9

Construction and Properties of Canonical Transforms

In this chapter we present a class of integral transforms which we shall call canonical transforms. These constitute a parametrized continuum of transforms which include the Fourier, Laplace, Gauss–Weierstrass, and Bargmann transforms as particular cases. As these have arisen quite recently, we shall include brief historical sketches of the developments which led to their recognition and refer the interested reader to the research literature for a more rigorous treatment. Section 9.1 deals with *real* linear canonical transforms, while Section 9.2 enlarges the set to *complex* ones. The former appeared a couple of times before Moshinsky and Quesne (1974) called attention to their significance in connection with canonical transformations in quantum mechanics. A particular case of the latter was developed by Segal (1963) and Bargmann (1961) in order to formalize Fok's boson calculus (1928). Section 9.3 shows that canonical transforms have a hyperdifferential operator realization in addition to the usual integral form. Several examples and exercises show the economy of concepts and computation introduced by this new technique.

9.1. Real Linear Canonical Transforms

There are several ways to introduce the subject matter of this chapter. We have chosen here the approach which constructs canonical transforms as those unitary transformations which map the operators $\mathbb{Q}$ and $\mathbb{P}$ of Section 7.2 into real linear combinations of themselves. One finds several instances in the mathematical physics literature where this problem has been tackled. Infeld and Plebañski (1955) and, later and independently, Moshinsky and

Quesne (1971a, 1971b) have constructed unitary transformation operators in quantum-mechanical phase space in a group-theoretical context, as have Itzykson (1969) and Bargmann (1970). We present here the integral transform aspects of this construction.

9.1.1. Posing the Operator Problem

One of the main properties of the Fourier transform is that the operator $\mathbb{Q}$ of multiplication by the argument [i.e., $(\mathbb{Q}\mathbf{f})(q) = qf(q)$, Eq. (7.55)] is transformed into the operator $\mathbb{P}$ of differentiation [i.e., $(\mathbb{P}\mathbf{f})(q) = -idf(q)/dq$, Eq. (7.56)] and vice versa with a minus sign [Eqs. (7.57)]. Such transformation properties are remindful of a rotation by $\pi/2$ in a "$\mathbb{Q}$–$\mathbb{P}$ phase-space" plane. This is actually the case, without quotation marks, in the Schrödinger quantum-mechanics formalism. On a purely mathematical basis, however, we propose to investigate linear operators $\mathbb{C}$ which turn $\mathbb{Q}$ and $\mathbb{P}$ into linear combinations of each other,

$$\mathbb{Q}' := \mathbb{C}\mathbb{Q}\mathbb{C}^{-1} = d\mathbb{Q} - b\mathbb{P}, \tag{9.1a}$$

$$\mathbb{P}' := \mathbb{C}\mathbb{P}\mathbb{C}^{-1} = -c\mathbb{Q} + a\mathbb{P}, \tag{9.1b}$$

where the constants a, b, c, and d are real—in this section. There is one restriction: the *commutator* of (9.1a) and (9.1b) [defined as in (7.59b) and using (7.65)] is

$$\begin{aligned}[\mathbb{Q}', \mathbb{P}'] &= [d\mathbb{Q} - b\mathbb{P}, -c\mathbb{Q} + a\mathbb{P}] = i(ad - bc)\mathbb{1} \\ &= \mathbb{C}[\mathbb{Q}, \mathbb{P}]\mathbb{C}^{-1} = i\mathbb{1}.\end{aligned} \tag{9.2}$$

The four parameters must therefore relate by

$$ad - bc = 1. \tag{9.3a}$$

The Fourier transform, we see immediately, corresponds to the particular case $a = 0 = d$, $b = 1 = -c$. The identity transformation corresponds to $a = 1 = d$, $b = 0 = c$. We shall label the transform operator as $\mathbb{C}_M$ by the unimodular *matrix*

$$\mathbf{M} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det \mathbf{M} = 1. \tag{9.3b}$$

This operator $\mathbb{C}_M$ can be made to act on an appropriate space of functions $\mathscr{B}_1$. The action is *linear*, i.e.,

$$\mathbb{C}_M(c_1\mathbf{f} + c_2\mathbf{g}) = c_1\mathbb{C}_M\mathbf{f} + c_2\mathbb{C}_M\mathbf{g}, \qquad c_1, c_2 \in \mathscr{C}, \mathbf{f}, \mathbf{g} \in \mathscr{B}_1, \tag{9.4}$$

and, due to (9.1), it has the following property: if the $\mathbb{C}_M$ transform of $f(q)$ is $f^M(q') = (\mathbb{C}_M\mathbf{f})(q')$, then the $\mathbb{C}_M$ transform of $qf(q)$ will be

$$\begin{aligned}(\mathbb{C}_M\mathbb{Q}\mathbf{f})(q') &= (\mathbb{C}_M\mathbb{Q}\mathbb{C}_M^{-1}\mathbf{f}^M)(q') = ((d\mathbb{Q} - b\mathbb{P})\mathbf{f}^M)(q') \\ &= dq'f^M(q') + ib\, df^M(q')/dq'.\end{aligned} \tag{9.4a}$$

Similarly, the $\mathbb{C}_M$ transform of $(\mathbb{P}\mathbf{f})(q) = -i\, df(q)/dq$ is

$$(\mathbb{C}_M\mathbb{P}\mathbf{f})(q') = (\mathbb{C}_M\mathbb{P}\mathbb{C}_M^{-1}\mathbf{f}^M(q') = \big((-c\mathbb{Q} + a\mathbb{P})f^M\big)(q')$$
$$= -cq'f^M(q') - ia\, df^M(q')/dq'. \tag{9.4b}$$

9.1.2. Integral Transform and Kernel

As a concrete realization of the linear operator $\mathbb{C}_M$ we can propose an *integral transform* with a *kernel* $C_M(q', q)$:

$$f^M(q') = (\mathbb{C}_M\mathbf{f})(q') := \int_{\mathscr{R}} dq f(q) C_M(q', q), \tag{9.5}$$

where the integration takes place over the full real line $\mathscr{R}$. Linearity is automatically satisfied by the integral form, while conditions (9.4) will determine the kernel function $C_M(q', q)$ up to an arbitrary multiplicative constant. The extreme members of (9.4a) and (9.4b) with (9.5) are

$$\int_{\mathscr{R}} dq\cdot qf(q)C_M(q', q) = \left(dq' + ib\frac{d}{dq'}\right)\int_{\mathscr{R}} dq f(q) C_M(q', q), \tag{9.6a}$$

$$\int_{\mathscr{R}} dq[-i\, df(q)/dq]C_M(q', q) = -\left(cq' + ia\frac{d}{dq'}\right)\int_{\mathscr{R}} dq f(q) C_M(q', q). \tag{9.6b}$$

A sufficient condition for (9.6) to hold is that $C_M(q', q)$ satisfy the following differential equations:

$$qC_M(q', q) = \left(dq' + ib\frac{\partial}{\partial q'}\right)C_M(q', q), \tag{9.7a}$$

$$i\frac{\partial}{\partial q}C_M(q', q) = -\left(cq' + ia\frac{\partial}{\partial q'}\right)C_M(q', q), \tag{9.7b}$$

where the second one was obtained by integration by parts of the first integral in (9.6b) under the assumption that

$$f(q)C_M(q', q)|_{q=-\infty}^{\infty} = 0. \tag{9.7c}$$

This will help in the selection of the space $\mathscr{B}_1$ to which $f(q)$ can belong.

Proposing a solution of the kind $\exp(Aq^2 + Bq'q + Cq'^2)$, we find upon replacement that $A = ia/2b$, $B = -i/b$, $C = i\, d/2b$. The integral kernel is thus

$$C_M(q', q) = \theta_M \exp[i(aq^2 - 2q'q + dq'^2)/2b]. \tag{9.8a}$$

The choice of the multiplicative constant

$$\theta_M = (2\pi b)^{-1/2}\exp(-i\pi/4) \tag{9.8b}$$

will be seen to be convenient later on.

For real parameters, the behavior of the kernel (9.8) is that of a function which oscillates strongly for large $|q|$ and $|q'|$ but whose modulus is fixed at $(2\pi|b|)^{-1/2}$. (The limit $b \to 0$ is interesting but will be deferred until later in this section.) The validity of the assumed boundary condition (9.7c) can be seen to be the same as for ordinary Fourier transforms. We shall thus let $\mathscr{B}_1$ be $\mathscr{L}^2(\mathscr{R})$.

9.1.3. Inversion

The inversion of the $\mathbb{C}_M$ transform (9.5) with *real* parameters is easily accomplished with the help of the Fourier transform. We shall show that

$$(\mathbb{C}_M^{-1}\mathbf{f}^M)(q) = \lim_{\varepsilon \to 0} \tfrac{1}{2}[f(q+\varepsilon) + f(q-\varepsilon)] = \lim_{L\to\infty} \int_{-L}^{L} dq' f^M(q') C_M(q', q)^*. \tag{9.9}$$

If we substitute (9.5) and (9.8) into (9.9) and use the Fourier integral theorem (7.3), we obtain

$$\begin{aligned}
&\lim_{L\to\infty} \int_{-L}^{L} dq' \left[\int_{\mathscr{R}} dq'' f(q'') C_M(q', q'')\right] C_M(q', q)^* \\
&= \lim_{L\to\infty} \int_{-L}^{L} dq' \int_{\mathscr{R}} dq'' f(q'')(2\pi|b|)^{-1} \\
&\quad \times \exp[i(aq''^2 - 2q'q'' + dq'^2 - aq^2 + 2q'q - dq'^2)/2b] \\
&= (2\pi|b|)^{-1} \exp(-iaq^2/2b) \lim_{L\to\infty} \int_{-L}^{L} dq' \int_{\mathscr{R}} dq'' \\
&\quad \times \exp(iaq''^2/2b) f(q'') \exp[iq'(q - q'')/b].
\end{aligned} \tag{9.10}$$

The last step leads to (9.9) by a simple change of variables for q'.

Exercise 9.1. Prove the *Parseval* identity for real linear canonical transforms

$$\int_{\mathscr{R}} dq f(q)^* g(q) = \int_{\mathscr{R}} dq' f^M(q')^* g^M(q'). \tag{9.11}$$

The second integral is meant to be taken as $\lim_{L\to\infty} \int_{-L}^{L}$. Note carefully that (9.11)—and (9.9)—are valid strictly for *real* parameters.

The function $\mathbf{f}^M = \mathbb{C}_M \mathbf{f}$ given by (9.5) is the real linear canonical $\mathbb{C}_M$ transform of $\mathbf{f}$. For each matrix $\mathbf{M}$ we have a corresponding transform. This class of transforms has been called *Moshinsky–Quesne* (1971a, 1971b, 1974) transforms since they were recognized as such at the Solvay conference in Brussels (1970) and in their contiguous 1971 papers. [In attributing names of living authors to their mathematical constructs, care and tact must be used. Thus it should be noted that as a group-theoretical problem similar formulas in a related but not identical context were derived by Infeld and Plebañski [1955, Eqs. (3.4)], Bargmann (1970, Section 3), and Kalnins and Miller (1974), Section 3. This representation has been called the *metaplectic* representation by Weil (1963); it is presented, for instance, by Burdet *et al.* (1978, Section 5).]

We have stated before that the Fourier transform is a *unitary* mapping of $\mathscr{L}^2(\mathscr{R})$ onto itself. This property holds for all real linear canonical transforms with the inner product defined—as usual—by

$$(\mathbf{f}, \mathbf{g})_1 := \int_{\mathscr{R}} dq f(q)^* g(q). \tag{9.12}$$

We would not bother to write (9.12) again were it not for the fact that a more general inner product will appear in Section 9.2, when the matrix parameters (9.3) are allowed to go complex. Thence the index "1" in (9.12).

9.1.4. Composition of Transforms

As we have a three-parameter continuum of canonical transforms, we may ask about the possibilities of *composition* of the elements of the set. Assume $\mathbb{Q}$ and $\mathbb{P}$ are transformed into $\mathbb{Q}'$ and $\mathbb{P}'$ by $\mathbb{C}_{M_1}$ as in (9.1) and that the latter are in turn transformed into $\mathbb{Q}''$ and $\mathbb{P}''$ by $\mathbb{C}_{M_2}$ by a similar action. The relation between the double-primed and the original operators is then

$$\begin{aligned} \mathbb{Q}'' &= \mathbb{C}_{M_2}\mathbb{Q}'\mathbb{C}_{M_2}^{-1} = \mathbb{C}_{M_2}\mathbb{C}_{M_1}\mathbb{Q}(\mathbb{C}_{M_2}\mathbb{C}_{M_1})^{-1} \\ &= \mathbb{C}_{M_2}(d_1\mathbb{Q} - b_1\mathbb{P})\mathbb{C}_{M_2}^{-1} = d_1\mathbb{C}_{M_2}\mathbb{Q}\mathbb{C}_{M_2}^{-1} - b_1\mathbb{C}_{M_2}\mathbb{P}\mathbb{C}_{M_2}^{-1} \\ &= (c_2b_1 + d_2d_1)\mathbb{Q} - (a_2b_1 + b_2d_1)\mathbb{P} =: \mathbb{C}_{M_{12}}\mathbb{Q}\mathbb{C}_{M_{12}}^{-1}, \end{aligned} \tag{9.13a}$$

$$\begin{aligned} \mathbb{P}'' &= \mathbb{C}_{M_2}\mathbb{P}'\mathbb{C}_{M_2}^{-1} = \mathbb{C}_{M_2}\mathbb{C}_{M_1}\mathbb{P}(\mathbb{C}_{M_2}\mathbb{C}_{M_1})^{-1} \\ &= \mathbb{C}_{M_2}(-c_1\mathbb{Q} + a_1\mathbb{P})\mathbb{C}_{M_2}^{-1} = -c_1\mathbb{C}_{M_2}\mathbb{Q}\mathbb{C}_{M_2}^{-1} + a_1\mathbb{C}_{M_2}\mathbb{P}\mathbb{C}_{M_2}^{-1} \\ &= -(c_2a_1 + d_2c_1)\mathbb{Q} + (a_2a_1 + b_2b_1)\mathbb{P} =: \mathbb{C}_{M_{12}}\mathbb{P}\mathbb{C}_{M_{12}}^{-1}, \end{aligned} \tag{9.13b}$$

c_1

where

$$\mathbb{C}_{M_{21}} := \mathbb{C}_{M_2}\mathbb{C}_{M_1} = \varphi\mathbb{C}_{M_2M_1}, \qquad \varphi \in \mathscr{C}, \tag{9.14a}$$

corresponding to

$$\begin{aligned} \mathbf{M}_2\mathbf{M}_1 &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix} \\ &=: \begin{pmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{pmatrix} = \mathbf{M}_{21}. \end{aligned} \tag{9.14b}$$

By thus acting on the operators $\mathbb{Q}$ and $\mathbb{P}$, *the composition of* $\mathbb{C}_{M_2}$ *and* $\mathbb{C}_{M_1}$ (so that $\mathbb{C}_{M_1}$ acts first and $\mathbb{C}_{M_2}$ second) *is a* $\mathbb{C}_{M_2M_1}$ *transform with the parameters of a matrix which is the product of the parameters of the matrices of the constituent transforms.* We have left a free parameter φ in (9.14a) since the similarity transformation, involving $\mathbb{C}_{M_{21}}$ and $\mathbb{C}_{M_{21}}^{-1}$, allows for $\mathbb{C}_{M_{21}}$ and

$\mathbb{C}_{M_2M_1}$ to differ by a constant factor. Whether or not this factor can be chosen to be unity will be seen as we explore now the composition of $\mathbb{C}_M$ transforms on function spaces.

Allowing for exchange of integration order, as the intermediate transformed functions are assumed to exist, we must prove the last equality in

$$\begin{aligned}(\mathbb{C}_{M_2}\mathbb{C}_{M_1}\mathbf{f})(q'') &= \int_{\mathscr{R}} dq' \left[\int_{\mathscr{R}} dq f(q) C_{M_1}(q', q) \right] C_{M_2}(q'', q') \\ &= \int_{\mathscr{R}} dq f(q) \int_{\mathscr{R}} dq' C_{M_2}(q'', q') C_{M_1}(q', q) \\ &\overset{?}{=} \varphi \int_{\mathscr{R}} dq f(q) C_{M_2M_1}(q'', q). \qquad (9.15)\end{aligned}$$

In performing the integration over q' we must deal with an integrand of the form $\exp[i(r^2q'^2 + sq')]$, which we now proceed to calculate for *complex* r and s. The added generality will serve us later. Completing squares, we obtain

$$\begin{aligned}I(r, s) &:= \int_{\mathscr{R}} dq' \exp[i(r^2q'^2 + sq')] \\ &= \exp(-is^2/4r^2) \int_{\mathscr{R}} dq' \exp[i(rq' + s/2r)^2]. \qquad (9.16a)\end{aligned}$$

In reducing the last integral to Euler's, which involves $\exp(-x^2)$, change variables to

$$x := \exp(-i\pi/4)(rq' + s/2r); \qquad (9.16b)$$

the integration path will be a line in the complex plane inclined at an angle $-\pi/4 + \arg r$ and passing through the point $x = \exp(-i\pi/4)s/2r$. As the integrand is entire analytic in the whole complex plane, we are allowed to shift the integration contour to pass through the origin of the complex x-plane. If r is in the first or third quadrant [Fig. 9.1(a)], the integration

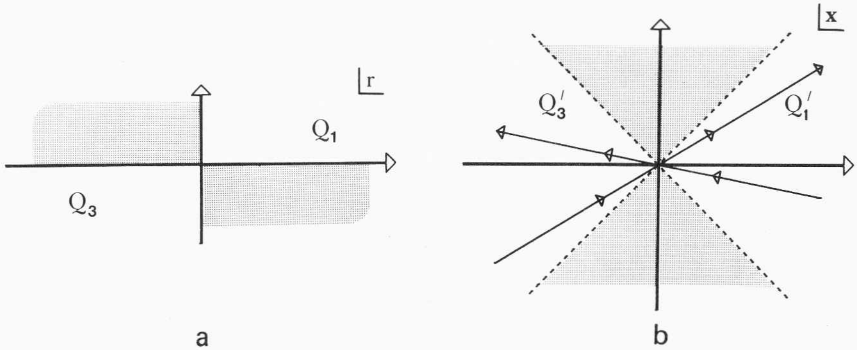

Fig. 9.1. (a) Allowed (unshaded) quadrants for the parameter r and (b) integration contours for the integral (9.16).

contour advances in the $\pi/2$ sector centered on the $+\operatorname{Re} x$ or the $-\operatorname{Re} x$ axes, respectively [Fig. 9.1(b)]. In these cases the integrand is a decreasing Gaussian function which is integrable by a standard Cauchy–Jordan argument which rotates the contour back to the real axis. If $\arg r$ is in the first quadrant, the integral reduces to Euler's $\pi^{1/2}$, while if $\arg r$ is in the third quadrant, there will be a reversal of the integral bounds, yielding $-\pi^{1/2}$. When r lies *on* the real or imaginary axes, the integral in (9.16a) exists only in the sense $\lim_{L\to\infty}\int_{-L}^{L}$, which can be evaluated to be the limit of the integral as r approaches these axes from the allowed regions. Hence,

$$I(r, s) = \varphi(r)\pi^{1/2}r^{-1}\exp(i\pi/4)\exp(-is^2/4r^2), \qquad |\arg r - \pi/4| \leqslant \pi/2, \tag{9.16c}$$

$$\varphi(r) := \begin{cases} 1 & \text{if } \arg r \in [0, \pi/2] (\operatorname{mod} 2\pi), \\ -1 & \text{if } \arg r \in [-\pi, -\pi/2] (\operatorname{mod} 2\pi). \end{cases} \tag{9.16d}$$

Convergence will be absolute when the strict inequality on $\arg r$ holds.

We can now establish the result we claimed in (9.15), as

$$\begin{aligned} \int_{\mathscr{R}} dq'\, C_{M_2}(q'', q')C_{M_1}(q', q) &= \theta_{M_2}\theta_{M_1}\exp[i(a_1q^2/2b_1 + d_2q''^2/2b_2)] \\ &\quad \times I\big((a_2/2b_2 + d_1/2b_1)^{1/2}, -(q''/b_2 + q/b_1)\big) \\ &= \theta_{M_{21}}\varphi(b_{21}/b_2b_1)\exp[i(a_{21}q^2 - 2q''q + d_{21}q''^2)/2b_{21}] \\ &= \varphi(b_{21}/b_2b_1)C_{M_{21}}(q'', q), \end{aligned} \tag{9.17}$$

$\varphi([b_{21}/b_2b_1]^{1/2})$

where we have collected the terms in the exponents and used the algebraic equalities in (9.14b) (in particular, note that $a_2/b_2 + d_1/b_1 = b_{21}/b_2b_1$). We see that the arbitrary constant φ has been made unity by the proper choice of modulus of θ_M in (9.8b), while the phase of (9.8b) assures us that, for $b_1, b_2, b_{21} > 0$ at least, the constant is unity. It turns out to be impossible to redefine the phases θ_M so as to get rid of φ in the composition formula (9.17). [In group theory, the kernels C_M are said to constitute a *ray* representation of the group $SL(2, \mathscr{R})$ of 2×2 unimodular real matrices.] Note carefully that we have proven the validity of the composition formula for *complex* values of the parameters. The only restriction has been that the integrand in (9.16a) be a bounded function.

Exercise 9.2. Show that the composition relation (9.17) is *associative*, namely, that $(\mathbb{C}_{M_3}\mathbb{C}_{M_2})\mathbb{C}_{M_1} = \mathbb{C}_{M_3}(\mathbb{C}_{M_2}\mathbb{C}_{M_1})$. This is slightly tedious, but the fact that no extra phase enters into this relation is important.

Having shown that the unitary linear real canonical transforms $\mathbb{C}_M$ multiply as the parameter matrices do—modulo a sign—we would like to see that the other two axioms of group composition (Section 1.4) also hold. These are the existence of an identity element and the inverse for every element in the set.

9.1.5. The Identity Transform Limit

The identity $\mathbb{C}_M$ transform, we should expect, corresponds to the unit matrix $\mathbf{M} = \mathbf{1}$. The integral kernel (9.8) looks peculiar in this case, as it does for the whole class of lower triangular matrices $\mathbf{M}$ which have the 1-2 element $b = 0$. We shall prove that for $b \to 0$ the integral kernel provides a sequence of oscillating Gaussians whose limit (in the *weak* sense of Section 7.3) is a Dirac δ. To this end we use (9.3a) to replace d in (9.8) and write

$$\begin{aligned} C_M(q', q) &= \theta_M \exp\{i[aq^2 - 2q'q + a^{-1}(1 + bc)q'^2]/2b\} \\ &= \theta_M \exp(icq'^2/2a) \exp[i(a^{1/2}q - a^{-1/2}q')^2/2b] \\ &= \beta \exp(icq'^2/2a) G_{|b|}(\beta(a^{1/2}q - a^{-1/2}q')), \end{aligned} \tag{9.18a}$$

where we have used the Gaussian function (7.20) and defined the phase

$$\beta := \exp[-i(\pi/2 + \arg b)/2]. \tag{9.18b}$$

Again, it will behoove us to work with complex parameters so as to understand the multivaluation features of the set of transforms. Note first that the integral transform (9.5) exists in $\mathscr{L}^2(\mathscr{R})$ only when (9.8a) is bounded. This means

$$\operatorname{Re}(ia/b) \leqslant 0 \quad \text{or} \quad \operatorname{Im}(a/b) \geqslant 0, \qquad \text{i.e.,} \quad \arg(a/b) \in [0, \pi] \pmod{2\pi}, \tag{9.19a}$$

and

$$\text{if } a = 0, \quad \text{then } \operatorname{Im} b = 0. \tag{9.19b}$$

The phase of the Gaussian's argument q is $\alpha := \arg(\beta a^{1/2}) = \frac{1}{2}\arg(a/b) - \pi/4$. Hence, if $\arg(a/b) \in [0, \pi]$, $\alpha \in [-\pi/4, \pi/4]$, while if $\arg(a/b) \in [-2\pi, -\pi]$ or $[2\pi, 3\pi]$, $\alpha \in [-5\pi/4, -3\pi/4]$ or $[3\pi/4, 5\pi/4]$. Thus in the first interval, $\beta a^{1/2}$ lies in the "forward" sector of Fig. 9.1(b), while in the last two it lies in the "backward" sector of the same figure. The upper and lower sectors are forbidden. Although the Gaussian in (9.18a) has a complex argument, we can relate its $|b| \to 0$ limit to a Dirac δ showing that (a) for any $q \neq a^{-1}q'$ the function either vanishes (for $\operatorname{Im} b \neq 0$) or oscillates with infinite rapidity (for $\operatorname{Im} b = 0$); (b) the integral of the function over $\mathscr{R}$ is finite:

$$\begin{aligned} \lim_{|b| \to 0} \int_{\mathscr{R}} dq G_{|b|}(\beta(a^{1/2}q - a^{-1/2}q')) &= \beta^{-1}a^{-1/2} \lim_{|b| \to 0} \int_{\alpha\mathscr{R}} dx G_{|b|}(x - \beta a^{-1/2}q') \\ &= \beta^{-1}a^{-1/2}\varphi((a/b)^{1/2}), \end{aligned} \tag{9.20a}$$

where we have made use of the integration contour deformation of Fig. 9.1(b) and introduced the phase function (9.16d). Hence

$$\lim_{|b|\to 0} G_{|b|}(\beta(a^{1/2}q - a^{-1/2}q')) = \beta^{-1}a^{-1/2}\varphi((a/b)^{1/2})\delta(q - a^{-1}q'). \quad (9.20b)$$

Substitution of (9.20b) into (9.18) yields, for $|b| \to 0$ and $(a/b)^{1/2}$ in the first complex quadrant,

$$\lim_{|b|\to 0} C_M(q', q) = a^{-1/2}\exp(icq'^2/2a)\delta(q - a^{-1}q'), \quad (9.21)$$

while if $(a/b)^{1/2}$ is in the third complex quadrant, a minus sign is necessary.

Near to the identity matrix, the parameter a is near to unity, c is near to zero, and if we agree to let b approach zero from the *lower* complex half-plane, including the real axis, then

$$\lim_{\mathbf{M}\to\mathbf{1}} C_M(q', q) = \delta(q - q'). \quad (9.22)$$

The integral kernel thus acts as the simple reproducing kernel for functions under integral transformations. Equation (9.22) thus constitutes the *identity* for the group of real linear canonical transforms. This result determined our choice of phase for θ_M in (9.8b).

Equation (9.21) also specifies the action of the operators for the class of lower-triangular matrices $\mathbf{M}$ ($b = 0$), as

$$(\mathbb{C}_{M(b=0)}\mathbf{f})(q) = a^{-1/2}\exp(icq^2/2a)f(a^{-1}q). \quad (9.23)$$

These will be called *geometric* transformations. They consist of dilatations by a and/or multiplication by an oscillating Gaussian of (imaginary) width a/ic. Note that the three parameters of $\mathbf{M}$ in (9.3b) have dropped to two, a and c, as $d = a^{-1}$.

Exercise 9.3. Compare the case of dilatations in (9.23) with (7.34) and with (7.71). Verify that the multivaluation "paradoxes" disappear in the complex parameter plane.

As to the inverse of the canonical transform $\mathbb{C}_M$, we remark that the inverse of the 2×2 unimodular matrix $\mathbf{M}$ in (9.3b) is

$$\mathbf{M}^{-1} := \begin{pmatrix} d & \exp(-i\pi)b \\ \exp(i\pi)c & a \end{pmatrix}. \quad (9.24)$$

We have used the sign definition since, insofar as matrices are concerned, the 1-2 matrix element is simply $-b$, the sheet in the complex b-plane being irrelevant. Due to the bivaluation property of the canonical transforms, $-b = \exp(in\pi)b$ for $n = \ldots, -1, 3, 7, \ldots$, etc., is needed. In this case, from (9.8) it follows that

$$C_{M^{-1}}(q', q) = C_M(q, q')^*. \quad (9.25)$$

Comparison with (9.9) shows that this is the proper transform kernel for the inverse of the real canonical transformation $\mathbb{C}_M$. Note that b and $\exp(-i\pi)b$ correspond to $(a/b)^{1/2}$ on the same parameter quadrant [Fig. 9.1(a)] for a close to unity in $\mathbb{C}_M$ and $\mathbb{C}_M^{-1}$.

9.1.6. One-Parameter Transform Subgroups

We have shown that the set of real linear canonical transforms $\mathbb{C}_M$ forms a *group* of unitary transformations of $\mathscr{L}^2(\mathscr{R})$ onto itself, in correspondence with (as a two-valued *ray homomorphism* of) the set of real 2×2 unimodular matrices $\mathbf{M}$. This group will be denoted, as is customary in the literature, by $SL(2, \mathscr{R})$, meaning special (unimodular) linear group in two real dimensions. It will prove worthwhile to examine in detail some of the one-parameter *sub*groups of $SL(2, \mathscr{R})$ and establish some connections with previously treated transformations. The fact that each set constitutes a group by itself is evident.

(a) *Dilatations* [recall Eqs. (7.34) and (7.71)], Eq. (9.23):

$$\mathbf{M}^d(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad (\mathbb{C}_M\mathbf{f})(q') = a^{-1/2}f(a^{-1}q') = (\mathbb{D}_a\mathbf{f})(q'). \tag{9.26}$$

(b) *Imaginary Gauss–Weierstrass transforms* [recall Eqs. (7.74), (7.75), and (8.90), setting the parameter ω or t to pure imaginary values]:

$$\mathbf{M}^f(-b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

$$(\mathbb{C}_M\mathbf{f})(q') = \theta_M \int_{\mathscr{R}} dq f(q) \exp[i(q-q')^2/2b] = (\mathbb{G}_{ib}\mathbf{f})(q'). \tag{9.27}$$

(c) *Multiplication by a Gaussian* of imaginary width, from (9.23):

$$\mathbf{M}^g(c) := \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad (\mathbb{C}_M\mathbf{f})(q') = \exp(icq'^2/2)f(q'). \tag{9.28}$$

Exercise 9.4. Because of the composition property (9.17), show that (*almost*) every $\mathbb{C}_M \in SL(2, \mathscr{R})$ can be written as the product of elements of the three subgroups (9.26)–(9.28), following the decomposition

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} = \mathbf{M}^g(c/a)\mathbf{M}^d(a)\mathbf{M}^f(-b/a). \tag{9.29}$$

This fails, however, for the special class of $\mathbf{M}$'s with $a = 0$, including the Fourier transform.

There is another triad of one-parameter subgroups of $SL(2, \mathscr{R})$ which avoids the failure in Exercise 9.4 and has by itself some interesting properties. The subgroups are

(α) *Hyperbolic* subgroup, defined as

$$\mathbf{M} = \begin{pmatrix} \cosh(\alpha/2) & -\sinh(\alpha/2) \\ -\sinh(\alpha/2) & \cosh(\alpha/2) \end{pmatrix}, \qquad \alpha \in \mathscr{R}. \tag{9.30}$$

(β) *Parabolic* subgroup, defined as the subgroup of dilatations but for $a = \exp(-\beta/2)$, $\beta \in \mathscr{R}$.

(γ) *Elliptic* subgroup, defined by

$$\mathbf{M} = \begin{pmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix}, \qquad \gamma \equiv \gamma \bmod 4\pi. \tag{9.31}$$

The last subgroup is interesting since for $\gamma = -\pi$ the corresponding integral kernel can be seen to be essentially the Fourier transform, but for a phase

$$\mathbb{C}_F = \exp(-i\pi/4)\mathbb{F}, \qquad \mathbf{F} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9.32}$$

[The reader familiar with Lie group theory will see immediately that the hyperbolic and elliptic subgroups can be used to decompose $SL(2, \mathscr{R}) \simeq SO(2, 1)$ uniquely and parametrize $\mathbb{C}_M$ by Euler angles on a hyperboloid.]

Exercise 9.5. A decomposition of the general canonical transform $\mathbb{C}_M$ which involves the Fourier transform can be made by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & b^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a/b & 1 \end{pmatrix}, \tag{9.33}$$

which means that a function is first multiplied by Gaussian (9.28) and then Fourier-transformed; finally it undergoes a "geometric" transformation (9.23).

Exercise 9.6. Using (9.1), show that the following self-adjoint operators are left *invariant* under the corresponding subgroups of canonical transforms:

$$\mathbb{J}_2 := \tfrac{1}{4}(\mathbb{Q}\mathbb{P} + \mathbb{P}\mathbb{Q}) =: \tfrac{1}{2}\mathbb{H}^d \qquad \text{[under (a) or } (\beta)\text{]}, \tag{9.34a}$$

$$\tfrac{1}{2}\mathbb{P}^2 =: \mathbb{H}^f \qquad \text{[under (b)]}, \tag{9.34b}$$

$$\tfrac{1}{2}\mathbb{Q}^2 =: \mathbb{H}^g \qquad \text{[under (c)]}, \tag{9.34c}$$

$$\mathbb{J}_1 := \tfrac{1}{4}(\mathbb{P}^2 - \mathbb{Q}^2) =: \tfrac{1}{2}\mathbb{H}^r \qquad \text{[under } (\alpha)\text{]}, \tag{9.34d}$$

$$\mathbb{J}_0 := \tfrac{1}{4}(\mathbb{P}^2 + \mathbb{Q}^2) =: \tfrac{1}{2}\mathbb{H}^h \qquad \text{[under } (\gamma)\text{]}. \tag{9.34e}$$

[Operators $\mathbb{J}_i$ have appeared before, in Eqs. (7.174).] Note that this implies that, under all $\mathbb{C}_M$ transforms of each subgroup, the eigenfunctions of the corresponding operator will transform into multiples of themselves, that is, they will be *self-reciprocal* under $\mathbb{C}_M$. Thus, the eigenfunctions of $\mathbb{J}_0$ are the harmonic oscillator wave functions of Section 7.5 which were built as self-reciprocal under the Fourier transform. They are also, therefore, self-reciprocal under any elliptic

$\mathbb{C}_M$ transform (9.31). The eigenfunctions of the repulsive oscillator are the $\chi_\lambda{}^\pm$ functions (7.203), self-reciprocal under hyperbolic $\mathbb{C}_M$ transforms (9.30). The free-particle Schrödinger eigenfunctions of $\mathbb{H}^f$, basis for the Fourier partial-wave decomposition, are self-reciprocal under imaginary—and real—Gauss–Weierstrass transforms. Eigenfunctions of $\mathbb{J}_2$ are the basis functions for the Mellin transform and are self-reciprocal under dilatations. Finally, the eigenfunctions of $\mathbb{H}^g$ are in general displaced Dirac δ's. Multiplication by a Gaussian obviously leaves them as multiples of themselves. A description of self-reciprocating functions can be found in Wolf (1977a, and the references therein).

Exercise 9.7. Note that $\mathbb{J}_1$ in (9.34d) can be transformed into $\mathbb{J}_2$ in (9.34a) by a linear, real canonical transform:

$$\mathbb{C}_A[\tfrac{1}{4}(\mathbb{P}^2 - \mathbb{Q}^2)]\mathbb{C}_A^{-1} = \tfrac{1}{4}(\mathbb{Q}\mathbb{P} + \mathbb{P}\mathbb{Q}), \qquad \mathbf{A} := 2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{9.35}$$

and thus the eigenfunctions $\chi_\lambda{}^\pm(q)$ of the repulsive oscillator Hamiltonian $\mathbb{H}^r$ are the $\mathbb{C}_A^{-1}$ transforms of the Mellin-basis functions $(2\pi)^{-1/2}q_\pm^{-i\lambda-1/2}$, eigenfunctions of $\mathbb{J}_2$. Show that this is precisely the way they were found in Eq. (7.203a), although the process appeared more circuitous there. Note also that [Eqs. (9.32) and (9.35)] $\mathbf{A}^2 = \mathbf{F}$. The transform $\mathbb{C}_A$ thus qualifies as the *square root* of the Fourier transform, i.e., $\boldsymbol{\chi}_\lambda{}^\pm = (2\pi)^{-1/2}\,\mathbb{F}^{-1/2}\mathbf{q}_\pm^{-i\lambda-1/2}$, but for a constant phase. [Watch out for a dummy change of scale in p in Eq. (7.203a).]

Exercise 9.8. Show that the *parity* of a function is preserved under linear canonical transforms.

Exercise 9.9. Study the $\mathbb{C}_M$ version of convolution. Let $f^M(q')$ and $g^M(q')$ be the $\mathbb{C}_M$ transforms of $f(q)$ and $g(q)$, respectively. Show that the real $\mathbb{C}_M$ transform of $h(q) := f(q)g(q)$ is

$$h^M(q') = \int_{\mathscr{R}} dq_1 \int_{\mathscr{R}} dq_2 f^M(q_1)g^M(q_2)C^{(M)}(q'; q_1, q_2), \tag{9.36a}$$

where

$$\begin{aligned} C^{(M)}(q'; q_1, q_2) &:= \int_{\mathscr{R}} dq C_M(q, q')C_M(q, q_1)^* C_M(q, q_2)^* \\ &= \exp(-i\pi/4)[2\pi b a^{1/2}]^{-1} \\ &\quad \times \exp\{i[a\, d(q^2 - q_1{}^2 - q_2{}^2) + (q - q_1 - q_2)^2]/ab\} \end{aligned} \tag{9.36b}$$

2

is the $\mathbb{C}_M$ *coupling* coefficient. [Recall the convolution structure and coupling coefficients for the finite case in Eqs. (3.3)–(3.4).] Verify that for the Fourier transform (9.32) the usual convolution formula (7.43) is regained.

Exercise 9.10. Show that the product of the dispersion Δ_f of a function $f(q)$ [Eq. (7.217)] times the dispersion Δ_{f^M} of its real $\mathbb{C}_M$ transform is given by

$$\Delta_f \Delta_{f^M} \geqslant b^2/4. \tag{9.37}$$

This generalizes Heisenberg's Fourier uncertainty relation (7.218).

The program of unifying several different integral transforms as particular cases of canonical transforms is quite incomplete up to this point. The Fourier transform and its inverse transform have been successfully incorporated into a continuous $SL(2, \mathscr{R})$ group, but other transforms such as those of Laplace and Gauss–Weierstrass are still outside the general case (9.8). If we allow ourselves to tamper with the reality of the parameters we see that for

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

we have the bilateral Laplace transform kernel and for

$$\begin{pmatrix} 1 & -ib \\ 0 & 1 \end{pmatrix}, \qquad b \geqslant 0,$$

the Gauss–Weierstrass kernel. In Section 9.2 we shall see that other important transforms appear, notably the *Bargmann* transform. We shall let the parameters go complex but shall be required to amend the inner product of the transform space so as to preserve unitarity. Comments on canonical transforms other than linear ones will be deferred until the end of Section 9.2.

9.2. Complex Linear Transforms and Bargmann Space

The creation and annihilation operators for the quantum harmonic oscillator presented in Section 7.5 have commutation relations [see Eqs. (7.163)] which suggest their representation as $\mathbb{Z} \mapsto d/dz$ and $\mathbb{Z}^\dagger \mapsto z$. This was done quite early by Fok (1928), following the lead of Schrödinger. The ensuing developments made use of the considerable algebraic simplifications which "boson calculus" brought both in second-quantized field theories and in many-body quantum systems which were based, in one form or another, on harmonic oscillator models [see Biedenharn and Louck (1979, Chapter 5)]. The fact remained, however, that the dagger of the $\mathbb{Z}^\dagger$ in the Fok representation did not mean the adjoint of $\mathbb{Z}$, as d/dz is not the adjoint of z under the usual $\mathscr{L}^2(\mathscr{R})$ inner product. Segal (1963) and Bargmann (1961, 1967) put Fok's representation on a proper mathematical frame by the introduction of a Hilbert space of analytic functions.

During a visit to Mexico in 1972, Professor Peter Kramer suggested some problems in nuclear cluster theory which could be solved by the complexification of the linear transformation parameters of Moshinsky and Quesne (1971a, 1971b). This was accomplished in 1973 (Kramer *et al.*, 1975; references to the original canonical transformation program can be found within). The present formulation was completed shortly thereafter (Wolf, 1974a) during the author's stay at the Centre de Recherches Mathématiques in Montréal, Canada. Further works will be cited as they arise.

9.2.1. Introducing an Inner Product

The original setup in Section 9.1 was to describe the $SL(2, \mathscr{R})$ canonical transformation operators $\mathbb{C}_M$ as those which transform linearly the pair of operators $\mathbb{Q}$ and $\mathbb{P}$ as

$$\begin{pmatrix} \mathbb{Q}' \\ \mathbb{P}' \end{pmatrix} = \mathbb{C}_M \begin{pmatrix} \mathbb{Q} \\ \mathbb{P} \end{pmatrix} \mathbb{C}_M^{-1} = \mathbf{M}^{-1} \begin{pmatrix} \mathbb{Q} \\ \mathbb{P} \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{9.38}$$

We are using here a rather obvious vector notation. When the parameters of $\mathbf{M}$ are allowed to go complex, we retain most of the results of Section 9.1 such as the transform kernel (9.8) with the restrictions (9.19) and its composition properties. What falls through is the unitarity of the transform [in mapping $\mathscr{L}^2(\mathscr{R})$ into itself] and consequently the inversion formula (9.9). The primed operators (9.38) are no longer self-adjoint under the $\mathscr{L}^2(\mathscr{R})$ inner product (9.12). Instead,

$$\begin{pmatrix} \mathbb{Q}'^{\dagger} \\ \mathbb{P}'^{\dagger} \end{pmatrix} = \mathbf{M}^{*-1} \begin{pmatrix} \mathbb{Q}^{\dagger} \\ \mathbb{P}^{\dagger} \end{pmatrix} = \mathbf{M}^{*-1} \begin{pmatrix} \mathbb{Q} \\ \mathbb{P} \end{pmatrix} = \mathbf{M}^{*-1}\mathbf{M} \begin{pmatrix} \mathbb{Q}' \\ \mathbb{P}' \end{pmatrix}, \tag{9.39}$$

where $\mathbf{M}^*$ is the matrix whose elements are the complex conjugates of those of $\mathbf{M}$. Of course $\mathbf{M}^{*-1}\mathbf{M} = \mathbf{1}$ if and only if $\mathbf{M}$ is real. In asking for self-adjointness, an inner product and a space of functions, domain of the operator, must be specified. Accordingly, if we are able to define a sesquilinear inner product $(\mathbf{f}^M, \mathbf{g}^M)_M$ such that $\mathbb{Q}'$ and $\mathbb{P}'$ *do* satisfy $(\mathbb{Q}'\mathbf{f}^M, \mathbf{g}^M)_M = (\mathbf{f}^M, \mathbb{Q}'\mathbf{g}^M)_M$ and correspondingly for $\mathbb{P}$, *then* unitarity of $\mathbb{C}_M$ could be upheld in the form

$$(\mathbf{f}, \mathbf{g})_1 = (\mathbb{C}_M\mathbf{f}, \mathbb{C}_M\mathbf{g})_M = (\mathbf{f}^M, \mathbf{g}^M)_M, \tag{9.40}$$

which would then be the new form of the Parseval identity. As we shall show in proceeding to implement this program, the inversion formula follows. We first characterize the function space $\mathscr{B}_M$ to which $(\mathbb{C}_M\mathbf{f})(q')$ belongs as determined from the transform kernel $C_M(q', q)$ in (9.8) with the restrictions (9.19) and $f(q) \in \mathscr{L}^2(\mathscr{R})$. We claim that (for b *not* real) $f^M(q')$ is an *entire analytic function.* This means that the functions have a convergent Taylor series in q' in the whole complex plane. Thus they have no singularities except at the point at infinity, and they are functions of q' only, and *not* of its complex conjugate q'^*. This property can be seen from the fact that the Taylor expansion $(\mathbb{C}_M\mathbf{f})(q')$ converges everywhere in the complex plane since the integrand contains a decreasing Gaussian in the integration variable and is entire analytic in q'. Note that as $|q'| \to \infty$ along a ray in the complex plane, a Gaussian $G_{\omega^{-1}}(q')$ is a *decreasing* function of $|q|$ along the direction where

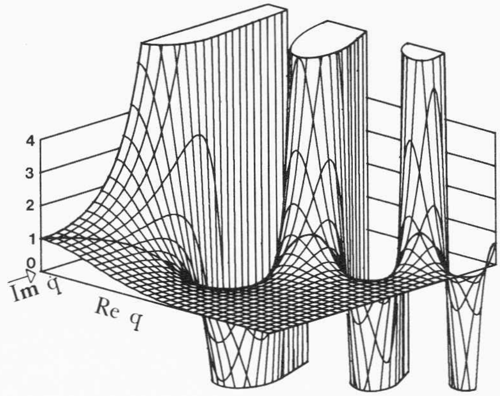

Fig. 9.2. Real part of the Gaussian function in the complex argument plane. The width is $\omega = \frac{1}{2}$, $\operatorname{Re} q$ and $\operatorname{Im} q$ are plotted in the interval $(0, 3)$, and the vertical scale is in units of $\pi^{-1/2}$. We have marked the bisector line $\operatorname{Re} q = \operatorname{Im} q$, where the function oscillates with increasing rapidity. This line separates the "convergent" from the "divergent" directions.

$\arg \omega + 2 \arg q = 0$, *increasing* at right angles to it and oscillating along the bisectors. See Fig. 9.2. It is for entire analytic functions of at most Gaussian growth that the inner product $(\cdot\,,\cdot)_M$ for $\mathscr{B}_M$ can be defined. Specification of this growth will be done below.

9.2.2. The Weight Function

Bargmann (1961) has proposed the following sesquilinear inner product form involving q and its complex conjugate q^*,

$$(\mathbf{f}, \mathbf{g})_M = \int_{\mathscr{C}} d^2\mu_M(q, q^*) f(q)^* g(q), \tag{9.41a}$$

where the integral is taken *over the complex q-plane* $\mathscr{C}$ with

$$d^2\mu_M(q, q^*) = \nu_M(q, q^*)\, d \operatorname{Re} q\, d \operatorname{Im} q =: \nu_M(q, q^*)\, d^2 q. \tag{9.41b}$$

The *weight factor* $\nu_M(q, q^*)$ will depend on $\mathbf{M}$ and on the independent variables q and q^*. This weight function will be found for $\mathscr{B}_M$ by requiring that $(\mathbb{Q}'\mathbf{f}, \mathbf{g})_M = (\mathbf{f}, \mathbb{Q}'\mathbf{g})_M$ and $(\mathbb{P}'\mathbf{f}, \mathbf{g})_M = (\mathbf{f}, \mathbb{P}'\mathbf{g})_M$. It will be easier to set up the determining equations from $\mathbb{Q}$ and $\mathbb{P}$ as

$$\begin{aligned}(\mathbb{Q}\mathbf{f}, \mathbf{g})_M &= \big((a\mathbb{Q}' + b\mathbb{P}')\mathbf{f}, \mathbf{g}\big)_M = a^*(\mathbf{f}, \mathbb{Q}'\mathbf{g})_M + b^*(\mathbf{f}, \mathbb{P}'\mathbf{g})_M \\ &= \big(\mathbf{f}, (u\mathbb{Q} + iv\mathbb{P})\mathbf{g}\big)_M,\end{aligned} \tag{9.42a}$$

$$\begin{aligned}(\mathbb{P}\mathbf{f}, \mathbf{g})_M &= \big((c\mathbb{Q}' + d\mathbb{P}')\mathbf{f}, \mathbf{g}\big)_M = c^*(\mathbf{f}, \mathbb{Q}'\mathbf{g})_M + d^*(\mathbf{f}, \mathbb{P}'\mathbf{g})_M \\ &= \big(\mathbf{f}, (iw\mathbb{Q} + u^*\mathbb{P})\mathbf{g}\big)_M,\end{aligned} \tag{9.42b}$$

where

$$u := a^*d - b^*c = \omega \exp(i\phi), \qquad \omega = |u|,\ \phi = \arg u, \tag{9.43a}$$

$$iv := b^*a - a^*b, \qquad \text{i.e.,}\quad v = 2 \operatorname{Im}(b^*a), \tag{9.43b}$$

$$iw := c^*d - d^*c, \qquad \text{i.e.,}\quad w = 2 \operatorname{Im}(c^*d). \tag{9.43c}$$

In (9.43b) we see that the restriction (9.19) implies that $v > 0$. Note that the unimodularity condition $\det \mathbf{M} = 1$, Eq. (9.3a), implies

$$\omega^2 + vw = 1. \tag{9.43d}$$

The extreme members of (9.42) thus read

$$\int_{\mathscr{C}} \nu_M(q, q^*)\, d^2q q^* f^c(q^*) g(q) = \int_{\mathscr{C}} \nu_M(q, q^*)\, d^2q f^c(q^*)(uq + v\, d/dq) g(q), \tag{9.44a}$$

$$\int_{\mathscr{C}} \nu_M(q, q^*)\, d^2q\, df^c(q^*)/dq^* g(q) = \int_{\mathscr{C}} \nu_M(q, q^*)\, d^2q f^c(q^*)(wq - u^*\, d/dq) g(q), \tag{9.44b}$$

where we have used the fact that $\mathbf{f}, \mathbf{g} \in \mathscr{B}_M$ are *analytic* functions in order to write $f(q)^* = f^c(q^*)$, f^c being a function of q^* only.

Exercise 9.11. Show that for $F = F(q, q^*)$,

$$B_F := \int_{\mathscr{C}} d^2q\, \partial F/\partial q^* = \frac{1}{2}\int_{-\infty}^{\infty} d\,(\mathrm{Im}\, q) F\Big|_{\mathrm{Re}\, q = -\infty}^{\infty} + \frac{i}{2}\int_{-\infty}^{\infty} d\,(\mathrm{Re}\, q) F\Big|_{\mathrm{Im}\, q = -\infty}^{\infty}. \tag{9.45a}$$

This is the boundary term for integration by parts for $F(q, q^*) = G(q, q^*) \cdot H(q, q^*)$:

$$\int_{\mathscr{C}} d^2q\, \partial G/\partial q^* H = B_{GH} - \int_{\mathscr{C}} d^2q G\, \partial H/\partial q^*. \tag{9.45b}$$

Integrating (9.44) by parts by (9.45b), noting that $\partial f^c(q^*)/\partial q = 0 = \partial g(q)/\partial q^*$, and assuming that the boundary term $B_{\nu f^* g}$ in (9.45a) vanishes, we find as a sufficiency condition a pair of partial differential equations,

$$q^* \nu_M(q, q^*) = (uq - v\partial/\partial q)\nu_M(q, q^*), \tag{9.46a}$$

$$-\partial \nu_M(q, q^*)/\partial q^* = (wq + u^*\partial/\partial q)\nu_M(q, q^*), \tag{9.46b}$$

which will determine the weight function.

This equation pair is of the same form as the pair which determined the transform kernel in (9.7). Its solution, with a proper choice of normalization, is a real function

$$\begin{aligned}\nu_M(q, q^*) &= (\pi v/2)^{-1/2} \exp[(uq^2 - 2qq^* + u^* q^{*2})/2v] \\ &= (\pi v/2)^{-1/2} \exp\{-\rho^2[1 - \omega \cos(\phi + 2\theta)]/v\}.\end{aligned} \tag{9.47}$$

The last expression uses the polar representation of u in (9.43a) and $q =: \rho \exp(i\theta)$. We note that as long as $v > 0$, the weight function is well defined.

The limit $v \to 0^+$, $\mathrm{Im}(a/b) \to 0^+$, which includes real matrices $\mathbf{M}$, will be examined later in this section.

The Parseval identity

$$(\mathbf{f}, \mathbf{g})_1 = \int_{\mathscr{R}} dq f(q)^* g(q) = \int_{\mathscr{C}} \nu_M(q, q^*)\, d^2q f^M(q)^* g^M(q) = (\mathbf{f}^M, \mathbf{g}^M)_M \tag{9.48}$$

will fix the normalization constant for the weight function (see Exercise 9.14).

Having found the explicit form of the weight function, we can verify that the boundary term $B_{\nu f^* g}$ vanishes for functions with a finite inner product (9.48). Let $f^M(q) =: \exp(uq^2/2v) f_B{}^M(q)$. Upon substituting (9.47) into (9.48), we see that the $\mathscr{C}$-integral of $\exp(-|q|^2/v)|f_B{}^M(q)|^2$ must be finite. The part $f_B{}^M(q)$ must therefore be of *growth* less than $(2, \frac{1}{2}v)$. [Recall the definition of growth in (8.2).] As the integrand is a Gaussian of finite width, in all directions in the complex plane the boundary integral (9.45a) is zero. We can thus characterize $\mathscr{B}_M$ as the space of entire analytic functions on $\mathscr{C}$ which are $\exp(uq^2/2v)$ times a function of growth less than $(2, 1/2v)$. Polynomials times $\exp(uq^2/2v)$, in particular, are seen to be elements of $\mathscr{B}_M$.

Exercise 9.12. To provide an example for $\mathbb{C}_M$ transforms, consider $\Psi_0(q)$, the lowest harmonic oscillator wave function (7.156), namely,

$$\Psi_0(q) := \pi^{-1/4} \exp(-q^2/2). \tag{9.49a}$$

Using (9.16) and the associated phase function, show its $\mathbb{C}_M$ transform to be

$$\begin{aligned}\Psi_0{}^M(q) &= \exp(-i\pi/4)\pi^{-3/4}(2b)^{-1/2} \exp(idq^2/2b) I([(a + ib)/2b]^{1/2}, -q/b) \\ &= [\pi^{1/2}(a + ib)]^{-1/2} \varphi([(a + ib)/2b]^{1/2}) \exp[-q^2(d - ic)/2(a + ib)].\end{aligned} \tag{9.49b}$$

Follow this function for the various complex one-parameter subgroups of Section 9.1. Check that for the Fourier transform it yields the correct result.

Exercise 9.13. Evaluate the integral, for real $\beta > 0$,

$$\begin{aligned}J(\alpha, \beta, \gamma, \delta, \varepsilon) &= \int_{\mathscr{C}} d^2q \exp(\alpha q^2 - \beta q q^* + \gamma q^{*2} + \delta q + \varepsilon q^*) \\ &= \pi(\beta^2 - 4\alpha\gamma)^{-1/2} \exp[(\alpha\varepsilon^2 + \gamma\delta^2 + \beta\delta\varepsilon)/(\beta^2 - 4\alpha\gamma)],\end{aligned} \tag{9.50a}$$

absolutely convergent if and only if

$$|\alpha + \gamma| < \beta. \tag{9.50b}$$

This integral will be useful later on. The calculation is tedious but straightforward: integrate over $\mathrm{Re}\, q$ and $\mathrm{Im}\, q$ using (9.16) twice. The condition for absolute convergence for this equation, Eq. (9.16c), leads to $|\arg(\alpha + \beta + \gamma)| < \pi$, which means, β being real and positive, that $|\alpha + \gamma| < \beta$. The two φ-factors are unity.

Exercise 9.14. Find the proper normalization constant for the weight function by requiring $(\Psi_0{}^M, \Psi_0{}^M)_M = 1$. Use (9.49), (9.50), and the identity $u/v - (d - ic)/(a + ib) = (a^* + ib^*)/v(a + ib)$.

9.2.3. Inversion

Having found a Parseval relation, we can suggest an inversion formula for the complex $\mathbb{C}_M$ transform which will then be proven. Although we have insisted on dealing with functions $\mathbf{f} \in \mathscr{B}_1 = \mathscr{L}^2(\mathscr{R})$, it is not difficult to argue that our transforms work—as the Fourier transforms do—for Dirac δ's and other generalized functions. If $\boldsymbol{\delta}_p$ is the Dirac δ sitting at $q = p$, its complex $\mathbb{C}_M$ transform will be, by placing the $\delta(q - p)$ in (9.5),

$$\delta_p{}^M(q') := (\mathbb{C}_M \boldsymbol{\delta}_p)(q') = C_M(q', p). \tag{9.51}$$

If this generalized function appears together with a continuous $\mathbf{f} \in \mathscr{B}_1$ in the Parseval identity (9.48), it tells us that

$$\begin{aligned} f(q) = (\boldsymbol{\delta}_q, \mathbf{f})_1 = (\boldsymbol{\delta}_q{}^M, \mathbf{f}^M)_M &= \int_{\mathscr{C}} \nu_M(q', q'^*)\, d^2q' f^M(q') C_M(q', q)^* \\ &= (\mathbb{C}_M^{-1}\mathbf{f}^M)(q), \end{aligned} \tag{9.52}$$

thereby providing an inversion formula for complex canonical transforms. It will be observed that the inverse transform involves the complex conjugate of the direct transform kernel as for the real case (9.9), but the integral is now that of the $\mathscr{B}_M$ inner product.

Equation (9.52) can be found as a limiting formula if we place a kernel $C_{M'}(q', q)^*$ with $\mathbf{M}'$ close to $\mathbf{M}$, asking only for the convergence of the integrals,

$$\begin{aligned} &\int_{\mathscr{C}} \nu_M(q', q'^*)\, d^2q' f^M(q') C_{M'}(q', q)^* \\ &\quad = \int_{\mathscr{C}} \nu_M(q', q'^*)\, d^2q' \left[\int_{\mathscr{R}} dq'' f(q'') C_M(q', q'')\right] C_{M'}(q', q)^* \\ &\quad = \int_{\mathscr{R}} dq'' f(q'') \left[\int_{\mathscr{C}} \nu_M(q', q'^*)\, d^2q' C_M(q', q'') C_{M'}(q', q)^*\right]. \end{aligned} \tag{9.53}$$

The expression between brackets, we suspect, should be $\delta(q - q'')$ for $\mathbf{M}' = \mathbf{M}$. Saving the reader some algebra, we calculate

$$\begin{aligned} &\int_{\mathscr{C}} \nu_M(q', q'^*)\, d^2q' C_M(q', q'') C_{M'}(q', q)^* \\ &\quad = (2\pi^3 vbb'^*)^{-1/2} \exp[i(aq''^2/2b - a'^*q^2/2b'^*)] \\ &\qquad \times J(b^*/2bv, 1/v, u^*/2v - i\, d'^*/2b'^*, -iq''/b, iq/b'^*) \\ &\quad = C_{M'^{*-1}M^*}(q'', q) \xrightarrow[M' \to M]{} \delta(q - q''), \end{aligned} \tag{9.54}$$

the primed parameters being the parameters of $\mathbf{M}'$, $J(\cdots)$ being the integral (9.50), and the conclusion being due to (9.22). Equation (9.54) substituted into (9.53) reproduces (9.52).

9.2.4. The Bargmann Transform and Space

A particular case of complex linear canonical transforms—which will serve as the Fourier transform did for the real transforms—is *Bargmann's* transform, defined as $\mathbb{C}_B$, where

$$\mathbf{B} = 2^{-1/2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \qquad -i = \exp(-i\pi/2). \tag{9.55a}$$

The transform kernel and weight function in $\mathscr{B}_B$-space are

$$C_B(q', q) = (2^{1/2}\pi)^{-1/2}\exp(-q^2/2 + 2^{1/2}qq' - q'^2/2), \tag{9.55b}$$

$$\nu_B(q', q'^*) = (2/\pi)^{1/2}\exp(-|q'|^2), \qquad u_B = 0, v_B = 1. \tag{9.55c}$$

This essentially defines the transform introduced by Bargmann in his 1961 paper. There, the inversion formula (9.52) is rigorously proven. The space $\mathscr{B}_B$ of entire analytic functions—called Segal–Bargmann or *Bargmann space* thereafter—is shown to be a *Hilbert* space, and $\mathbb{C}_B$ is thereby established as a *unitary* operator. The Bargmann transform, as it appeared originally and is currently used in most of the literature, has normalization coefficients different from ours: The constant factor in the transform kernel (9.55b) is $\pi^{-1/4}$ in place of our $(2^{1/2}\pi)^{-1/2}$, and in the weight function (9.55c) it is π^{-1} instead of our $(2/\pi)^{1/2}$. We uphold our choice of constants by the argument that the normalization for all $\mathbb{C}_M$ transforms provides the correct composition formulas and limits for arbitrary parameter values.

9.2.5. Properties of the Harmonic Oscillator Wave Functions

The important feature of the Bargmann transform is that, from (9.1) and (7.160),

$$\mathbb{C}_B\mathbb{Z}^\dagger = \mathbb{Q}\mathbb{C}_B, \tag{9.56a}$$

$$\mathbb{C}_B\mathbb{Z} = i\mathbb{P}\mathbb{C}_B. \tag{9.56b}$$

Hence

$$(\mathbb{C}_B\mathbb{Z}^\dagger\mathbf{f})(q') := 2^{-1/2}\big(\mathbb{C}_B(\mathbb{Q} - i\mathbb{P})\mathbf{f}\big)(q') = (\mathbb{Q}\mathbb{C}_B\mathbf{f})(q') = q'f^B(q'), \tag{9.57a}$$

$$(\mathbb{C}_B\mathbb{Z}\mathbf{f})(q') := 2^{-1/2}\big(\mathbb{C}_B(\mathbb{Q} + i\mathbb{P})\mathbf{f}\big)(q') = (i\mathbb{P}\mathbb{C}_B\mathbf{f})(q') = df^B(q')/dq'. \tag{9.57b}$$

In other words, application of the harmonic oscillator raising operator $\mathbb{Z}^\dagger$ on $f(q)$ multiplies its Bargmann transform $f^B(q')$ by q', while the lowering operator $\mathbb{Z}$ on $f(q)$ transforms into d/dq' acting on $f^B(q')$. Furthermore, under the $\mathscr{B}_B$ inner product, z is the adjoint of d/dz and vice versa, as we can easily see:

$$\begin{aligned}(\mathbb{Q}\mathbf{f}^B, \mathbf{g}^B)_B &= (\mathbb{Q}\mathbb{C}_B\mathbf{f}, \mathbb{C}_B\mathbf{g})_B = (\mathbb{C}_B\mathbb{Z}^\dagger\mathbf{f}, \mathbb{C}_B\mathbf{g})_B = (\mathbb{Z}^\dagger\mathbf{f}, \mathbf{g})_1 = (\mathbf{f}, \mathbb{Z}\mathbf{g})_1 \\ &= (\mathbb{C}_B\mathbf{f}, \mathbb{C}_B\mathbb{Z}\mathbf{g})_B = (\mathbb{C}_B\mathbf{f}, i\mathbb{P}\mathbb{C}_B\mathbf{g})_B = (\mathbf{f}^B, \nabla\mathbf{g}^B)_B. \end{aligned} \tag{9.58}$$

As a consequence, the harmonic oscillator wave functions (Section 7.5) will have a particularly simple Bargmann transform. The oscillator ground state $\Psi_0(q)$ can be seen to transform into the *constant* $(2\pi)^{-1/4}$ [Eqs. (9.49) for the parameters of $\mathbf{B}$ in (9.55a)]. Since we generated the orthonormal set $\{\Psi_n(q)\}_{n=0}^{\infty}$ as powers of $\mathbb{Z}^\dagger$ acting on the ground state, we immediately deduce from (9.57a) that

$$\Psi_n^B(q') = [(2\pi)^{1/2}n!]^{-1/2}q'^n. \tag{9.59}$$

As we are assured that the Bargmann transform is unitary and we know that $\{\Psi_n(q)\}_{n=0}^{\infty}$ constitutes a complete and orthonormal basis for $\mathscr{L}^2(\mathscr{R})$, it follows that *the set of power functions* (9.59) *is a complete and orthonormal basis for Bargmann space*. The mathematics of systems described in terms of harmonic oscillator wave functions is particularly streamlined in $\mathscr{B}_B$ as these wave functions involve only power functions. Applying operators generally means applying multiplication and differentiation, plus some combinatorics.

Exercise 9.15. Verify directly that the power functions (9.59) are an orthonormal set in $\mathscr{B}_B$. Calculate $(\mathbf{\Psi}_n{}^B, \mathbf{\Psi}_n{}^B)_B$ using the polar representation of the complex q'-plane: $d^2q' = |q'|\, d|q'|\, d\arg q'$. The angular integral will provide the $\delta_{n,n'}$-factor, and the radial integral is Euler's representation of the Γ-function (Appendix A).

Exercise 9.16. Show that the harmonic oscillator Hamiltonian operator $\mathbb{H}^h = 2\mathbb{J}_0$ in (9.34e) is represented in Bargmann space essentially as $\mathbb{J}_2$ in (9.34a); i.e., $\mathbb{C}_B\mathbb{J}_0\mathbb{C}_B^{-1} = i\mathbb{J}_2$. Accepting the hyperdifferential representation (7.197) as the Fourier transform $\mathbb{F} = \exp(i\pi/4)\exp(-i\pi\mathbb{J}_0)$, show that *Fourier transformation in Bargmann space appears as a dilatation by a factor* $i = \exp(i\pi/2)$ [as in (7.71) and (9.26)], namely, $\mathbb{C}_B\mathbb{F}\mathbb{C}_B^{-1} = \exp(i\pi/4)\mathbb{D}_i$. Using (7.72), verify the self-reciprocity relation (7.167) of the harmonic oscillator wave functions. Note that, as *matrices* [Eqs. (9.26), (9.32), and (9.55a)],

$$\mathbf{BFB}^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

9.2.6. Transform and Reproducing Kernels as Generating Functions

The transform kernel $C_B(q', q)$ has a rather transparent series expression as a *generating function* relating two orthonormal bases:

$$\sum_{n=0}^{\infty} \Psi_n^B(q')\Psi_n(q)^* = (2\pi)^{-1/4} \sum_{n=0}^{\infty} (n!)^{-1/2} q'^n \Psi_n(q) = (2\pi)^{-1/4} G_\psi(2^{1/2}q', q)$$
$$= C_B(q', q) = 2^{-1/4}\pi^{-1/2} \exp[-q^2/2 + 2^{1/2}qq' - q'^2/2], \tag{9.60}$$

where we have used the harmonic oscillator generating function (7.178).

Exercise 9.17. The Bargmann transform maps $\mathscr{L}^2(\mathscr{R})$ functions onto functions with convergent Taylor expansions in $\mathscr{C}$. Relate the harmonic oscillator "partial waves" of a given function (7.180) with the Taylor coefficients of its Bargmann transform. Equation (9.60) provides a very handy relation.

Exercise 9.18. The representation of transform kernels as generating functions relating orthonormal bases can be applied to other cases; for example, the Fourier transform kernel [Eq. (9.32)] under the integral sign can be represented as

$$(2\pi)^{-1/2} \exp(ipq) = \sum_{n=0}^{\infty} i^n \Psi_n(p)\Psi_n(q) = \exp(i\pi/4)C_F(p, q). \tag{9.61}$$

Exercise 9.19. Perform the Bargmann transform of (9.61) and show that

$$(2\pi)^{-1/2}[\mathbb{C}_B \, \mathbf{exp}(ip\cdot)](q') = \sum_{n=0}^{\infty} \Psi_n(p)\Psi_n^B(iq') = C_B(iq', p). \tag{9.62}$$

Notice that this holds in spite of the fact that the oscillating exponential function is "just" outside $\mathscr{L}^2(\mathscr{R})$. Correspondingly, the Bargmann transform kernel behaves as $\exp(q'^2/2)$, i.e., it is "just" outside $\mathscr{B}_B$. The treatment of generalized functions has been developed in Bargmann's second paper (1967).

Generating functions built out of pairs of orthonormal bases have a deep relation with transform kernels. This is suggested by (9.60) and (9.61). In $\mathscr{L}^2(\mathscr{R})$ when a generating function has orthonormal basis functions in two different arguments, a Dirac δ results as in Eq. (7.180d). In Bargmann space the orthonormal basis (9.59), for two different arguments, yields

$$K_B(q_1, q_2) := \sum_{n=0}^{\infty} \Psi_n^B(q_1)\Psi_n^B(q_2)^* = (2\pi)^{-1/2} \sum_{n=0}^{\infty} (n!)^{-1} q_1^n q_2^{*n}$$
$$= (2\pi)^{-1/2} \exp(q_1 q_2^*). \tag{9.63}$$

This well-defined function is the *reproducing* kernel under the Bargmann inner product integral, since for any $\mathbf{f}^B \in \mathscr{B}_B$,

$$\int_{\mathscr{C}} \nu(q, q^*) \, d^2q f^B(q) K_B(q', q) = f^B(q'), \tag{9.64}$$

as its Taylor series converges. The role of the Dirac δ in $\mathscr{L}^2(\mathscr{R})$ is thus taken by K_B in $\mathscr{B}_B$.

Exercise 9.20. Prove that

$$\int_{\mathscr{C}} \nu(q_2, q_2^*)\, d^2q_2 K_B(q_1, q_2) K_B(q_2, q_3) = K_B(q_1, q_3) \tag{9.65}$$

is the analogue of the convolution relation between Dirac δ's in Exercise 7.31.

9.2.7. The Coherent-State Basis

There are two generalized bases of the Dirac kind for $\mathscr{L}^2(\mathscr{R})$ and $\mathscr{B}_B$ on which we would like to comment briefly. First, there is *the* Dirac basis which "expands" a function in terms of a continuum of Dirac δ's sitting on points $q' \in \mathscr{R}$ with continuous linear combination coefficients $f(q')$ as suggested by Eq. (7.86). Correspondingly, Eq. (9.64) expands a function $f^B(q')$ in a continuum of functions $K_B(q, q')$ for $q \in \mathscr{C}$ with linear combination coefficients $f^B(q)$. This set is *overcomplete* in the sense that the functions are linearly independent but not orthogonal [Eq. (9.65)]. Second, we can point out that the inverse and direct Bargmann transforms are

$$f(q) = (2\pi)^{1/4} \int_{\mathscr{C}} \nu_B(q', q'^*)\, d^2q' f^B(q') \Upsilon_{q'}(q), \qquad q \in \mathscr{R}, \tag{9.66a}$$

$$f^B(q') = (2\pi)^{1/4} \int_{\mathscr{R}} dq f(q) \Upsilon_q(q'), \qquad q' \in \mathscr{C}, \tag{9.66b}$$

where we have introduced the *coherent states* (7.188), which are, note, essentially the Bargmann transform kernel. Equation (9.66a) can be interpreted as *the expansion of a function in a complex continuum of coherent states.* This basis is again overcomplete, although a strictly complete subset can be found [see Bargmann *et al.* (1971)]. Conversely, Eq. (9.66b) expands a function $\mathbf{f}^B \in \mathscr{B}_B$ in a real continuum of the same states. Several other generalized bases, complete and orthonormal, can be found for Bargmann space, including, strangely enough, the repulsive oscillator wave-function basis, which happens to be self-reciprocal under $\mathbb{C}_B$ [see Wolf (1977a) and Exercise 9.29].

9.2.8. The Gauss–Weierstrass Transform as a Complex Canonical Transform

The main features of the Bargmann transform can be extended to all other permissible complex canonical transforms. We shall not burden the reader with generalities, however. We *are* interested, nevertheless, in two other important particular cases of $\mathbb{C}_M$ transforms, the Gauss–Weierstrass "diffusive" transform and the bilateral Laplace transform.

The Gauss–Weierstrass transform (for time t) arises for the matrix parameters

$$\mathbf{W}(t) = \begin{pmatrix} 1 & -2it \\ 0 & 1 \end{pmatrix}, \qquad t > 0, \ -i = \exp(-i\pi/2). \tag{9.67}$$

The general $\mathbb{C}_{w(t)}$ kernel (9.8) then becomes the Gauss–Weierstrass kernel (8.90) which is the diffusion equation Green's function for time t. The applications of canonical transforms to the study of the diffusion equation will occupy Section 10.1. The main points we want to emphasize here are the following: (a) The $\mathbb{C}_{w(t)}$ transform provides us with the known results on analyticity of the heat equation's solution [see Widder (1975)]. (b) Diffusive time evolution has, as the energy in the wave equation, a *sesquilinear* invariant associated with a conserved inner product $(\cdot, \cdot)_{w(t)}$ in $\mathscr{B}_{w(t)}$. Previously, only *total heat*, a linear invariant, was counted. (c) The problem of the *backward* time evolution takes a new aspect as a $\mathbb{C}_{w(t)}^{-1}$ operator acting on $\mathscr{B}_{w(t)}$. [In this connection, recall the discussion of the Gauss–Weierstrass transform in Section 8.5 and its inversion. The problem has been tackled by Doetsch (1928, 1936) and Tricomi (1936). More modern treatments can be found in the work of Bilodeau (1961) and Rooney (1957, 1958, 1963). Its relevance in physics is connected with the quantum mechanics of unstable particles; see the articles by Horwitz *et al.* (1971) and Sinha (1972).]

9.2.9. The Collapse of Bargmann to $\mathscr{L}^2(\mathscr{R})$-Spaces, the Laplace Transform

The bilateral Laplace transform kernel will now be seen to arise as the particular case of (9.8) for the parameter values

$$\mathbf{L} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad i = \exp(i\pi/2). \tag{9.68}$$

The problem with (9.68), however, is that when setting up a $\mathscr{B}_L$-space we are faced with an apparently singular weight function in (9.47), since for $a = 0$ the values of the parameters (9.43) are $u = -1$, $v = 0 = w$. We shall examine in general the behavior of the real weight function $\nu_M(q, q^*)$ as $v \to 0$. Basically, we shall see that the integral over the complex plane in the inner product and the inversion formula becomes a $\int_{\mathscr{R}_\phi} dq \cdots$ in the complex plane along a ray depending in the phase of u. In the case when $\mathbf{M}$ becomes real, the integration contour becomes the real axis. To follow the expression for $\nu_M(q, q^*)$ as $v \to 0^+$ we use the polar form given by the last member of Eq. (9.47). From (9.43d) as $v \to 0^+$, $\omega = (1 - vw)^{1/2} \simeq 1 - vw/2$, so that $\nu_M(q, q^*)$ has its maximum along the line in the complex q-plane where $\cos(\phi + 2\theta) = 1$, i.e., for $\arg q = \theta = -\phi/2 = -\frac{1}{2}\arg u$ and for $\theta = -\phi/2 + \pi$. This approximation and a trigonometric identity allow us to write, for $v \to 0^+$,

$$\nu_M(q, q^*) \simeq (\pi v/2)^{-1/2} \exp(-w\rho^2/2) \exp[-2\rho^2 \sin^2(\theta + \phi/2)/v]. \tag{9.69}$$

In the limit $v \to 0^+$ we can use a sequence of Gaussians of decreasing width which define the Dirac δ in (7.83), writing, as a *weak* limit,

$$\nu_M(q, q^*) \xrightarrow[v\to 0^+]{} 2^{-1/2} \exp(-w\rho^2/2)\delta\big(2^{1/2}\rho \sin(\theta + \phi/2)\big). \tag{9.70}$$

Since this appears under the double integral $\int_0^\infty \rho\, d\rho \int_{-\pi}^{\pi} d\theta$, the point $\rho = 0$ is immaterial insofar as the δ is concerned and only the integration over θ is reduced to the integrand's value at the roots of the sine function: $\theta = -\phi/2$ and $\theta = \pi - \phi/2$, the former with a plus and the latter with a minus sign. If we define $x := \rho \exp(-i\phi/2)$ and $-x := \rho \exp[i(\pi - \phi/2)]$, the limit of the inner product will be a *line* integral,

$$\lim_{v\to 0^+} \int_{\mathscr{C}} \nu_M(q, q^*)\, d^2q f^M(q)^* g^M(q) = \int_{\mathscr{R}_\phi} \exp(-w|x|^2/2)\, dx f^M(x)^* g^M(x), \tag{9.71}$$

along $\mathscr{R}_\phi := \mathscr{R} \exp(-i\phi/2)$, tilted by $-\frac{1}{2} \arg u$ with respect to the real axis. When the matrix $\mathbf{M}$ is real, $u = 1$, $\phi = 0$, and $w = 0$; hence $(\mathbf{f}^M, \mathbf{g}^M)_M = (\mathbf{f}^M, \mathbf{g}^M)_1$, and the defining inner product for $\mathscr{B}_M$ is that of $\mathscr{B}_1 = \mathscr{L}^2(\mathscr{R})$.

We return now to the canonical transform $\mathbb{C}_L$ determined by (9.68). The transform kernel (9.8) is $-(2\pi)^{-1/2} i \exp(-qq')$, which is the bilateral Laplace kernel (although unfortunately off by a factor and phase). The corresponding $\mathscr{B}_L$-space has an inner product (9.71) for $w = 0$, and, since $u = -1$, $\phi = \pm\pi$. The inner product therefore involves *integration up along the imaginary axis.* (See Exercise 9.21.) This agrees with (8.1). The inverse Laplace transform integrates, along this contour, the function in company with the complex conjugate kernel: $(2\pi)^{-1/2} i \exp(xq)$. As x is the value of q' for pure imaginary values, complex conjugation produces the correct inverse transform kernel.

It should be noted that the Laplace transform has been correctly described in spite of the fact that $\mathbf{L}$ in (9.68) does *not* satisfy the integrability conditions (9.19)—in fact, we know that the bilateral Laplace transform is not defined on all of $\mathscr{L}^2(\mathscr{R})$ but only on a *dense* subset: causal functions. We shall comment briefly on this below.

Exercise 9.21. The Laplace transform matrix (9.68) has $u = -1$ [Eq. (9.43a)]. This could mean $u = \exp(in\pi)$ for any odd integer n. Concentrate on $n = \pm 1$. Show that the two cases give rise to the same integral: $x = \rho \exp(\mp i\pi/2)$ along the ray at $\mp i\pi/2$ minus $x = \rho \exp(\pm i\pi/2)$ along the ray at $\pm i\pi/2$.

9.2.10. Further Extensions and References

To sum up, we have shown that the class of complex linear transforms $\mathbb{C}_M$ is a set of unitary transformations between $\mathscr{L}^2(\mathscr{R})$ and Bargmann-like spaces $\mathscr{B}_M$, which properly include the Bargmann, Gauss–Weierstrass, Moshinsky–Quesne (real linear), and—somewhat improperly—bilateral Laplace transforms.

Certain further developments will be left for the interested reader to look up in the literature. The first one concerns the composition of complex linear canonical transforms, generalizing the real case discussed in Section 9.1 and proceeding essentially from the composition property (9.54). Second, we can repeat our program in N dimensions, dealing with $2N$-dimensional *symplectic* matrices $\mathbf{M}$ and defining integral transforms in N-space (Wolf, 1974a). Third, there is the question of existence of transforms *violating*—as the Laplace transform—the integrability conditions (9.19). It turns out that the *unitary* canonical transforms constitute a *subsemigroup* of the group of $2N \times 2N$ symplectic matrices. [See Kramer and Schenzle (1973) and Brunet and Kramer (1976).] It *also* turns out *apropos* that many of the nuclear cluster model calculations need transforms which lie outside this subsemigroup. [See Zahn (1975), Seligman (1976), and Seligman and Zahn (1976a).] Fourth, N-dimensional transforms invite the consideration of *complex radial transforms* (Moshinsky *et al.*, 1972; Wolf, 1974b), which include the Hankel transform and—corresponding to the Bargmann case—the *Barut–Girardello* transform [see Barut and Girardello (1971)]. [See also Kramer *et al.* (1975, Section VI), where Girardello's name has been unjustly left out, and Seligman and Zahn (1976b).] The role of canonical transformations in quantum mechanics has suggested several generalizations (Moshinsky, 1973; Mello and Moshinsky, 1975). Finally, studies in group representations have been done with the aid of canonical transforms. [See Boyer and Wolf (1975, 1976) and Wolf (1977b).]

9.3. Canonical Transforms by Hyperdifferential Operators

We have a parametrized continuum of integral transforms, one for each complex 2×2 unimodular matrix. Since these matrices form a *group*, and integral transforms compose following matrix multiplication—up to a sign—we have at our disposal the powerful results of Lie theory to define and solve many questions. Our aim here does not require the full use of group theory language; rather, we shall phrase the subject of integral transforms in the following terms.

9.3.1. Operators Generating Transforms

Given a one-parameter integral transform family

$$(\mathbb{C}_{M(\tau)}\mathbf{f})(q') = \int_{\mathscr{R}} dq f(q) C_{M(\tau)}(q', q) =: f(q', \tau), \tag{9.72}$$

which includes the identity for $\tau = 0$ [i.e., $\mathbf{M}(0) = \mathbf{1}$ and $f(q, 0) = f(q)$], we want to find a differential operator $\mathbb{H}$ which we can write as a function of q and d/dq such that

$$(\mathbb{C}_{M(\tau)}\mathbf{f})(q) = \exp(i\tau\mathbb{H})f(q) =: \sum_{n=0}^{\infty} (n!)^{-1}(i\tau\mathbb{H})^n f(q). \tag{9.73}$$

Operators of this kind have appeared before mainly in connection with the time evolution of the wave and diffusion equations, translations, and dilatations. They involve arbitrarily high derivatives so their domain must be a subset of the $\mathscr{C}^\infty$ functions (although we have seen that *weakly* their action can be defined on larger function spaces).

The operator $\mathbb{H}$ can be formally obtained from (9.73) by differentiating with respect to τ and setting $\tau = 0$:

$$\mathbb{H}f(q') = -i\frac{\partial}{\partial\tau}\int_{\mathscr{R}} dq f(q) C_{M(\tau)}(q', q)|_{\tau=0}. \tag{9.74}$$

The operator $\mathbb{H}$ will be said to *generate* the integral transform family (9.72). The theory of Lie groups assures us that once we have found the operator $\mathbb{H}$ by (9.74) its exponentiated (and properly defined) action is that of (9.72)–(9.73).

The following one-parameter subgroups of the group $SL(2, \mathscr{R})$ are of particular interest:

$$\mathbf{M}^d(e^{-\tau}) := \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix},$$

$$\begin{aligned}\frac{\partial}{\partial\tau} C_M(q', q)\bigg|_{\tau=0} &= \frac{1}{2} C_M(q', q) - \exp\left(\frac{\tau}{4}\right) q'\delta'\left(q - \exp\left(\frac{\tau}{2}\right) q'\right)\bigg|_{\tau=0} \\ &= \frac{1}{2}\delta(q - q') - q'\frac{\partial}{\partial q}\delta(q - q') \\ &= \left(\frac{1}{2} - q\frac{\partial}{\partial q}\right) C_M(q', q)\bigg|_{\tau=0},\end{aligned} \tag{9.75a}$$

$$\mathbf{M}^f(\tau) := \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix},$$

$$\begin{aligned}\frac{\partial}{\partial\tau} C_M(q', q)\bigg|_{\tau=0} &= -(2\tau)^{-1} C_M(q', q) + i\frac{1}{2}\tau^{-2}(q - q')^2 C_M(q', q)|_{\tau=0} \\ &= -i\frac{1}{2}\frac{\partial^2}{\partial q^2} C_M(q', q)\bigg|_{\tau=0},\end{aligned} \tag{9.75b}$$

$$\mathbf{M}^g(\tau) := \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix},$$

$$\frac{\partial}{\partial\tau} C_M(q', q)\bigg|_{\tau=0} = i\frac{1}{2}q'^2 C_M(q', q)\bigg|_{\tau=0} = i\frac{1}{2}q^2\delta(q - q')$$

$$= i\frac{1}{2}q^2 C_M(q', q)\bigg|_{\tau=0}, \tag{9.75c}$$

$$\mathbf{M}^r(\tau) := \begin{pmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix},$$

$$\frac{\partial}{\partial\tau} C_M(q', q)\bigg|_{\tau=0} = \left[\frac{1}{2}\coth\tau + i\frac{1}{2}(q^2 - 2qq'\cosh\tau + q'^2)/\sinh^2\tau\right]$$

$$\times\ C_M(q', q)|_{\tau=0}$$

$$= i\frac{1}{2}\left(-\frac{\partial^2}{\partial q^2} - q^2\right)C_M(q', q)\bigg|_{\tau=0}, \tag{9.75d}$$

$$\mathbf{M}^h(\tau) := \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix},$$

$$\frac{\partial}{\partial\tau} C_M(q', q)\bigg|_{\tau=0} = \left[\frac{1}{2}\cot\tau - i\frac{1}{2}(q^2 - 2qq'\cos\tau + q'^2)/\sin^2\tau\right]$$

$$\times\ C_M(q', q)|_{\tau=0}$$

$$= i\frac{1}{2}\left(-\frac{\partial^2}{\partial q^2} + q^2\right)C_M(q', q)\bigg|_{\tau=0}. \tag{9.75e}$$

We have used the labeling (9.26)–(9.28), (9.30), and (9.31) for the matrices and formulas (9.8) and (9.21) for the transform kernels. The final expression for each case is put in a form where integration by parts can be readily implemented so as to have the differential operators acting—with a minus sign—on the integrand function $f(q)$ in (9.74). The generating operators are thus found to be, respectively,

$$\mathbb{H}^d := \tfrac{1}{2}(\mathbb{QP} + \mathbb{PQ}) = 2\mathbb{J}_2, \tag{9.76a}$$

$$\mathbb{H}^f := \tfrac{1}{2}\mathbb{P}^2 = \mathbb{J}_0 + \mathbb{J}_1, \tag{9.76b}$$

$$\mathbb{H}^g := \tfrac{1}{2}\mathbb{Q}^2 = \mathbb{J}_0 - \mathbb{J}_1, \tag{9.76c}$$

$$\mathbb{H}^r := \tfrac{1}{2}(\mathbb{P}^2 - \mathbb{Q}^2) = 2\mathbb{J}_1, \tag{9.76d}$$

$$\mathbb{H}^h := \tfrac{1}{2}(\mathbb{P}^2 + \mathbb{Q}^2) = 2\mathbb{J}_0, \tag{9.76e}$$

where we have introduced the three $\mathbb{J}_i$ operators from Eqs. (7.174) and (9.34).

Lie theory customarily works with the exponentiation of first-order differential operators such as $\mathbb{P}$ [generating translations: Eq. (7.69)], $\mathbb{J}_2$ [generating dilatations: Eq. (7.71)], $\mathbb{Q}$, or $\mathbb{Q}^2$ (multiplying the function by an exponential or Gaussian). Here we are exponentiating *second*-order differential operators of which only $\mathbb{P}^2$ has been seen before [in Eqs. (7.74)–(7.75)]. When substituted into (9.73), Eqs. (9.76) lead to

$$\exp(i\beta\mathbb{J}_2) = \mathbb{C}\begin{pmatrix} \exp(-\beta/2) & 0 \\ 0 & \exp(\beta/2) \end{pmatrix}, \tag{9.77a}$$

$$\exp(ib\mathbb{P}^2/2) = \mathbb{C}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \tag{9.77b}$$

$$\exp(ic\mathbb{Q}^2/2) = \mathbb{C}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \tag{9.77c}$$

$$\exp(i\alpha\mathbb{J}_1) = \mathbb{C}\begin{pmatrix} \cosh(\alpha/2) & -\sinh(\alpha/2) \\ -\sinh(\alpha/2) & \cosh(\alpha/2) \end{pmatrix}, \tag{9.77d}$$

$$\exp(i\gamma\mathbb{J}_0) = \mathbb{C}\begin{pmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix}. \tag{9.77e}$$

The matrix subindex of $\mathbb{C}_M$ has been written, for clarity, as $\mathbb{C}(\mathbf{M})$. If we allow complex parameters in the $SL(2, \mathscr{R})$ matrices, we can denote the resulting group by $SL(2, \mathscr{C})$: complex two-dimensional special (unimodular) linear transformations. We can now state that *linear canonical transforms are generated by all operators constructed out of quadratic expressions in* $\mathbb{Q}$ *and* $\mathbb{P}$. It is easy to see that (9.76) constitute bases for *all* second-order operators in $\mathbb{Q}$ and $\mathbb{P}$ and, only slightly less easy, that the one-parameter subgroups generated by (9.76) exhaust, by multiplication, all of $SL(2, \mathscr{R})$. [See (9.29), (9.33), or the "Euler angle" decomposition involving (9.77d) and (9.77e).] It does, however, require a good amount of mathematical finesse to fully condition and justify that the correspondence between hyperdifferential operators and integral transforms does hold over a continuous parameter range as suggested by our deceptively simple approach. Some of the aspects related to the domains of the hyperdifferential and integral forms of operators have been indicated at the end of Section 7.2. In dealing interchangeably with the two forms, we shall not encounter any major pitfalls and shall be able to simplify rather messy calculations to simple 2×2 matrix algebra. The results can always be verified by the more traditional methods.

We now have *three* sets of mathematical objects at our disposal: (a) integral transforms $\mathbb{C}(\mathbf{M})$, (b) hyperdifferential operators $\exp(i\tau\mathbb{H})$, and (c) 2×2 matrices $\mathbf{M}$. For every element in one there are corresponding elements in the other two, and this correspondence is preserved under composition, sum, and multiplication save for a possible sign in the composition of integral transforms. As we shall see, the key element for many applications is that all $b = 0$ "integral" transforms do not involve integration at all but are purely *geometric* transformations such as (9.23).

Exercise 9.22. Verify that the Fourier transform hyperdifferential form suggested in (7.197) holds as it belongs to the family (9.77e) for $\gamma = -\pi$. Show that $\mathbb{F}^2$ is the inversion operator (9.77a) times $\exp(i\pi/2)$ and $\mathbb{F}^4 = \mathbb{1}$.

Exercise 9.23. Verify that the square of the Bargmann transform is the inverse of the bilateral Laplace transform.

Exercise 9.24. Let $\mathbb{A}$ and $\mathbb{H}$ be operators. Prove the following relation:

$$\exp(\theta\mathbb{H})\mathbb{A}\exp(-\theta\mathbb{H}) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} [\mathbb{H}, [\mathbb{H}, \cdots [\mathbb{H}, \mathbb{A}] \underbrace{\cdots]]}_{n}. \tag{9.78}$$

This can be seen for the first few powers of θ and then by induction on n.

Exercise 9.25. Use Eq. (9.78) in order to verify that the exponentiated second-order operators (9.77) indeed transform $\mathbb{Q}$ and $\mathbb{P}$ as in (9.38). The subgroups (9.77b) and (9.77c) lead to a terminating series. The series obtained from (9.77a) can be summed. The subgroups (9.77d) and (9.77e) require a recursion argument.

9.3.2. Baker–Campbell–Hausdorff Formulas

Consider the following matrix identity:

$$\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} = \begin{pmatrix} 1 & -\tanh\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \operatorname{sech}\theta & 0 \\ 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tanh\theta & 1 \end{pmatrix}, \tag{9.79a}$$

i.e., a decomposition along the lines of (9.29). Using the correspondence (9.77) with hyperdifferential operators, we are led to

$$\begin{aligned} \exp[-i\theta\tfrac{1}{2}(d^2/dx^2 + x^2)] = {} & \exp(-i\tfrac{1}{2}\tanh\theta\, d^2/dx^2) \\ & \times \exp[\tfrac{1}{2}\ln\cosh\theta\,(x\, d/dx + d/dx\cdot x)] \\ & \times \exp(-i\tfrac{1}{2}\tanh\theta x^2). \end{aligned} \tag{9.79b}$$

This is a *Baker–Campbell–Hausdorff* relation between hyperdifferential operators. [See, for instance, Wilcox (1967) and Eriksen (1968).] By substituting the value $\theta = i\pi/4$ into (9.79), this becomes the Bargmann transform matrix [Eqs. (9.55)], as $\cosh(i\pi/4) = 2^{-1/2} = -i\sinh(i\pi/4)$. This implies that, acting on a function,

$$\begin{aligned} (\mathbb{C}_B\mathbf{f})(q) &= \exp[(\pi/8)(d^2/dq^2 + q^2)]f(q) \\ &= 2^{-1/4}\exp(\tfrac{1}{2}\, d^2/dq^2)[\exp(q^2/4)f(2^{-1/2}q)], \end{aligned} \tag{9.80}$$

i.e., reading from right to left, $f(q)$ is multiplied by $\exp(q^2/2)$, unitarily rescaled by a factor $2^{-1/2}$, and finally subjected to a unit Gauss–Weierstrass transform. This produces the Bargmann transform of the function.

Exercise 9.26. Verify that the three operations in (9.80) transform the harmonic oscillator wave functions $\Psi_n(q)$ in (7.166) into power functions. In particular, you will use the inverse of (7.193) in the last step. Conversely, this equation can be proven by (9.80).

Exercise 9.27. Use the decomposition (9.33)—the leftmost factor separated in two—to write another Baker–Campbell–Hausdorff formula for the first member of (9.79).

Exercise 9.28. Use the result of Exercise 9.27 to express the Bargmann transform as (a) multiplication by a decreasing Gaussian of unit width, (b) Fourier transformation, (c) multiplication by another Gaussian, and (d) change of scale.

Exercise 9.29. Find the Bargmann transform of the *repulsive* oscillator functions (7.203). You can either resort to a table of integrals or make use of the eigenfunction equation (7.198) together with the hyperdifferential expression for $\mathbb{C}_B$ in (9.80). You have thus found a new generalized basis for Bargmann space. [See Wolf (1977a).]

9.3.3. Time-Evolution Operators as Generating Canonical Transforms

The hyperdifferential operator realization for canonical transforms will serve us now to bring out the relation between the transform kernels and the Green's functions for a set of quantum-mechanical systems.

Consider partial differential equations of the form

$$\mathbb{H}\psi(q, t) = -i\frac{\partial}{\partial t}\psi(q, t), \tag{9.81}$$

where $\mathbb{H}$ is a differential operator in q only. This is the *diffusion* equation when $\mathbb{H}$ is $-2i\mathbb{H}^f$ as defined in (9.76b). It is Schrödinger's equation for the quantum free particle or the repulsive or the harmonic oscillator when $\mathbb{H}$ is $\mathbb{H}^f$, $\mathbb{H}^r$, or $\mathbb{H}^h$, respectively [Eqs. (9.76b), (9.76d), or (9.76e)—thence the superscript labeling]. The solution to (9.81) at time t can be expressed in terms of the initial or boundary data at time $t = 0$:

$$\psi(q, t) = \exp(it\mathbb{H})\psi(q, 0). \tag{9.82a}$$

This corresponds to a family of canonical transforms parametrized by t,

$$\psi(q, t) = (\mathbb{C}_{M(t)}\psi(\cdot, 0))(q) = \int_{\mathscr{R}} dq'\psi(q', 0)C_{M(t)}(q, q'), \tag{9.82b}$$

where $\mathbf{M}(t)$ is the one-parameter matrix subgroup associated to the generator

$\mathbb{H}$. The canonical transform $\mathbb{C}_{M(t)}$ is the *time-evolution or Green's operator* for the system governed by (9.81). If the initial $\psi(q, 0)$ is a Dirac δ sitting at q' [i.e., $\delta_{q'}(q) := \delta(q - q')$], then clearly the solution to (9.82) is

$$G_{q'}(q, t) := (\mathbb{C}_{M(t)}\boldsymbol{\delta}_{q'})(q) = C_{M(t)}(q, q'), \tag{9.83}$$

which is the Green's function for the system.

Exercise 9.30. Verify independently that (9.83) is the Green's function of (9.81) for the five one-parameter subgroups $\mathbf{M}(t)$ which we have been handling, since (a) Eqs. (9.75) show that it is a solution to the differential equation, and (b) it is such that $G_{q'}(q, 0) = \delta(q - q')$.

The economy of using matrices to represent canonical transforms is readily apparent when the initial conditions are themselves given as integral transforms. Consider the time development of a real Gaussian wave function, of width ω centered at q' and normalized to unity, under a quantum harmonic oscillator potential, i.e.,

$$\begin{aligned}\psi(q, 0) = G_\omega(q - q') &= \left[\mathbb{C}\begin{pmatrix}1 & \exp(-i\pi/2)\omega\\ 0 & 1\end{pmatrix}\boldsymbol{\delta}_{q'}\right](q)\\ &= (2\pi\omega)^{-1/2}\exp[-(q - q')^2/2\omega].\end{aligned} \tag{9.84}$$

The time evolution is given by the $\mathbf{M}^h(t)$ subgroup of transforms (9.75e)–(9.76e)–(9.77e):

$$\begin{aligned}\psi(q, t) &= \left[\mathbb{C}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\boldsymbol{\psi}(\cdot, 0)\right](q)\\ &= \left[\mathbb{C}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\mathbb{C}\begin{pmatrix}1 & -i\omega\\ 0 & 1\end{pmatrix}\boldsymbol{\delta}_{q'}\right](q)\\ &= \left[\mathbb{C}\begin{pmatrix}\cos t & -\sin t - i\omega\cos t\\ \sin t & \cos t - i\omega\sin t\end{pmatrix}\boldsymbol{\delta}_{q'}\right](q).\end{aligned} \tag{9.85}$$

The result is thus a C_M function which can be written by substituting the entries of the last matrix into (9.8). No integration is needed. In Figs. 9.3, 9.4, and 9.5 is the time development of a Gaussian wave function under free, repulsive, and harmonic oscillator potentials. The diffusion equation Green's function is obtained from that of the free-particle Schrödinger case by the simple replacement $t \to 2it$.

Exercise 9.31. Note that the solution (9.85) is periodic with period 2π. This is to be expected, as, classically, the oscillation period of a harmonic oscillator is independent of its energy, and thus the wave function returns to its original shape after that time. At half that period, show that the wave function is inverted. At one-quarter the period, you have the inverse Fourier transform of the initial wave function. See Fig. 9.5.

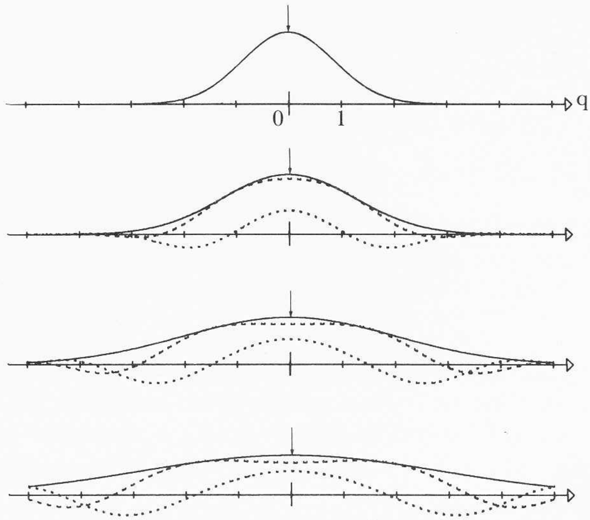

Fig. 9.3. Time development of a Gaussian wave function (of width 0.75, centered at the origin) under the free-particle Schrödinger equation. Real, imaginary, and absolute values are plotted in heavy dotted, light dotted, and continuous lines. The time intervals between two graphs are $\pi/4$. The small arrows indicate the (fixed) peak of the broadening, complex Gaussian.

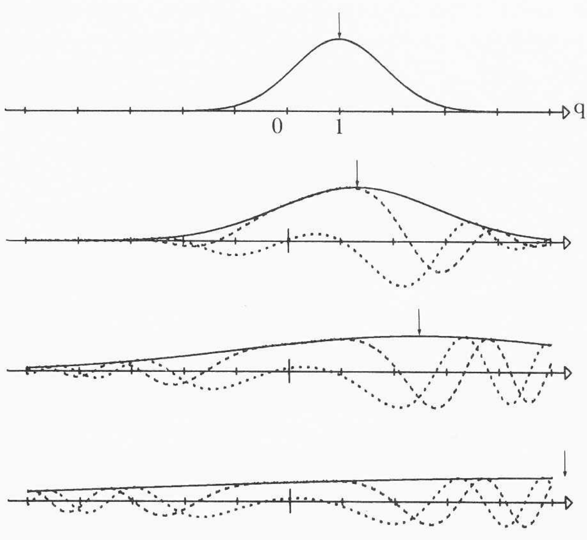

Fig. 9.4. Time development of a Gaussian wave function (of width 0.75, centered at $q = 1$) under the repulsive oscillator Schrödinger equation. Plot marks and parameters are as in Figure 9.3. The peak of the spreading Gaussian moves as a classical particle would under the same potential.

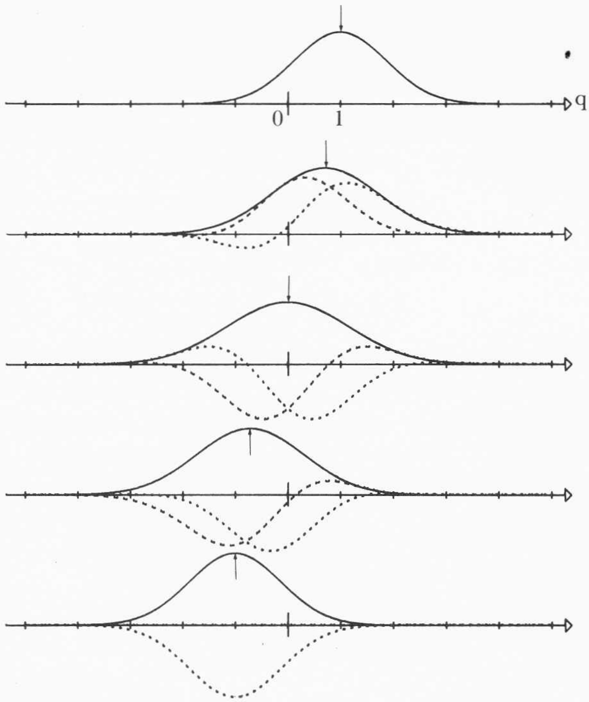

Fig. 9.5. Time development of a Gaussian wave function under the harmonic oscillator Schrödinger equation (9.85). Figure characteristics are as in Fig. 9.4. Here, the Gaussian peak performs the harmonic motion characteristic of coherent states.

9.3.4. Quantum Normal Modes

Wave functions generally lose their original shape under the influence of a quantum potential. One set of functions which *do* preserve their form, remindful of the normal modes of the elastic media, is the *eigenfunctions* of $\mathbb{H}$. If $\Psi_\lambda(q)$ is an eigenfunction of $\mathbb{H}$ with eigenvalue λ, then for $\mathbf{M}(t)$ generated by $\mathbb{H}$,

$$\Psi_\lambda(q, t) := \int_{\mathscr{R}} dq'\Psi_\lambda(q')C_{M(t)}(q, q') = \exp(it\mathbb{H})\Psi_\lambda(q) = \exp(it\lambda)\Psi_\lambda(q), \tag{9.86}$$

i.e., it is *self-reproducing* under all $\mathbb{C}_{M(t)}$ and a *separated* function of q and t. Detailing: as the operators which we are interested in are $\mathbb{H}^f$, $\mathbb{H}^r$, and $\mathbb{H}^h$, for the purposes of notational uniformity we shall denote their eigenfunctions by

$\Psi^{\omega}_{\lambda,\sigma}(q)$, $\omega = f, r, h$, with an extra label σ to resolve degeneracy when necessary. These are

$$\mathbb{H}^f:\quad \Psi^f_{\lambda,\sigma}(q) = (2\pi)^{-1/2}\exp[i\sigma(2\lambda)^{1/2}q],$$
$$\lambda \in \mathscr{R}^+,\ \sigma = \pm, \tag{9.87a}$$

$$\mathbb{H}^r:\quad \Psi^r_{\lambda,\sigma}(q) = \chi_\lambda{}^\sigma(q),$$
$$\lambda \in \mathscr{R},\ \sigma = \pm \text{ [Eqs. (7.203)]}, \tag{9.87b}$$

$$\mathbb{H}^h:\quad \Psi_\lambda{}^h(q) = \Psi_n(q),$$
$$\lambda = n + 1/2,\ n = 0, 1, 2, \ldots \text{ [Eq. (7.166)]}. \tag{9.87c}$$

Ordinary or Dirac orthonormality and completeness hold for these functions. Hence, multiplying (9.86) by $\Psi_\lambda(q'')^*$ and summing or integrating, as the case may be, over the label set (λ, σ), from the second and last members we obtain

$$C_{M^\omega(t)}(q, q'') = \underset{\lambda,\sigma}{\mathsf{S}} \exp(it\lambda)\Psi^\omega_{\lambda,\sigma}(q)\Psi^\omega_{\lambda,\sigma}(q'')^*, \tag{9.88}$$

where the symbol S is meant to stand appropriately for integration or sum in each case. It becomes

$$\sum_{\sigma=\pm}\int_0^\infty (2\lambda)^{-1/2}\,d\lambda \quad (f \text{ case}), \qquad \sum_{\sigma=\pm}\int_{-\infty}^\infty d\lambda \quad (r \text{ case}),$$

$$\sum_{\substack{n=0\\ \lambda=n+1/2}}^{\infty} \quad (h \text{ case}).$$

Equation (9.88) reduces, in the harmonic oscillator case, to (7.180d) for $t = 0$ and (9.61) for $t = \pi/2$. For the free-particle case, $t = 0$ reproduces the known integral representation of the Dirac δ [Eq. (7.93) for $n = 0$], and similarly for the repulsive oscillator case. The transform kernel is shown by (9.88) to generalize the above completeness relations.

Exercise 9.32. Show that the general $\mathbb{C}_M$ transforms of the harmonic oscillator wave functions are

$$(\mathbb{C}_M\Psi_n)(q) = \left[\left(2\frac{a+ib}{a-ib}\right)^n n!\pi^{1/2}(a+ib)\right]^{-1/2}\exp\left(-\frac{d-ic}{a+ib}\frac{q^2}{2}\right)$$
$$\times\ H_n\big((a^2+b^2)^{-1/2}q\big). \tag{9.89}$$

Verify the result for Fourier and Bargmann transforms. In the latter case, as $a^2 + b^2 = 0$, it turns out that only the polynomial leading term (of coefficient 2^n) survives. Show that (9.89) is defined for almost all complex M. Equation (9.89) can be proven (a) by straightforward integration, (b) by (9.49) and the appropriate raising operators, or, best, (c) by decomposing $\mathbf{M}$ as $\mathbf{M}^g\mathbf{M}^d\mathbf{M}^h$ [Eqs. (9.75c), (9.75a), and (9.75e)] and noting that under the rightmost factor $\Psi_n(q)$ is only multiplied by a phase.

9.3.5. Coherent States and Their Time Evolution

Another set of states which preserve their shape while under the action of a quantum potential is termed *coherent states* and is particularly important for the harmonic oscillator case. Coherent states $\Upsilon_c(q)$ were defined in (7.188) either as the result of acting with the exponentiated oscillator raising operator on the ground state $\Psi_0^h(q)$ or as eigenfunctions of the oscillator lowering operator. They are essentially displaced Gaussian of unit width and centered at $2^{1/2}c$, which reappeared briefly in (9.66), where we noted that they happen to be basically Bargmann's transform kernel (9.55b). They are

$$\begin{aligned}\Upsilon_c(q) &= \pi^{-1/4}\exp(c^2/2)\exp[-(q-2^{1/2}c)^2/2] = (2\pi)^{-1/4}C_B(q, c)\\ &= (2\pi)^{-1/4}\left[\mathbb{C}\begin{pmatrix} 2^{-1/2} & -i2^{-1/2}\\ -i2^{-1/2} & 2^{-1/2}\end{pmatrix}\boldsymbol{\delta}_c\right](q). \qquad (9.90)\end{aligned}$$

The time evolution of the coherent states (9.90) under the harmonic oscillator potential is thus

$$\begin{aligned}\Upsilon_c(q, t) &= (2\pi)^{-1/4}\left[\mathbb{C}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\mathbb{C}\begin{pmatrix} 2^{-1/2} & -i2^{-1/2}\\ -i2^{-1/2} & 2^{-1/2}\end{pmatrix}\boldsymbol{\delta}_c\right](q)\\ &= (2\pi)^{-1/4}\left[\mathbb{C}\begin{pmatrix} 2^{-1/2}e^{it} & -2^{-1/2}ie^{-it}\\ -2^{-1/2}ie^{it} & 2^{-1/2}e^{-it}\end{pmatrix}\boldsymbol{\delta}_c\right](q)\\ &= (2\pi)^{-1/4}\left[\mathbb{C}\begin{pmatrix} 2^{-1/2} & -i2^{-1/2}\\ -i2^{-1/2} & 2^{-1/2}\end{pmatrix}\mathbb{C}\begin{pmatrix}e^{it} & 0\\ 0 & e^{-it}\end{pmatrix}\boldsymbol{\delta}_c\right](q)\\ &= \exp(it/2)\Upsilon_{c'(t)}(q), \qquad c'(t) := c\exp(it). \qquad (9.91)\end{aligned}$$

The key step has been to write the matrix product $\mathbf{M}^h(t)\mathbf{B}$ as $\mathbf{B}\mathbf{M}^d(-it)$. The last matrix denotes a *dilatation* canonical transform (9.26) which changes $\delta(q-c)$ into $\exp(it/2)\delta(q-e^{it}c)$. The subsequent $\mathbb{C}_B$ transform completes the result. The absolute value of $\Upsilon_{c'(t)}(q)$ is thus a Gaussian of unit width with an oscillating center at $2^{1/2}c\cos t$ representing the motion of a classical point particle moving with harmonic motion under an oscillator potential. As we saw in Section 7.6, coherent states exhibit the minimum dispersion product compatible with Heisenberg's uncertainty relation. Coherent states thus qualify as the *closest* quantum analogue of classical point particles under a harmonic oscillator potential.

Coherent states for Hamiltonians $\mathbb{H}$ other than the harmonic oscillator are of some interest. Our procedure in Eq. (9.91) suggests their definition and calculation. Let $\mathbb{H}$ generate the time-evolution operator $\mathbb{C}_{M(t)}$, and let $\mathbf{A}$ *diagonalize* the matrix subgroup $\mathbf{M}(t)$; then, as $\mathbf{M}(t)\mathbf{A} = \mathbf{A}\mathbf{M}^d(f(t))$, $f(t)$ being some function of t, $\Upsilon_c(q) := (\mathbb{C}_A\boldsymbol{\delta}_c)(q)$ qualifies as a generalized coherent state for $\mathbb{H}$. Its time evolution will be given, following (9.91), by $\varphi\Upsilon_{c'(t)}(q)$, where $c'(t) = cf(t)$ is the "classical" motion of the wave packet and $\varphi = [f(t)]^{1/2}$.

Exercise 9.33. Implement this definition and calculation for the repulsive oscillator (9.76d)–(9.77d). Show that the diagonalizing matrix **A** is given by (9.35) and that $f(t) = \exp t$, which is the classical motion of a particle pushed away by a repulsive oscillator. Unfortunately these repulsive coherent states are oscillating Gaussians, and their absolute values do not peak. The free-particle Hamiltonian generates a triangular-matrix time-evolution operator. This is not diagonalizable, and hence this potential does not possess coherent states.

The coherent-state construction can be carried further for potentials involving "centrifugal" barriers ($\sim q^{-2}$). Thus Barut and Girardello (1971) built eigenfunctions of the second-order lowering operator $\mathbb{J}_-$, the analogue of (7.174c) in more than one dimension. This can also be obtained by a hyperdifferential operator calculus involving $\mathbb{Q}^{-2}$ (Wolf, 1974b). Not surprisingly, the Barut–Girardello coherent states are the precise analogues of (9.90).