical harmonics [for solutions to the angular part of the N-dimensional Laplacian, see Eqs. (8.77)]. For $\gamma = \frac{1}{2}$, $\omega(x) = 1$, we have the *Legendre* polynomials $P_n(x)$, which appear in three-dimensional spherical coordinate separation. The second case, half-infinite intervals $(0, \infty)$, leads to *Laguerre* polynomials $L_n^{(\alpha)}(x)$ when $\omega(x) = x^{\alpha} \exp(-x)$. The Coulomb and radial harmonic oscillator quantum systems are solved in terms of these functions. Finally, infinite intervals require *Hermite* polynomials for $\omega(x) = \exp(-x^2)$. These are present in the harmonic oscillator wave functions in Cartesian coordinates.

The three families of *classical* orthogonal polynomials are further related, with some rather technical restrictions, to the existence of a *Rodrigues* differential recursion formula as shown by Tricomi (1955) [this can be seen, in simplified version, in the text by Dennery and Krzywicki (1967, Section III-10)]. A result of this recursion formula is that the interval (a, b) determines uniquely the weight function $\omega(x)$ and the differential equation satisfied by the polynomials. For the above intervals, the Jacobi, Laguerre, and Hermite families satisfy (6.44)–(6.45) with $p(x) = 1 - x^2$, x, and 1, respectively.

Books on the series expansion in terms of orthogonal polynomials include the classic by Szegő (1939), Rainville (1960), and Boas and Buck (1964). The general subject of eigenfunction expansions including many concrete examples can be seen in the two-volume work by Titchmarsh (1946, 1958). A more readable account can be found in Yoshida (1960).

6.4.4. Two- and Three-Variable Series Expansions

The wave equation, once the time dependence has been "factored off" and replaced by a $-\omega^2$ term, is a Helmholtz equation such as (6.4). In two-dimensional space, the Helmholtz equation is known to have separable solutions in (*only*) four coordinate systems: Cartesian, polar, parabolic, and elliptic. If the boundary conditions are given following these coordinate lines, the solutions will be given in terms of circular and Bessel functions in the first two cases and *parabolic cylinder* and *Mathieu* functions in the last two. Corresponding orthonormal and complete sets of normal modes can be obtained for elastic media enclosed in such boundaries, except that in the last two cases these remain as two-variable u-v function expansions of the form $U_{mn}(u)V_{mn}(v)$, where the separation constants, related to m and n, are *coupled* and do *not* simplify to single-variable series. Only Cartesian and polar coordinates have this property. A variety of problems involving the two-dimensional wave equation with various boundary conditions can be found in Morse and Feshbach (1953, Sections 5-1 and 11-2). For parabolic cylinder and Mathieu functions we have to turn to more specialized literature (see below). The latter are given concise treatment in the textbook by Hochstadt (1966).

The wave equation in three dimensions leads to further special functions, since the corresponding Helmholtz equation separates now in 11 coordinate systems. The four orthogonal coordinate systems which separate the two-dimensional Helmholtz equation yield, under translations along a normal, Cartesian, circular, parabolic, and elliptic cylinder coordinates. Under rotation around an axis in the plane they generate spherical, parabolic, and prolate and oblate spheroidal coordinates. In addition, there are the conical, ellipsoidal, and paraboloidal systems. The surfaces defined by these coordinates which can serve as boundaries are three-dimensional conic surfaces. New functions appear: associated Legendre polynomials and spheroidal, Lamé, Jacobian elliptic, and ellipsoidal functions. The three-variable normal mode expansions do not simplify to two- or single-variable series except for those obtained by translation. The spherical coordinates are somewhat special in that their normal modes have the structure $R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\phi)$. For $m = 0$ they yield the Legendre polynomial series in $\cos\theta$ and the Bessel–Fourier series in r.

The higher transcendental functions appearing in the solution of the Helmholtz equation in two and three dimensions are given a full discussion in volumes such as those by Hobson (1931), McLahlan (1947), Meixner and Schärfke (1954), and especially Arscott (1964). Of particular importance, the spherical harmonics, orthogonal functions on the surface of a sphere, have many interesting group-theoretical properties. These have been presented in books by Edmonds (1957) and Rose (1957). Last, it should be mentioned that the theory of Lie algebras and groups offers a powerful method for the determination of certain operator eigenfunctions and their completeness. The works of Maurin (1968) and Olevskiĭ (1975) develop this field.