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The Heisenberg–Weyl Ring in Quantum Mechanics

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I. Introduction

Nearly 50 years have elapsed since quantum mechanics was introduced as a mathematical structure differing in an essential way from the mathematics known by the physicists of the preceding years. True, function analysis, differential equations, Sturm–Liouville theory, and many other branches of mathematics were used and continue to be used in order to derive results. The fundamental objects of study, however, have changed. Matrix mechanics as introduced by Heisenberg (*1*), Born and Jordan (*2*), Born *et al.* (*3*), and Dirac (*4*) describes nature in a nonnumerical representation which was recognized by Weyl (*5*) to be the framework of group theory. A review of the ensuing developments is not attempted here: the subject is too wide and very much of it is familiar to the reader. In this article we want to concentrate on a particular mathematical construct, the Heisenberg commutation relation

$$[\mathscr{Q}, \mathscr{P}] = i\hbar 1 \tag{1.1}$$

between the quantum operators of position $\mathscr{Q}$ and momentum $\mathscr{P}$ (1 stands for the unit operator and $\hbar = h/2\pi$, h being the Planck constant), and consider it as the defining bracket of a Lie algebra. Closely related to (1.1) is the Weyl commutation relation

$$\exp(ix\mathscr{Q}) \exp(iy\mathscr{P}) = \exp(iy\mathscr{P}) \exp(ix\mathscr{Q}) \exp(-\hbar xy), \tag{1.2}$$

where $x, y \in \mathfrak{R}$, the real field, which is equivalent to (1.1) if certain conditions are met, and allows the structure of a Lie group.

In Section II we define a Lie algebra $\mathscr{W}$ and group W which we call by the names of Heisenberg and Weyl. It is the simplest nilpotent Lie group and is to the general nilpotent groups what $SO(3)$ is to the semisimple groups. It is thus a bit surprising that it is relatively little known among physicists. It is not the symmetry group of any physical system and only appears in a rather trivial way as a dynamical algebra for the one-dimensional harmonic oscillator. The purpose of this article is to show, however, that it lies at the very root of quantum mechanics. In Section III we construct the group ring $\mathfrak{W}$, which contains the ring of all quantum mechanical operators $\overline{\mathscr{W}}$ (defined as all linear combinations and products of $\mathscr{Q}$ and $\mathscr{P}$). This seems to be the proper framework to examine, in Section IV, the questions pertaining to the general connection between classical and quantum mechanics: the classical limit, the quantization process, the freedom in choosing a quantization scheme, and the statements which can be made independently of this choice. Section V is devoted to the subject of canonical transformations as defined in classical mechanics and, quantum mechanically, as the automorphisms of the algebra (1.1). The two definitions do not in general agree except, specifically, for linear and point canonical transformations. Here, classical and quantum mechanics follow each other. The formulation of quantum mechanics on a compact space (i.e., a circle) through a similar algebraic structure has been somewhat elusive. In Section VI we present a variant of the Heisenberg–Weyl group $W^\star$ which allows such a program.

On the global level, our purpose is to present an orderly mathematical structure for quantum mechanics (*6*). We do not intend an "improvement" upon the existing theory which has had within its range of validity a definitive success. All the results of the one-dimensional nonrelativistic theory are regained. Rather, we are emphasizing the importance of the Heisenberg–Weyl algebra as the basic building unit for quantum mechanics. It is only fair to mention, however, that there are several alternative approaches to quantum mechanics. One is to start with the symmetry and dynamical algebra which describe a system (*7*) and to look for operators within the enveloping algebra which form canonically conjugate pairs and which transform under the generated group in the proper way to qualify them as position and momentum operators (*8–11*). These approaches are certainly not equivalent, since not all representations of the higher groups can be realized on a homogeneous space restricted by the dimensionality of physical space, while in some representations of the higher algebras, position and momentum operators are not to be found. The first approach has a classical limit built in, but poses the problem of quantization, while the latter has no quantization problems but a classical limit is not always defined. This is no obstacle for quantum characteristics with no classical analog (e.g., spin), in fact, it may seem as a welcome feature of the theory. On the other hand, we may be at a loss on how to set up a quantum system to reproduce a classical one and, when dealing with canonical transformations, it may not be clear how to implement them without recourse to the classical correspondence.

A word about references: In a field as broad as quantum mechanics it is difficult to do full justice to the credits of discovery, so no claim of completeness is made. Nevertheless, the author has derived great pleasure in reading some of the better known classics, both for the boldness of the emerging ideas and for the doubts and mistakes of the pioneers. The main point of references, however, is to establish connections with recent work performed in related areas. It is hoped that the omissions are not too grievous.

II. The Heisenberg–Weyl Group

A. The Algebra $\mathscr{W}$ and Covering Algebra $\overline{\mathscr{W}}$

We define the Lie algebra of generators $\mathscr{Q}$, $\mathscr{P}$, and $\mathscr{H}$ with the commutator bracket

$$[\mathscr{Q},\mathscr{P}] = i\mathscr{H}, \tag{2.1a}$$

$$[\mathscr{Q},\mathscr{H}] = 0, \tag{2.1b}$$

$$[\mathscr{P},\mathscr{H}] = 0, \tag{2.1c}$$

as the *Heisenberg–Weyl* algebra $\mathscr{W}$. Out of $\mathscr{W}$ we can construct the covering algebra $\overline{\mathscr{W}}$ of all linear combinations and formal products of the elements of $\mathscr{W}$. Since the commutator bracket has the derivation property (i.e., $[\mathscr{A},\mathscr{B}\mathscr{C}] = \mathscr{B}[\mathscr{A},\mathscr{C}] + [\mathscr{A},\mathscr{B}]\mathscr{C}$ for any three elements in the algebra), $\overline{\mathscr{W}}$ itself is an algebra under the commutator bracket, with an infinity of generators. Any element of $\overline{\mathscr{W}}$ will be a linear combination of *monomials* which we can choose to be of the *standard form* $\mathscr{H}^l\mathscr{Q}^m\mathscr{P}^n$ ($l, m, n \in \mathfrak{Z}^+$, the set of nonnegative integers). We identify $\mathscr{Q}^0$, $\mathscr{P}^0$, and $\mathscr{H}^0$ with 1, the right and left unit under multiplication. Any other *form* of the generators (i.e., with some factors of $\mathscr{Q}$ to the right of some factors of $\mathscr{P}$) can be brought to a sum of terms of the standard form through the use of the relation

$$[\mathscr{Q}^m,\mathscr{P}^n] \qquad \sum_{k=1}^{(m,n)} \binom{m}{k}\binom{n}{k} k!(-i\mathscr{H})^k \mathscr{Q}^{m-k}\mathscr{P}^{n-k} \tag{2.2}$$

[where (m, n) is the smaller of m and n], which can be proved by induction out of (2.1). The position of $\mathscr{H}$ in the standard form is immaterial. When we consider formal power series $\mathscr{F}$ as an entire function in one of the generators, we come to use

$$[\mathscr{F}(\mathscr{Q}),\mathscr{P}] = i\mathscr{H}\dot{\mathscr{F}}(\mathscr{Q}) \tag{2.3}$$

where $\dot{\mathscr{F}}$ is the formal derivative of $\mathscr{F}$, obtained as if the argument of $\mathscr{F}$ were a real variable. Clearly, general functions of both $\mathscr{Q}$ and $\mathscr{P}$ must include the specification of the form, that is, the order of appearance of the generators. This can always be referred to the standard form. We will occasionally refer also to the *antistandard* form: all $\mathscr{P}$'s to the left of all $\mathscr{Q}$'s.

The multiplication operation in $\overline{\mathscr{W}}$ is associative, has a unit 1, but no inverse. The elements of $\overline{\mathscr{W}}$ thus constitute a *ring* (with unit element) under multiplication. This can be identified with the ring of all quantum mechanical operators when we associate, as the notation suggests, the observables of position and momentum with the generators $\mathscr{Q}$ and $\mathscr{P}$, $\mathscr{H}$ corresponding to $\hbar\mathbf{1}$. Commutators between classes of elements in $\mathscr{W}$ give rise to formulas of the Baker–Campbell–Hausdorff type which can be found in (*12*, *13*).

For construction purposes it is convenient to have a representation of the generators of $\mathscr{W}$ by finite matrices $\mathbf{Q}$, $\mathbf{P}$, and $\mathbf{H}$. One such representation is

$$\mathbf{Q} = \begin{pmatrix} 0 & -i & -i \\ i & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{2.4a}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix} \tag{2.4b}$$

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix} \tag{2.4c}$$

Familiarity with quantum mechanics would lead us to expect that $\mathscr{H}$, being the center of the algebra, can be represented by a multiple of the unit matrix. This can only be true, however, for infinite-dimensional representations of $\mathscr{W}$. When the dimension is finite, the trace of the left-hand side of (2.1a) is zero and hence tr $\mathbf{H} = 0$. Thus $\mathbf{H}$ cannot be a multiple of $\mathbf{1}$. Even simpler looking representations of $\mathscr{W}$ can be built [e.g., (*14*, p. 235)], but (2.4) generalizes most easily to n dimensions (Section II,C). The matrices (2.4), however, do not constitute a *faithful* representation of the covering algebra $\mathscr{W}$, since $\mathbf{Q}^2 = \frac{1}{2}\mathbf{H} = \mathbf{P}^2$ and $\mathbf{Q}^n = \mathbf{0} = \mathbf{P}^n$ for $n \geq 3$, but, as the Pauli matrices for the rotation group, they lend themselves for a convenient exponentiation to build the associated Lie group.

B. Construction of the Group W

We can construct the general element g of the *Heisenberg–Weyl group* W exponentiating the Lie algebra (2.1) in its matrix representation (2.4) with a definite choice of parameter, writing

$$g(x, y, z) = \exp i(x\mathscr{Q} + y\mathscr{P} + z\mathscr{H}) \tag{2.5a}$$

$$= \exp(ix\mathscr{Q}) \exp(iy\mathscr{P}) \exp(i[z + \tfrac{1}{2}xy]\mathscr{H}) \tag{2.5b}$$

$$= \exp(iy\mathscr{P}) \exp(ix\mathscr{Q}) \exp(i[z - \tfrac{1}{2}xy]\mathscr{H}). \tag{2.5c}$$

In the representation (2.4), this

$$g(x, y, z) \quad \begin{pmatrix} 1 & x+iy & x+iy \\ -x+iy & +2iz-\tfrac{1}{2}(x^2+y^2) & 2iz-\tfrac{1}{2}(x^2+y^2) \\ x-iy & -2iz+\tfrac{1}{2}(x^2+y^2) & 1-2iz+\tfrac{1}{2}(x^2+y^2) \end{pmatrix} \tag{2.5d}$$

The group multiplication law follows:

$$g(x_1, y_1, z_1)g(x_2, y_2, z_2) \quad g(x_1 \; x_2, y_1+y_2, z_1+z_2 + \tfrac{1}{2}[y_1x_2 - x_1y_2]), \tag{2.6}$$

the group identity being $e = g(0, 0, 0)$ and the inverse $g(x, y, z)^{-1} = g(-x, -y, -z)$. Inequivalent subgroups are generated by $\mathscr{Q}$, $\mathscr{P}$, and $\mathscr{H}$: $W_Q = \{g \in W, g(x, 0, 0)\}$, $W_P = \{g \in W, g(0, y, 0)\}$, and $W_H = \{g \in W, g(0, 0, z)\}$. The latter is the central normal and commutator subgroup of W. All parameters range over the real line $\mathfrak{R}$ and the group manifold of W is thus isomorphic with $\mathfrak{R}^3$, and can be shown to be simply connected (*15*). The right- and left-invariant Haar measure can be seen to be simply

$$d\omega(g) = dx\, dy\, dz.$$

C. Other Versions of the Heisenberg–Weyl Group

1. *An n-Dimensional Version of W*

The group W_n of $(n \quad 2) \times (n+2)$ matrices

$$g(\boldsymbol{\xi}, z) = \quad \begin{pmatrix} & \boldsymbol{\xi} & \boldsymbol{\xi} \\ \boldsymbol{\xi}^\dagger & +2iz-\tfrac{1}{2}|\boldsymbol{\xi}|^2 & 2iz-\tfrac{1}{2}|\boldsymbol{\xi}|^2 \\ \boldsymbol{\xi}^\dagger & -2iz+\tfrac{1}{2}|\boldsymbol{\xi}|^2 & -2iz+\tfrac{1}{2}|\boldsymbol{\xi}|^2 \end{pmatrix} \tag{2.8}$$

where $\boldsymbol{\xi}$ is a complex n-dimensional vector of components $\xi_j = x_j + iy_j$ $(j = 1, 2, \ldots, n)$, $\boldsymbol{\xi}^\dagger$ its transpose conjugate, and z real, has the following group composition law:

$$g(\boldsymbol{\xi}_1, z_1)g(\boldsymbol{\xi}_2, z_2) \quad g(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, z_1 + z_2 \quad \tfrac{1}{2}\,\mathrm{Im}\, \boldsymbol{\xi}_1{}^\dagger\boldsymbol{\xi}_2). \tag{2.9}$$

The group W_n is an n-dimensional version of W in the sense that the generators defined through

$$g(\mathbf{x} + i\mathbf{y}, z) = \exp i(\textstyle\sum x_j\mathscr{Q}_j + \sum y_j\mathscr{P}_j + z\mathscr{H}) \tag{2.10}$$

have the commutation relations

$$[\mathscr{Q}_j, \mathscr{P}_k] = i\,\delta_{jk}\mathscr{H}, \qquad [\mathscr{Q}_j, \mathscr{H}] = 0 = [\mathscr{P}_j, \mathscr{H}], \tag{2.11a}$$

$$[\mathscr{Q}_j, \mathscr{Q}_k] = 0 = [\mathscr{P}_j, \mathscr{P}_k], \tag{2.11b}$$

and clearlv $W_1 \equiv W$.

2. *A Group of Translations in a Magnetic Field*

Another group of the Heisenberg–Weyl type which has been used to describe the kinematics of Bloch electrons in a crystal in the presence of a magnetic field $\mathbf{B}$ is given by *(16)*

$$g'(\mathbf{v}_1, u_1)g'(\mathbf{v}_2, u_2) = g'(\mathbf{v}_1 + \mathbf{v}_2, u_1 \quad u_2 \quad \boldsymbol{\beta} \times \mathbf{v}_1 \cdot \mathbf{v}_2), \tag{2.12}$$

where $\boldsymbol{\beta} = -e\mathbf{B}/4\pi c\hbar$ and $\mathbf{v}$ is a three-dimensional translation vector whose components, due to the presence of the magnetic field, do not commute. It has a matrix realization

$$g'(\mathbf{v}, u) = \left. \begin{array}{ccc} & \mathbf{v} & \mathbf{v} \\ \widetilde{-\boldsymbol{\beta} \times \mathbf{v}} & +u & u \\ \widetilde{\boldsymbol{\beta} \times \mathbf{v}} & u & u \end{array} \right) \tag{2.13}$$

where $\tilde{\mathbf{v}}$ is the transpose of $\mathbf{v}$. The corresponding Lie algebra can be obtained from

$$g'(\mathbf{v}, u) = \exp i(\textstyle\sum v_j \mathscr{V}_j + u\mathscr{U}), \tag{2.14}$$

and is

$$[\mathscr{V}_j, \mathscr{V}_k] \quad i\varepsilon_{jkl}\beta_l \mathscr{U}, \tag{2.15a}$$

$$[\mathscr{V}_j, \mathscr{U}] = 0. \tag{2.15b}$$

The properties and representations of this group and algebra have been studied in *(17–21)*.

D. Representations of the Group W

1. *A Multiplier Representation*

Consider the $\mathfrak{R}^3$ manifold of the group W and functions $f(g)$ over it. Under the action of W from the left, these transform as

$$f(g) \xrightarrow{g'(L)} \mathscr{T}^{\mathrm{L}}(g')f(g) = f(g'^{-1}g), \tag{2.16a}$$

while under action from the right

$$f(g) \xrightarrow{g'(R)} \mathscr{T}^{\mathrm{R}}(g')f(g) = f(gg'). \tag{2.16b}$$

From (2.5a) we can then realize the generators of $\mathscr{W}$ as differential operators on the group manifold as

$$\mathscr{Q}^{\mathrm{L}} = i\left(\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}\right), \qquad \mathscr{Q}^{\mathrm{R}} = -i\left(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial z}\right), \tag{2.17a}$$

$$\mathscr{P}^{\mathrm{L}} = i\left(\frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}\right), \qquad \mathscr{P}^{\mathrm{R}} = -i\left(\frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial z}\right), \tag{2.17b}$$

$$\mathscr{H}^{\mathrm{L}} = i\frac{\partial}{\partial z} = -\mathscr{H}^{\mathrm{R}},$$

where we can verify that the generators of left transformations as well as those of right transformations follow the commutation relations (2.1). Furthermore, one set commutes with the other.

We introduce now the decomposition of an arbitrary element in W as

$$g(x, y, z) = g(0, y, z - \tfrac{1}{2}xy)g(x, 0, 0) \tag{2.18}$$

and consider functions f_c on the space of left cosets W/W_Q (i.e., functions only of y and $u \equiv z - \frac{1}{2}xy$) and their transformations under left action (2.16a) by the group

$$f_c(y, u) \xrightarrow{g'(L)} \mathscr{T}^{\mathrm{L}}(g')f_c(y, u) = f_c(y - y', u - z' + yx' - \tfrac{1}{2}x'y'). \tag{2.19}$$

We can now write f_c in an $\mathscr{H}^{\mathrm{L}}$-eigenbasis decomposition through the Fourier transformation

$$f_c(y, u) = \int_{-\infty}^{\infty} d\lambda\, \phi^\lambda(y)e^{-i\lambda u}. \tag{2.20}$$

This yields the multiplier representation of W

$$\phi^\lambda(y) \xrightarrow{g'} \mathscr{T}(g')\phi^\lambda(y) = \exp(i\lambda[z' + \tfrac{1}{2}x'y' - x'y])\phi^\lambda(y - y'), \tag{2.21}$$

which can be seen to follow the group multiplication law. The operator $\mathscr{T}(g')$ is Hermitian under the measure dy stemming from (2.7) and it is easy to verify that (2.21) are eigenfunctions of $\mathscr{H}$ with the eigenvalue λ, so that in the representation (2.21), $\mathscr{H} = \lambda 1$.

2. Representations with $\mathscr{P}$ Diagonal

Out of the Hilbert space $\mathfrak{L}^2(-\infty, \infty)$ of square integrable functions $\phi^\lambda(y)$ on which W acts through the multiplier representation (2.21) we can build a representation of W choosing a complete, orthonormal basis $\{\psi_n{}^\lambda(y)\}_{n\in\mathfrak{I}}$ ($\mathfrak{I}$ some index set) as

$$D^\lambda_{nn'}(g') = (\psi_n{}^\lambda, \mathscr{T}(g')\psi^\lambda_{n'})$$
$$= \int_{-\infty}^{\infty} dy\, \psi_n{}^\lambda(y)^* \exp(i\lambda[z' + \tfrac{1}{2}x'y' - x'y])\psi^\lambda_{n'}(y - y'), \tag{2.22}$$

and be assured that they follow the group multiplication law under the appropriate summation or integration over $\mathfrak{I}$ given by the completeness relation of the basis. For λ real, the representation matrices will be unitary.

One such (Dirac) basis is

$$\chi_p{}^\lambda(y) = (2\pi)^{-1/2} \exp(-ipy), \qquad p \in \mathfrak{R} \tag{2.23}$$

which are eigenfunctions of $\mathscr{P}$ with eigenvalue p. Direct calculation yields

$$D^\lambda_{pp'}(g(x, y, z)) = \delta(\lambda x - [p - p']) \exp i(\lambda z + \tfrac{1}{2}y[p + p']). \tag{2.24}$$

These are unitary representations of W reduced with respect to the W_P subgroup and the multiplication property

$$\int_{-\infty}^{\infty} dp'\, D^\lambda_{pp'}(g_1)D^\lambda_{p'p''}(g_2) = D^\lambda_{pp''}(g_1g_2) \tag{2.25}$$

holds.

3. Representations with $\mathscr{Q}$ Diagonal

A second (Dirac) basis is given by

$$\tilde{\chi}_q{}^\lambda(y) = |\, y\,|^{1/2}\, \delta(q + \lambda y), \qquad q \in \mathfrak{R}, \tag{2.26}$$

which are eigenfunctions of $\mathscr{Q}$ with eigenvalue q. Again, direct calculation yields the unitary representation of W reduced with respect to the W_Q subgroup

$$D^\lambda_{qq'}(g(x, y, z)) = \delta(\lambda y - [q' - q]) \exp i(\lambda z + \tfrac{1}{2}x[q + q']), \tag{2.27}$$

with a multiplication property parallel to (2.25). As functions on the W manifold, $D^\lambda_{pp'}(g)$ is an eigenfunction of $\mathscr{H}^{\mathrm{L}}$, $\mathscr{P}^{\mathrm{L}}$, and $\mathscr{P}^{\mathrm{R}}$, and $D^\lambda_{qq'}(g)$ of $\mathscr{H}^{\mathrm{L}}$, $\mathscr{Q}^{\mathrm{L}}$, and $\mathscr{Q}^{\mathrm{R}}$. They can also be constructed in this way.

4. *Mixed Representations*

Two other "mixed subgroup" matrices which will be useful later on are

$$D^{\lambda}_{qp}(g(x, y, z)) = (\tilde{\chi}_q{}^{\lambda}, \mathscr{T}(g)\chi_p{}^{\lambda})$$
$$= (2\pi \mid \lambda \mid)^{-1/2} \exp i(\lambda[z + \tfrac{1}{2}xy] + qx + py + pq/\lambda)$$
$$= (2\pi \mid \lambda \mid)^{-1/2} \int dq'\, D^{\lambda}_{qq'}(g(x, y, z)) \exp(iq'p/\lambda), \qquad (2.28a)$$

$$D^{\lambda}_{pq}(g(x, y, z)) = (\chi_p{}^{\lambda}, \mathscr{T}(g)\tilde{\chi}_q{}^{\lambda})$$
$$= (2\pi \mid \lambda \mid)^{-1/2} \exp i(\lambda[z - \tfrac{1}{2}xy] + qx + py - pq/\lambda)$$
$$= (2\pi \mid \lambda \mid)^{-1/2} \int dq'\, D^{\lambda}_{q'q}(g(x, y, z)) \exp(-iq'p/\lambda). \qquad (2.28b)$$

We are using the same letter (D) for the four matrices (2.24), (2.27), and (2.28). They are distinguished by the indices, however, and no confusion will result.

5. *Orthogonality and Completeness Relations*

We can verify directly that the unitary representations constructed above are orthogonal in the sense of Dirac under the Haar measure (2.7)

$$\int_W d\omega(g) D^{\lambda_1}_{r_1 r_1'}(g)^* D^{\lambda_2}_{r_2 r_2'}(g) = \frac{4\pi^2}{\mid \lambda_1 \mid} \delta(\lambda_1 - \lambda_2)\, \delta(r_1 - r_2)\, \delta(r_1' - r_2'), \qquad (2.29)$$

where r stands for q and/or p. The completeness relation, usually difficult to prove for noncompact groups, can be seen from the simple form of the representation matrices and (2.29) which provides the Plancherel measure

$$d\hat{\omega}(\lambda) = \frac{\mid \lambda \mid}{4\pi^2}\, d\lambda \qquad (2.30)$$

on the representation space $\hat{W}$ of W labeled by λ ranging over the full real line, i.e.,

$$\int_{\hat{W}} d\hat{\omega}(\lambda) \operatorname{tr}[D^{\lambda}(g_1)^{\dagger} D^{\lambda}(g_2)] = \delta(x_1 - x_2)\, \delta(y_1 - y_2)\, \delta(z_1 - z_2). \qquad (2.31)$$

Equations (2.32) and (2.34) thus tell us that any appropriately well-behaved function over the group can be expanded in the complete orthonormal set of functions $D^{\lambda}_{rr'}(g)$.

6. *The Coordinate Basis*

The realizations of the algebra generators we obtain through (2.21) as differential operators on the y manifold are not the usual ones in quantum mechanics. There, the eigenvalue q of $\mathscr{Q}$ is regarded as the configuration space coordinate. Functions ψ of the coordinates can be built as

$$\psi^\lambda(q) = (\tilde{\chi}_q{}^\lambda, \phi^\lambda) \qquad \lambda\,|^{-1/2}\phi^\lambda(q/\lambda). \tag{2.32}$$

In this space we have the action of the group W given by

$$\psi^\lambda(q) \xrightarrow{g} \mathscr{E}(g)\psi^\lambda(q) = \quad dq'\, D^\lambda_{qq'}(g)\psi^\lambda(q'), \tag{2.33}$$

and the familiar Schrödinger representation on the space of differentiable functions

$$\mathscr{Q}\psi^\lambda(q) \quad q\psi^\lambda(q), \tag{2.34a}$$

$$\mathscr{P}\psi^\lambda(q) \qquad \cdot i\lambda \frac{d}{da}\,\psi^\lambda(q), \tag{2.34b}$$

$$\mathscr{H}\psi^\lambda(q) = \lambda\psi^\lambda(q). \tag{2.34c}$$

This induces a differential operator realization of $\mathscr{W}$. The operators (2.34) have no common invariant subspace (*22*, Pt. III; *23*) and thus the group representations (2.24) and (2.27) are irreducible. Also, it is known that any other unitary representation of $\mathscr{W}$ is equivalent to (2.34) (*22*, *23*). Finally, the scalar product becomes

$$(\psi_1{}^\lambda, \psi_2{}^\lambda) \qquad \int_{-\infty}^{\infty} dq\, \psi_1{}^\lambda(q)^* \psi_2{}^\lambda(q). \tag{2.35}$$

In this way the link is established with the usual quantum mechanical formalism, states being represented by functions on the real q line, elements of the Hilbert space $\mathfrak{L}^2(-\infty, \infty)$ with respect to the scalar product (2.35). The momentum representation is obtained through the Fourier transformation

$$\tilde{\psi}^\lambda(p) = (2\pi\,|\,\lambda\,|)^{-1/2} \int_{-\infty}^{\infty} dq\, \psi^\lambda(q) \exp(-ipq/\lambda). \tag{2.36}$$

As stated in the introduction, however, most of our concern is with the operators and their representations.

7. The Harmonic Oscillator Basis

The bases we choose in order to label the row and column indices of the representation matrices (2.22) need not be related with any subgroup chain [see, e.g., (*24*) on the rotation group and nonsubgroup decompositions]. For W we can choose any basis for $\mathfrak{L}^2(-\infty, \infty)$. One particularly interesting denumerable orthonormal basis is provided by the eigenfunctions of the harmonic oscillator Hamiltonian

$$\tfrac{1}{2}(\mathscr{P}^2 + \mathscr{Q}^2)\psi_n^\lambda(q) = \tfrac{1}{2}\left(\quad \lambda^2 \frac{d^2}{dq^2} + q^2\right)\psi_n^\lambda(q)$$
$$= \lambda(n + \tfrac{1}{2})\psi_n^\lambda(q), \qquad n \in \mathfrak{Z}^+,$$

in the Schrödinger realization (2.34). These eigenfunctions are well known to be

$$\psi_n^\lambda(q) \quad (2^n n!)^{-1/2}(\pi\lambda)^{-1/4} \exp(-q^2/2\lambda) H_n(\lambda^{-1/2}q) \tag{2.38}$$

for $\lambda > 0$. The representation matrix (2.22) can be built, using (2.38), as

$$\begin{aligned} &D^\lambda_{mn}(g(x, y, z)) \\ &\quad = (\psi_m^\lambda, \mathscr{T}(g)\psi_n^\lambda) \\ &\quad = (2^{m+n} m! n! \pi\lambda)^{-1/2} \int_{-\infty}^{\infty} dq \exp(-q^2/2\lambda) H_m(\lambda^{-1/2}q) \\ &\qquad \times \exp i(\lambda[z + \tfrac{1}{2}xy] + xq) \exp[-(q + \lambda y)^2/2\lambda] H_n(\lambda^{-1/2}[q + \lambda y]). \end{aligned} \tag{2.39}$$

We can now use the generating function for the Hermite polynomials in order to write (*25*, Eqs. 8.957.1 and 3.323.2)

$$\begin{aligned} &\sum_{m,n=0}^{\infty} (2^{m+n} m! n!)^{1/2} D^\lambda_{mn}(g(x, y, z)) s^m t^n / m! n! \\ &\quad = \exp(-i\lambda z \quad \tfrac{1}{4}\lambda[x^2 \quad y^2]) \exp(2st + t\lambda^{1/2}[y + ix] + s\lambda^{1/2}[-y + ix]). \end{aligned} \tag{2.40}$$

Now, out of the generating function for the associated Laguerre polynomials (*25*, Eq. 8.975.2)

$$e^{-uv}(1 + v)^m = \sum_{n=0}^{\infty} L_n^{(m-n)}(u) v^n, \tag{2.41}$$

setting $u = cd$ and $v = b/c$, multiplying both sides of (2.41) by $c^m a^m / m!$, and summing over all $m \in \mathfrak{Z}^+$, we obtain the generating function

$$\exp(ab + ac - bd) = \sum_{m,n=0}^{\infty} L_n^{(m-n)}(cd)a^m b^n c^{m-n}/m! \tag{2.42}$$

which seems to be new [compare with Miller (*26*, Eq. 4.124), setting $c = d$]. Put now $a = \sqrt{2}\,s$, $b = \sqrt{2}\,t$, $c = (\lambda/2)^{1/2}(-y + ix) = d^*$, and upon comparing with (2.37), collect the coefficients in $s^m t^n$ to get

$$\begin{aligned} D^\lambda_{mn}(g(x, y, z)) &= \exp(\lambda iz + \tfrac{1}{4}\lambda[x^2 + y^2])(n!/m!)^{1/2} \\ &\quad \times ([\lambda/2]^{1/2}[-y + ix])^{m-n} L_n^{(m-n)}(\tfrac{1}{2}\lambda[x^2 + y^2]), \end{aligned} \tag{2.43a}$$

which is valid, by the definition of the associated Laguerre polynomials, for $m \geq n$. Similarly, setting $a = \sqrt{2}\,t$, $b = \sqrt{2}\,s$, and $c = (\lambda/2)^{1/2}(y + ix) = d^*$ in (2.42) and exchanging the dummy indices, we obtain

$$\begin{aligned} D^\lambda_{mn}(g(x, y, z)) &= \exp(\lambda iz + \tfrac{1}{4}\lambda[x^2 + y^2])(m!/n!)^{1/2} \\ &\quad \times ((\lambda/2)^{1/2}[y + ix])^{n-m} L_m^{(n-m)}(\tfrac{1}{2}\lambda[x^2 + y^2]), \end{aligned} \tag{2.43b}$$

valid for $m \leq n$. Unitarity of the representation (2.43),

$$D^\lambda_{mn}(g(-x, -y, -z)) = D^\lambda_{nm}(g(x, y, z))^*, \tag{2.44}$$

can be seen to hold, irreducibility has been proved (*26*, Lemma 4.5), and the representation property follows from construction.

The representations for $\lambda < 0$ are meaningless in the context of the construction in this section; they can be defined, however, irrespective of the bases through

$$D^{-\lambda}(g(x, y, z)) = D^\lambda(g(x, -y, -z)) \tag{2.45}$$

which holds for the algebra and for the representations built in former sections. The representations of the generators of $\mathscr{W}$ in the harmonic oscillator basis can be obtained from (2.43) considering the one-parameter subgroups. We obtain

$$\mathbf{Q} = \left(\frac{\lambda}{2}\right)^{1/2} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{2.46a}$$

$$\mathbf{P} = i\left(\frac{\lambda}{2}\right)^{1/2} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \tag{2.46b}$$

$$\mathbf{H} \quad \lambda\mathbf{1}, \tag{2.46c}$$

which are the more familiar infinite-dimensional representations of the Heisenberg–Weyl algebra [e.g., (*27*, Eqs. (1.34))].

E. Discussion

Equation (2.1a), with which we started this section rather abruptly, is originally due to Heisenberg, although it does not appear as such in his original paper (*1*), the idea of noncommuting operators associated to classical observables being clarified in his association with Born and Jordan (*2, 3*). Equation (2.1a) was written, however, with a unit matrix on the right-hand side (with a factor of $-i\hbar$) and thus used subsequently as, for example, in Dirac's classic book (*28*). The algebraic and group theoretic structure of (2.1) was brought out by Weyl (*29*) who, however, treated the kinematics of a physical system as expressed by a *ray* representation of a two-dimensional Abelian group of translations in phase space. His formulation is usually phrased in (2.5b) and (2.5c), known as the Weyl commutation relations, which, if the group is continuous, leads to Heisenberg's formulation (*29*, pp. 275–276). Weyl's commutator is preferable in a sense because it involves only bounded operators. He also mentions the possibility of having a cyclic, discrete, or even finite group. This does not seem to have been developed since.

Detailed descriptions of the Heisenberg–Weyl algebra and group are not abundant in the mathematical physics literature. We can refer the reader to chapters in the books by Hermann (*30*), Talman (*14*), and Miller (*26*). The mathematics literature has a definitive work by Kirillov (*15*) on general nilpotent groups. On commutation relations between operators see papers (*31, 32*) and books (*33*; *34*, Sect. 6.1; *35*).

Parametrizations other than ours are used: Talman (*14*) has a change of sign in z with respect to ours. Miller (*26*, Sects. 4-1, 4-11) defines

$$g\{a, b, c\} \quad \exp(ia\mathscr{H}) \exp(-ib\mathscr{P}) \exp(ic\mathscr{Q})$$

with $\mathscr{J}^- \leftrightarrow i\mathscr{Q}$, $\mathscr{J}^+ \leftrightarrow -i\mathscr{P}$, $\mathscr{E} \leftrightarrow i\mathscr{H}$, and $\tau \equiv 0$, which relates to ours as $g(x, y, z) = g\{z - \frac{1}{2}xy, -y, x\}$ and $g\{a, b, c\} = g(c, -b, a - \frac{1}{2}bc)$. The connection with his complex group S_4 (*26*, Eq. (4.117)) is achieved through $\alpha \equiv 0, x' \leftrightarrow \omega^*/2, y' \leftrightarrow -i\omega/2, z' \leftrightarrow \delta, \lambda \leftrightarrow \mu$. Still another parametrization is used in a paper by Itzykson (*36*). The Heisenberg–Weyl algebra and group can also be profitably seen as the Inönü–Wigner contraction of other three-dimensional algebras. Indeed, all but two of the 10 three-dimensional algebras contract to $\mathscr{W}$ (*37*), among them the rotation group algebra. The latter's representations were used by Talman (*14*) to find the harmonic oscillator basis representations of W. Miller (*26*, Sect. 4-21) studies the contraction of the two-dimensional Euclidean group to W. The only contraction of W, however, is to the three-dimensional Abelian group. This can be seen to be the limit $\lambda \to 0$ for the representations studied in this section.

The covering algebra $\mathscr{W}$ interested physicists originally in the context of the solution of quantum mechanical systems through the Schrödinger eigenvalue or Heisenberg matrix approaches, and more recently, however, through the construction of invariance and transition operator algebras: in constructing realizations of Lie algebras out of the enveloping algebras of a given Lie algebra (*38–45*). A theorem by Joseph (*45*, Theorem 4.5) states that only semisimple Lie algebras of rank at most n can be realized in W_n in terms of finite polynomials in the generators. For $\mathscr{W}$ this means that only the set of up-to-second-order polynomials in $\mathscr{Q}$ and $\mathscr{P}$ will close into an algebra $\mathscr{W} \wedge \mathscr{SO}(2, 1)$ in a realization where $\mathscr{H}$ is represented by a multiple of the identity operator. Although the connection with classical mechanics is made in Section IV, it should be mentioned here that the problems posed above have been examined in (*46–53*), where the Poisson bracket, being the algebra bracket, gives rise to a differential equation framework for finding elements of the covering algebra which will close into a subalgebra.

III. The Heisenberg–Weyl Ring $\mathfrak{W}$

A. Construction and Properties

Given an operator representation $\mathscr{E}(g(x, y, z))$ of the group W in some basis, construct the operators

$$\mathscr{A} = \int_W d\omega(g) A(g) \mathscr{E}(g) \tag{3.1}$$

where $A(g)$ is any distribution over the group manifold of W. We call $A(g)$ the *group representative* of $\mathscr{A}$. The operators (3.1) can be multiplied by complex numbers and summed: If $A(g)$ and $B(g)$ are the group representatives of $\mathscr{A}$ and $\mathscr{B}$, then $c_1A(g) + c_2B(g)$ will be the representative of $c_1\mathscr{A} + c_2\mathscr{B}$ ($c_1, c_2 \in \mathfrak{C}$ the complex field). The product of two operators (3.1) is induced by the product in W: If $\mathscr{A}\mathscr{B} = \mathscr{C}$, then

$$\begin{aligned}\mathscr{C} &= \int_W d\omega(g) C(g)\mathscr{E}(g) \\ &\left(\int_W d\omega(g_1)A(g_1)\mathscr{E}(g_1)\right)\left(\int_W d\omega(g_2)B(g_2)\mathscr{E}(g_2)\right) \\ &= \int_W d\omega(g)\left(\int_W d\omega(g_1)A(g_1)B(g_1^{-1}g)\right)\mathscr{E}(g),\end{aligned}$$

and hence the group representative of the product is given in terms of the convolution over the group of the representatives of the factors as

$$C(g) = \int_W d\omega(g_1)A(g_1)B(g_1^{-1}g). \tag{3.3}$$

The Dirac δ over the group $\delta_W(g) = \delta(x)\,\delta(y)\,\delta(z)$ represents, when placed in (3.1), an operator which acts as the identity element under multiplication. The inverse to an operator, however, is not always defined. The set of operators (3.1) thus constitute a ring, which we shall call the *Heisenberg–Weyl ring* $\mathfrak{W}$ [see (*54*)].

Notice that when we parametrize the group element g in $\mathscr{E}(g)$ as (2.5), if the group representative $A(g(x, y, z))$ is a product of Dirac deltas or their (finite) derivatives in x, y, and z, the ring element $\mathscr{A}$ will be a product of $\mathscr{Q}$'s, $\mathscr{P}$'s, and $\mathscr{H}$'s, an element of $\mathscr{W}$, the covering algebra of the Heisenberg–Weyl algebra. Concretely, to

$$A_s(g(x, y, z)) = i^{l+m+n}\,\delta^{(m)}(x)\,\delta^{(n)}(y)\,\delta^{(l)}(z + \tfrac{1}{2}xy) \tag{3.4a}$$

corresponds, through (2.5b), the standard form ring element

$$\mathscr{A}_s = \mathscr{H}^l\mathscr{Q}^m\mathscr{P}^n$$

while to

$$A_a(g(x, y, z)) = i^{l+m+n}\,\delta^{(m)}(x)\,\delta^{(n)}(y)\,\delta^{(l)}(z - \tfrac{1}{2}xy)$$

corresponds, through (2.5c), the antistandard form

$$\mathscr{A}_a = \mathscr{H}^l\mathscr{P}^n\mathscr{Q}^m.$$

The ring $\mathfrak{W}$ thus contains $\mathscr{W}$ (see Table I).

We can define an involution operation in $\mathfrak{W}$ to be the adjunction induced by that of the representation $\mathscr{E}(g)$. If the representation is unitary (and it will always be),

$$\mathscr{A}^{\dagger} = \left(\int_{W} d\omega(g) A(g) \mathscr{E}(g)\right)^{\dagger}$$

$$= \int_{W} d\omega(g) A(g^{-1})^{*} \mathscr{E}(g), \tag{3.6}$$

while, since $A^{\dagger}(g)$ is the group representative of $\mathscr{A}^{\dagger}$ we have

$$A^{\dagger}(g) = A(g^{-1})^{*}. \tag{3.7}$$

A *Hermitian* ring element can thus be defined in terms of its group representative having the property $A(g^{-1}) = A(g)^{*}$.

B. Representations of the Ring

When the operator representation of $\mathscr{E}(g)$ in (3.1) is taken in the coordinate basis of Section II,D,6, the action of an element of the ring $\mathfrak{W}$ on the space of wave functions on coordinate or momentum space is given by

$$\psi^{\lambda}(r) \xrightarrow{\mathscr{A}} \mathscr{A}\psi^{\lambda}(r) = \int_{W} d\omega(g) A(g) \int dr'\, D^{\lambda}_{rr'}(g) \psi^{\lambda}(r')$$

$$= \int dr'\, A^{\lambda}(r, r') \psi^{\lambda}(r'), \tag{3.8}$$

where

$$A^{\lambda}(r, r') = \int_{W} d\omega(g) A(g) D^{\lambda}_{rr'}(g) \tag{3.9}$$

is another representative of the ring element $\mathscr{A}$ in a basis given by two coordinate labels r and r' which can be q and/or p. It will be distinguished from $A(g)$ by its arguments.

Given $A^{\lambda}(r, r')$ we can reconstitute $A(g)$ and hence $\mathscr{A}$ using the orthogonality and completeness relations (2.32) to (2.34) to write

$$A(g) = \int_{\hat{W}} d\hat{\omega}(\lambda)\, \mathrm{tr}[A^{\lambda} D^{\lambda}(g^{-1})] \tag{3.10}$$

in terms of the new representatives. Linear combinations of ring elements $c_1\mathscr{A} + c_2\mathscr{B}$ correspond to linear combinations of their representatives $c_1A^{\lambda}(r, r') + c_2B^{\lambda}(r, r')$. Moreover, the multiplication of two ring elements

$\mathscr{A}\mathscr{B} \quad \mathscr{C}$ goes over, through (3.3), (3.9), and (3.10), into

$$C^{\lambda}(r, r') \quad \int dr'' \, A^{\lambda}(r, r'') B^{\lambda}(r'', r'), \tag{3.11}$$

which shows that the representative (3.10) behaves as a matrix with row r and column r' and the ring unit is represented by $\delta(r - r')$. Hermitian conjugation means, for the new representatives,

$$A^{\dagger\lambda}(r, r') = A^{\lambda}(r', r)^*,$$

as can be seen from (3.7) and (3.9).

C. Phase Space Representative Functions

We are particularly interested in the representatives $A^{\lambda}(r, r')$ obtained when (r, r') is (q, p) or (p, q) or combinations thereof, since in this way we can introduce the phase space coordinates and have functions over these represent elements of the ring $\mathfrak{W}$ and, in particular, elements of the algebra $\mathscr{W}$ of quantum mechanical operators.

Using the mixed subgroup representation matrix $D_{qp}^{\lambda}(g)$ in (2.28a) we can find $A^{\lambda}(q, p)$ given through (3.8). In particular, the standard form ring element (3.4) is represented by

$$A_{\mathrm{s}}{}^{\lambda}(q, p) = (2\pi \mid \lambda \mid)^{-1/2} \lambda^l q^m p^n \exp(ipq/\lambda) \tag{3.13}$$

while, using $D_{pq}^{\lambda}(g)$ in (2.28b), we can find $A^{\lambda}(p, q)$. The ring element to consider now is the antistandard form (3.5) which is represented by

$$A_{\mathrm{a}}{}^{\lambda}(p, q) = (2\pi \mid \lambda \mid)^{-1/2} \lambda^l q^m p^n \exp(-ipq/\lambda) \tag{3.14}$$

(see Table I). Equations (3.13) and (3.14) suggest the definition of the *phase space representative functions*

$$a_{\mathrm{s}}{}^{\lambda}(q, p) \equiv (2\pi \mid \lambda \mid)^{1/2} A^{\lambda}(q, p) \exp(-ipq/\lambda) \tag{3.15a}$$

$$a_{\mathrm{a}}{}^{\lambda}(q, p) \equiv (2\pi \mid \lambda \mid)^{1/2} A^{\lambda}(p, q) \exp(ipq/\lambda), \tag{3.15b}$$

with the property that $a_{\mathrm{s}}{}^{\lambda}(q, p) = \lambda^l q^m p^n$ represents the standard form (3.4b) while $a_{\mathrm{a}}{}^{\lambda}(q, p)$ with the same functional form represents the antistandard form (3.4b). These two functions or linear combinations thereof can therefore be expected to be related to the classical phase space observable translated to quantum mechanics. Two characteristic operators and their representatives are given in Table I.

TABLE I

Some Representative Functions for two Typical Elements of the Heisenberg–Weyl Ring $\mathfrak{W}$

Representative	$\mathscr{H}^l\mathscr{Q}^m\mathscr{P}^n$	$\mathscr{H}^l\mathscr{P}^n\mathscr{Q}^m$
$\mathfrak{a}(g(x, y, z))$	$i^{l+m+n}\, \delta^{(m)}(x)\, \delta^{(n)}(y)\, \delta^{(l)}(z + \frac{1}{2}xy)$	$i^{l+m+n}\, \delta^{(m)}(x)\, \delta^{(n)}(y)\, \delta^{(l)}(z - \frac{1}{2}xy)$
$\mathfrak{a}^\lambda(q, q')$	$\lambda^l q^m(-i\lambda)^n\, \delta^{(n)}(q - q')$	$\lambda^l q'^m(-i\lambda)^n\, \delta^{(n)}(q - q')$
$\mathfrak{a}^\lambda(p, p')$	$\lambda^l(-i\lambda)^m\, \delta^{(m)}(p' - p)p'^n$	$\lambda^l(-i\lambda)^m\, \delta^{(m)}(p' - p)p^n$
$\mathfrak{z}^\lambda{}_{\mathrm{s}}(q, p)$	$\lambda^l q^m p^n$	$\sum_{k=0}^{(m,n)} \binom{m}{k}\binom{n}{k}k!\,i^k\lambda^{l+k}q^{m-k}p^{n-k}$
$\mathfrak{z}^\lambda{}_{\mathrm{a}}(q, p)$	$\sum_{k=0}^{(m,n)} \binom{m}{k}\binom{n}{k}k!(-i)^k\lambda^{l+k}q^{m-k}p^{n-k}$	$\lambda^l q^m p^n$

The two phase space representatives are given in terms of the group representative through

$$a^{\lambda}_{\{\mathrm{a}}^{\mathrm{s}}(q,p) = \int dx\, dy\, dz\, A(g(x,y,z)) \exp i(\lambda[z \pm \tfrac{1}{2}xy] + qx + py), \tag{3.16a}$$

$$A(g(x,y,z)) = (2\pi)^{3} \int d\lambda\, dq\, dp\, a^{\lambda}_{\{\mathrm{a}}^{\mathrm{s}}(q,p) \exp[-i(\lambda[z \pm \tfrac{1}{2}xy] + qx + py)]. \tag{3.16b}$$

They are not independent, but

$$a_{\mathrm{a}}{}^{\lambda}(q,p) = \int dq'\, dp'\, a_{\mathrm{s}}{}^{\lambda}(q',p') \exp(i[p - p'][q - q']/\lambda), \tag{3.16c}$$

and moreover, if the ring element $\mathscr{A}$ is Hermitian, its two phase space representatives are related as

$$a_{\mathrm{a}}{}^{\lambda}(q,p) = a_{\mathrm{s}}{}^{\lambda}(q,p)^{*}. \tag{3.17}$$

In particular, one-half of the sum of the standard and antistandard forms of any operator is Hermitian and its phase space representatives satisfy (3.16c) and (3.17).

D. Commutators and Poisson Brackets

The multiplication of two ring elements $\mathscr{A}\mathscr{B} = \mathscr{C}$ can be written in terms of the standard phase space representatives (3.15), dropping the indices λ and s, as

$$c(q,p) = (2\pi \mid \lambda \mid)^{-1} \int dq'\, dp'\, a(q,p')b(q',p) \exp(-i[q - q'][p - p']/\lambda). \tag{3.18}$$

If we assume that the functions involved have a Taylor expansion and we make the change of variables $u = \lambda^{-1/2}(q' - q)$, $v = \lambda^{-1/2}(p' - p)$, we can write

$$\begin{aligned} c(q,p) &= (2\pi)^{-1} \int du\, dv\, a(q, p + \sqrt{\lambda}\, v) b(q + \sqrt{\lambda}\, u, p) e^{-iuv} \\ &= \sum_{m,n=0}^{\infty} \frac{\partial^m a}{\partial p^m} \frac{\partial^n b}{\partial q^n} \frac{\lambda^{(m+n)/2}}{m!n!} \frac{1}{2\pi} \int du\, dv\, v^m u^n e^{-iuv} \\ &= \sum_{m=0}^{\infty} \frac{(-i\lambda)^m}{m!} \frac{\partial^m a}{\partial p^m} \frac{\partial^m b}{\partial q^m} \\ &= \exp\left(-i\lambda \frac{\partial^2}{\partial q_1\, \partial p_1}\right) a(q,p_1) b(q_1,p) \Big|_{\substack{q_1 = q \\ p_1 = p}}. \end{aligned} \tag{3.19}$$

For the antistandard representatives $c_a^\lambda(q, p)$, etc., the same expressions hold exchanging λ with $-\lambda$.

We can now immediately see that the commutator of two ring elements

$$\mathscr{D} = \mathscr{A}, \mathscr{B} \tag{3.20a}$$

has a standard phase space representative which can be written in terms of those of the factors as

$$\begin{aligned} d(q, p) &= \left[\exp\left(-i\lambda \frac{\partial^2}{\partial q_2\, \partial p_1}\right) - \exp\left(-i\lambda \frac{\partial^2}{\partial q_1\, \partial p_2}\right)\right] a(q_1, p_1) b(q_2, p_2)\Big|_{\substack{q_1=q_2=q \\ p_1=p_2=p}} \\ &= -\sum_{k=1}^{\infty} \frac{(-i\lambda)^k}{k!} \left[\frac{\partial^k a}{\partial q^k} \frac{\partial^k b}{\partial p^k} - \frac{\partial^k a}{\partial p^k} \frac{\partial^k b}{\partial q^k}\right] \\ &= i\lambda\{a(q, p), b(q, p)\} + \mathscr{O}(\lambda^2) \end{aligned} \tag{3.20b}$$

where

$$\{a(q, p), b(q, p)\} = \frac{\partial a(q, p)}{\partial q} \frac{\partial b(q, p)}{\partial p} - \frac{\partial a(q, p)}{\partial p} \frac{\partial b(q, p)}{\partial q}. \tag{3.21}$$

is the classical Poisson bracket for the observables $a(q, p)$ and $b(q, p)$. A similar expression is valid for the antistandard representatives $d_a^\lambda(q, p)$, etc., exchanging λ with $-\lambda$ [see, e.g., Goldstein (*55*)]. If $\mathscr{A}$ and $\mathscr{B}$ are Hermitian, $\mathscr{D}$ will be anti-Hermitian and hence the expressions (3.20) for d_s and d_a are not independent. They are related through (3.17).

E. Discussion

Our construction thus far has been purely mathematical. The connection between the phase space representatives $a_s^\lambda(q, p)$ and $a_a^\lambda(q, p)$ and a classical counterpart to the operator $\mathscr{A}$ has been suggested but not established. This is done in the next section. Here we would like to comment on the bracket introduced in (3.20b) which seems to have appeared first in a short article by McCoy (*56*). Since it follows from the commutator bracket (3.20a), it must satisfy the conditions on a Lie bracket, namely (i) antisymmetry under exchange of the arguments, (ii) linearity in each argument, and (iii) the Jacobi identity. It satisfies the derivation property only to first approximation in λ, i.e., when we can neglect all terms beyond the Poisson bracket in the last member of (3.20b). It is interesting to notice that Mehta (*57*) has proved that the only bracket of the general form

$$[f(\partial^2/\partial q_2\, \partial p_1) + g(\partial^2/\partial q_1\, \partial p_2)] a(q_1, p_1) b(q_2, p_2)\Big|_{\substack{q_1=q_2=q \\ p_1=p_2=p}} \tag{3.22}$$

which satisfies the conditions of a Lie bracket is (3.20b), i.e., (3.21) with $f = g = \exp$. Similarly, he has shown that the only solution for a bracket of the form

$$h(\partial^2/\partial q_2\,\partial p_1 - \partial^2/\partial q_1\,\partial p_2)a(q_1, p_1)b(q_2, p_2)\Big|_{\substack{q_1=q_2=q\\ p_1=p_2=p}} \qquad (3.23)$$

to be a Lie bracket is for $h = \sin$. This is Moyal's sine bracket (*58*). The bracket (3.20b) has been shown again by Mehta (*57*) to be the composition law for the Margenau–Hill (*59*) phase space distribution function, while Moyal's sine bracket is the composition law (*57, 60*) for the Wigner phase space function. This corresponds to different quantization rules and will be further elaborated in the next section.

IV. The Quantization Process

A. The Classical Limit

We can define the *classical* representation of the Heisenberg–Weyl algebra $\mathscr{W}$ as that for which the representatives of the generators $\mathscr{Q}$ and $\mathscr{P}$ commute. This is the $\lambda = 0$ representation. In the group W, the elements $g(x, y, z)$ are then represented by matrices independent of z which are thus isomorphic to the representations of a two-dimensional Abelian group in the first two parameters. For the ring $\mathfrak{W}$ at $\lambda = 0$ we can see from (3.20) that all operators commute and $\mathscr{H}$ is represented by zero. When we consider the *limit* $\lambda \to 0$ we can make use of Table I for the case $l = 0$, when no factors of $\mathscr{H}$ are present, to see that the structure of the phase space representatives of a Hermitian ring element $\mathscr{A}$ is

$$a^\lambda(q, p) = a_c(q, p) + a^\lambda{}_Q(q, p) \qquad (4.1a)$$

where $a_c(q, p)$ is a real function of the arguments and $a^\lambda{}_Q(q, p)$ is such that

$$\lim_{\lambda\to 0} a_Q{}^\lambda(q, p) = 0. \qquad (4.1b)$$

The function $a_c(q, p)$, being common to both $a_s{}^\lambda(q, p)$ and $a_a{}^\lambda(q, p)$, can be identified as the *classical function* associated with the ring element $\mathscr{A}$ and can be read off $\mathscr{A}$ simply replacing the factors of $\mathscr{Q}$ and $\mathscr{P}$ by (commuting) q and p.

B. The Quantization Scheme Problem

1. *Statement of the Problem*

The inverse to the classical limit is the *quantization process*: "Given a function $a_c(q, p)$ build a ring element $\mathscr{A}(\mathscr{Q}, \mathscr{P}, \mathscr{H})$ such that $a_c(q, p)$ is its classical function." The solution is certainly not unique, since any other ring element $\mathscr{A}(\mathscr{Q}, \mathscr{P}, \mathscr{H}) + \mathscr{H}\mathscr{B}(\mathscr{Q}, \mathscr{P}, \mathscr{H})$ has the same property. The freedom in finding a solution is only slightly curtailed when we ask the ring element $\mathscr{A}$ to be Hermitian, so that it represents a proper quantum mechanical observable. Such an operator can be determined, from (3.11) and (4.1), as

$$a_s^\lambda(q, p) = a_c(q, p) + a_Q(q, p) \tag{4.2a}$$

$$a_a^\lambda(q, p) = a_c(q, p) + a_Q(q, p)^*, \tag{4.2b}$$

but while $a_c(q, p)$ is given and fixed, the only restrictions on $a^\lambda{}_Q(q, p)$ are the consistency relation (3.16c) and the limit (4.1b). This is not enough to fix $a^\lambda{}_Q(q, p)$.

A *quantization scheme* can be defined as a unique rule by which we can associate a function $a^\lambda{}_Q(q, p)$ to every $a_c(q, p)$ such that (3.16c) and (4.1b) are satisfied. Whether such is imposed by physics is still an open question but certainly not an irrelevant one: If the operator to be quantized is, for instance, the Hamiltonian, terms in $\hbar^2$ could result as a consequence of different choices for the quantization scheme (*61*). Some of the better known quantization schemes are now reviewed.

2. *The Born–Jordan Rule*

Introduced in 1925, this rule (*2*) gives the quantization of a classical function $q^m p^n$ as

$$q^m p^n \overset{Q}{\Longrightarrow} \frac{1}{m+1} \sum_{k=0}^{m} \mathscr{Q}^k \mathscr{P}^n \mathscr{Q}^{m-k} \qquad \frac{}{n+1} \sum_{k=0}^{n} \mathscr{P}^k \mathscr{Q}^m \mathscr{P}^{n-k}$$

$$\sum_{m,n=0}^{(m,n)} \binom{m}{k} \binom{n}{k} \frac{k!}{k+1} \qquad i\mathscr{H})^k \mathscr{Q}^{m-k} \mathscr{P}^{n-k} \tag{4.3}$$

[the symbol $\overset{Q}{\Longrightarrow}$ stands for "is quantized to," and (m, n) is the smaller of m and n]. The idea around this rule is that we should take all possible orderings of $\mathscr{Q}$'s around a central $\mathscr{P}^n$ and attach the same weight to each by summing over the $m + 1$ possible arrangements. This happens to be equal to the same process around a central $\mathscr{Q}^m$. For purposes of comparison, note that the Born–Jordan rule yields $q^2p^2 \overset{\mathbf{Q}}{\Longrightarrow} \mathscr{Q}^2\mathscr{P}^2 - 2i\mathscr{H}\mathscr{Q}\mathscr{P} - \frac{2}{3}\mathscr{H}^2$.

3. *The Dirac–von Neumann Construction*

Also in 1925, Dirac (*4*; *28*, Sect. 2.1) and later von Neumann (*23*) proposed a constructive approach, quantizing q and p by $\mathscr{Q}$ and $\mathscr{P}$ satisfying the well-known commutation relations and, for any functions of these, $f(q) \overset{\mathbf{Q}}{\Longrightarrow} f(\mathscr{Q})$ and $g(p) \overset{\mathbf{Q}}{\Longrightarrow} g(\mathscr{P})$. Then, for any $a(q, p)$ and $b(q, p)$ quantized to $\mathscr{A}(\mathscr{Q}, \mathscr{P})$ and $\mathscr{B}(\mathscr{Q}, \mathscr{P})$, $a + b \overset{\mathbf{Q}}{\Longrightarrow} \mathscr{A} + \mathscr{B}$ and the Poisson bracket being quantized as $\{a, b\} \overset{\mathbf{Q}}{\Longrightarrow} (i\mathscr{H})^{-1}[\mathscr{A}, \mathscr{B}]$. This construction gives a unique prescription for the quantization of the class $P_{\leq 2}$ of up-to-second-order polynomial functions in q and p. A basis for $P_{<2}$ is

$$P_{\leq 2} \quad \{q, p, q^2, qp, p^2\} \overset{\mathbf{Q}}{\Longrightarrow} \{\mathscr{Q}, \mathscr{P}, \mathscr{Q}^2, \tfrac{1}{2}(\mathscr{Q}\mathscr{P} + \mathscr{P}\mathscr{Q}), \mathscr{P}^2\} \tag{4.4a}$$

Dirac's construction is also unique for the classes

$$P_q : \{f(q)p \quad g(q)\} \overset{\mathbf{Q}}{\Longrightarrow} \{\tfrac{1}{2}[f(\mathscr{Q})\mathscr{P} + \mathscr{P}f(\mathscr{Q})] \quad g(\mathscr{Q})\}, \tag{4.4b}$$

$$P_p \quad \{f(p)q + g(p)\} \overset{\mathbf{Q}}{\Longrightarrow} \{\tfrac{1}{2}[f(\mathscr{P})\mathscr{Q} \quad \mathscr{Q}f(\mathscr{P})] + g(\mathscr{P})\}, \tag{4.4c}$$

where f and g are arbitrary. Dirac's construction is *not* unique, however, for general polynomial functions of q and p. As an example of such an ambiguity, consider the quantization of q^2p^2. Indeed $\frac{1}{9}\{q^3, p^3\} = q^2p^2 = \frac{1}{3}\{pq^2, p^2q\}$. Now all the operators inside the brackets belong to the classes of functions mentioned above and are unique; however, the prescriptions for the operators yield

$$(9i\mathscr{H})^{-1}[\mathscr{Q}^3, \mathscr{P}^3] = \mathscr{Q}^2\mathscr{P}^2 - 2i\mathscr{H}\mathscr{Q}\mathscr{P} \quad \tfrac{2}{3}\mathscr{H}^2$$

le

$$(12i\mathscr{H})^{-1}[\mathscr{P}\mathscr{Q}^2 + \mathscr{Q}^2\mathscr{P}, \mathscr{P}^2\mathscr{Q} + \mathscr{Q}\mathscr{P}^2] = \mathscr{Q}^2\mathscr{P}^2 \quad 2i\mathscr{H}\mathscr{Q}\mathscr{P} - \tfrac{1}{3}\mathscr{H}^2.$$

4. *The Weyl–McCoy Scheme*

In 1927 Weyl (*5*; *29*, Sect. IV-14) postulated the following unique rule to produce a Hermitian operator $\mathscr{A}(\mathscr{Q}, \mathscr{P})$ out of a classical function $a_c(q, p)$: First obtain the Fourier transform

$$A(x, y) \quad (2\pi)^{-1} \int dq\, dp\, a_c(q, p) \exp(-i[xq + yp]), \tag{4.5}$$

and second, obtain the inverse Fourier transform with the particular operator kernel

$$\exp i(x\mathscr{Q} + y\mathscr{P}), \qquad (4.6)$$

that is,

$$\mathscr{A}(\mathscr{Q},\mathscr{P},\mathscr{H}) = (2\pi)^{-1} \int dx\, dy\, A(x, y) \exp i(x\mathscr{Q} + y\mathscr{P}).$$

Subsequently, McCoy (*62*) derived the more practical rule to show that functions of the kind $q^m p^n$ are quantized as

$$\begin{aligned} q^m p^n \overset{\mathrm{Q}}{\Longrightarrow} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \mathscr{P}^{n-k} \mathscr{Q}^m \mathscr{P}^k &= \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \mathscr{Q}^{m-k} \mathscr{P}^n \mathscr{Q}^k \\ &= \sum_{k=0}^{(m,n)} \binom{m}{k}\binom{n}{k} k!\, 2^{-k} (-i\mathscr{H})^k \mathscr{Q}^{m-k} \mathscr{P}^{n-k} \\ &= \exp\left(\frac{i}{2} \mathscr{H} \frac{\partial^2}{\partial\mathscr{Q}\, \partial\mathscr{P}} \right) \mathscr{Q}^m \mathscr{P}^n. \end{aligned} \qquad (4.8)$$

The last member is meant, after formal differentiation, to be taken in the standard form. Weyl's scheme is equivalent to taking all permutations of $\mathscr{Q}$'s and $\mathscr{P}$'s considered as individual objects, attaching equal weight to all configurations, and summing over them. For comparison, note that the Weyl–McCoy rule gives $q^2 p^2 \overset{\mathrm{Q}}{\Longrightarrow} \mathscr{Q}^2 \mathscr{P}^2 - 2i\mathscr{H}\mathscr{Q}\mathscr{P} + \frac{1}{2}\mathscr{H}^2$.

5. *The Symmetrization Rule*

In a short letter, Rivier (*63*) gave some arguments dealing with canonical transformations in classical and quantum mechanics to uphold the symmetrization rule which can be stated either through the quantization of the fundamental term

$$\begin{aligned} q^m p^n &\overset{\mathrm{Q}}{\Longrightarrow} \tfrac{1}{2}(\mathscr{Q}^m \mathscr{P}^n + \mathscr{P}^n \mathscr{Q}^m) \\ &= \sum_{k=0}^{(m,n)} \binom{m}{k}\binom{n}{k} k!\, \frac{\delta_{k,0} + 1}{2} (-i\mathscr{H})^k \mathscr{Q}^{m-k} \mathscr{P}^{n-k}, \end{aligned}$$

or through the Weyl construction with an operator kernel

$$\begin{aligned} &\tfrac{1}{2}[\exp(ix\mathscr{Q}) \exp(iy\mathscr{P}) + \exp(iy\mathscr{P}) \exp(ix\mathscr{Q})] \\ &\quad = \cos(\tfrac{1}{2}xy\mathscr{H}) \exp i(x\mathscr{Q} + y\mathscr{P}) \end{aligned}$$

in (4.7). The example we have been handling yields $q^2 p^2 \overset{\mathrm{Q}}{\Longrightarrow} \mathscr{Q}^2 \mathscr{P}^2$ $2i\mathscr{H}\mathscr{Q}\mathscr{P} + \mathscr{H}^2$.

6. Normal Ordering

From quantum field theory, Mehta (*57*) and Mehta and Wolf (*64, 65*) took the classical variables $a = 2^{-1/2}(q + ip)$ and $a^* = 2^{-1/2}(q - ip)$, introducing them in the observable to be quantized and replacing them with the quantum operators $\mathscr{A}$ and $\mathscr{A}^\dagger$ in such a way that all destruction operators are to the right of all creation operators. It is to be noted that for real classical functions this gives rise to Hermitian operators.

7. The Feynman Formulation

The Feynman formulation of quantum mechanics through path integrals (*66*) appeared to give a unique quantization rule (*67, 68*), but this was disproved by Cohen (*69*) and Testa (*70*).

8. Cohen's Scheme Function

A formulation of the quantization scheme arbitrariness has been given by Cohen (*71*) in a work related to phase space distributions. It makes use of the Weyl approach in first obtaining the Fourier transform of the classical function (4.5), but the inverse transform, which gives the scheme ordering through the kernel operator [(4.7) for the Weyl scheme, (4.10) for the symmetrization scheme], is allowed to have an arbitrary order introduced through a function f in $xy\mathscr{H}$:

$$\mathscr{A}(\mathscr{Q},\mathscr{P},\mathscr{H}) = (2\pi)^{-1} \quad dx\, dy\, A(x, y) f(xy\mathscr{H}) \exp i(x\mathscr{Q} \quad y\mathscr{P}) \tag{4.11}$$

with the restriction $f(0) = 1$.

This yields for the basic monomial

$$q^m p^n \xrightarrow{Q} f\left(\quad \mathscr{H} \frac{\partial^2}{\partial\mathscr{Q}\,\partial\mathscr{P}}\right) \exp\left(\quad \frac{i}{2} \mathscr{H} \frac{\partial^2}{\partial\mathscr{Q}\,\partial\mathscr{P}}\right) \mathscr{Q}^m \mathscr{P}^n, \tag{4.12}$$

which is, as the last member in (4.8), to be taken in the standard form after formal differentiation. The Born–Jordan, Weyl–McCoy, and symmetrization rules are special cases of Cohen's scheme for the functions $f(u) = \sin \frac{1}{2}u / \frac{1}{2}u$, 1, and $\cos \frac{1}{2}u$. Indeed, (4.12) corresponds to the Weyl quantization of

$$f\left(\quad \hbar \frac{\partial^2}{\partial q\, \partial p}\right) q^m p^n \quad \sum_{k=0}^{(m,n)} \binom{m}{k}\binom{n}{k} k!\, f^{(k)}(0)(-\hbar)^k q^{m-k} p^{n-k}, \tag{4.13a}$$

which is

$$q^m p^n \overset{Q}{\Longrightarrow} \sum_{k=0}^{(m,n)} \binom{m}{k}\binom{n}{k} k!\,2^{-k} f_k (-i\mathscr{H})^k \mathscr{Q}^{m-k} \mathscr{P}^{n-k}, \tag{4.13b}$$

with

$$f_k = \sum_{l=0}^{k} \binom{k}{l} (-2i)^l f^{(l)}(0), \tag{4.13c}$$

which allows us to pass from the Weyl–McCoy scheme to any other. For real f, (4.13) is Hermitian. The standard and antistandard phase space representatives can be read off to be

$$\begin{aligned} a_{\mathrm{s}}^{\lambda}(q,p) &= a_{\bar{\mathrm{s}}}^{\lambda}(q,p)^* \\ &= q^m p^n + \sum_{k=1}^{(m,n)} \binom{m}{k}\binom{n}{k} k!\,2^{-k} f_k (-i\lambda)^k q^{m-k} p^{n-k}. \end{aligned} \tag{4.14}$$

9. Quantization-Scheme-Independent Statements

A few remarks on the Cohen function (*71*) are in order so that we can determine a convenient definition of the freedom in choosing a quantization scheme and to know what statements can be made which do not depend on this choice. We have, in principle, for $a_{\mathrm{c}}(g,p) \overset{Q}{\Longrightarrow} \mathscr{A}$ the choice in scheme to produce $a_{\mathrm{c}}(q,p) \overset{Q}{\Longrightarrow} \mathscr{A} + \mathscr{H}\mathscr{B}$ for arbitrary $\mathscr{B}$ in $\mathfrak{W}$. If we want the freedom of scheme to stem only from the operator *ordering* choice, $\mathscr{B}$ must have at least one power of $\mathscr{Q}$ *and* $\mathscr{P}$ less than $\mathscr{A}$, and Cohen's general scheme (4.11) gives what appears to be all possible choices. Since the Taylor expansion of f around zero must have a leading term unity, the quantization of q^m or of p^n is scheme independent. This is *not* true, however, for qp: If we want to agree with Dirac's construction, we must also impose that the second term in the Taylor expansion of f be zero, i.e., $f'(0) = 0$. This holds for the Born–Jordan, Weyl–McCoy, and symmetrization rules as their f's are even functions of the argument. Thus Cohen's function with the restrictions $f(0) = 1$ and $f'(0) = 0$ is taken to give the true freedom in choosing a quantization scheme. The statement which can now be made is that, within this freedom, the quantization of the classes of functions in $P_{\leq 2}$, P_q, and P_p [Eqs. (4.4)] is unique and scheme independent.

C. Discussion

The quantization of the more general functions of q and p does not appear in the traditional Schrödinger quantum mechanics and we might be tempted to say that it is therefore physically irrelevant. The statistical approach to quantum mechanics introduced by Wigner (*72*), however, reaches to a host of systems formulated through a semiclassical approximation. There, the relevant concept is that of a phase space distribution function to describe the state of a system and, to describe an observable, a "Wigner equivalent function," which in later presentations is a series development in powers of $\hbar$. To first order, it is the classical observable as a function of phase space. Expectation values for an observable for a system in a given state are then calculated as the integral over phase space of the product of the two functions given above. In this framework, Moyal (*58*) and Bartlett (*73*) worked with oscillator systems and the method has been applied with various degrees of success to transport theory (*74*), neutron scattering (*75*), black-body coherence, and correlation (*57, 59, 64, 65, 76*) and scattering theory (*77*). The construction, however, is quantization scheme dependent (*57, 71*), as is the case when velocity-dependent potentials are used (*78*). Comparison with experiment becomes then the criterion of choice.

It should also be mentioned that an approach which generalizes the canonical commutation relations by others dealing with higher derivatives more appropriate for a Lagrangian description of a system has been proposed [see, e.g. (*79, 80*), corrected in (*81*)]. The subject of Lagrangian quantum theory has been developed further by Bloore and collaborators (*82–86*).

V. Canonical Transformations

A. Classical Canonical Transformations

1. Definition

Let q and p be a pair of classical conjugate observables and consider a mapping of points in phase space (q, p) as given by

$$\mathscr{T} \quad q \to \bar{q} = \varphi(q, p) \tag{5.1a}$$
$$p \to \bar{p} = \psi(q, p), \tag{5.1b}$$

such that the functions φ and ψ are real, differentiable everywhere and the Poisson bracket (3.21) is conserved:

$$\{q, p\} = \{\varphi(q, p), \psi(q, p)\} \tag{5.1c}$$

The transformation (5. is then said to be canonical in the classical sense [e.g., (*55*)]. Equivalent definitions can be given in terms of Lagrange brackets, Pfaffians (*87*), and conservation of measure in phase space (*60, 88, 89*).

The product of two canonical transformations (5.1) is again a canonical transformation since the composite function obtained out of (5.1) is differentiable everywhere and (5.1c) holds. Such a product is associative and the unit transformation $1 : q \to \bar{q} = q$, $1 : p \to \bar{p} = p$ is clearly canonical. We are not assured, however, that every canonical transformation can be inverted, i.e., that there exist corresponding inverse functions φ^{-1} and ψ^{-1} such that $q = \varphi^{-1}(\bar{q}, \bar{p})$ and $p = \psi^{-1}(\bar{q}, \bar{p})$. In the following, however, we assume such a property in considering the set of all *invertible* transformations. Such a set forms a group. We shall consider the action of this group on the space $\mathfrak{S}$ of real entire functions over phase space given by

$$\mathscr{T} : a(q, p) \to \bar{a}(q, p) = a(\bar{q}, \bar{p}) \tag{5.2}$$

where $\bar{a}$ again belongs to $\mathfrak{S}$.

2. *The Classical Group and Its Generators*

The group of classical canonical transformations is a function group whose elements, though well defined, cannot be labeled by a finite or countable number of parameters. It is thus not, strictly speaking, a Lie group. We can introduce a Lie structure, however, when we construct, for every element $z(q, p) \in \mathfrak{S}$, the first-degree operator (*52, 90*)

$$z_{\text{op}} \equiv \frac{\partial z(q, p)}{\partial q} \frac{\partial}{\partial p} - \frac{\partial z(q, p)}{\partial p} \frac{\partial}{\partial q}, \tag{5.3a}$$

which has the property that, for every $a \in \mathfrak{S}$,

$$z_{\text{op}} a = \{z, a\} \tag{5.3b}$$

The function $z(q, p)$ can then be used to generate a one-parameter group of transformations $\mathscr{T}_z(\tau)$ of $\mathfrak{S}$ on itself through

$$\mathscr{T}_z(\tau) : a \to \bar{a} \quad \exp(\tau z_{\text{op}}) a \quad a + \tau\{z, a\} + \frac{\tau^2}{2!}\{z, \{z, a\}\} + \tag{5.4}$$

The linearity of the operator z_{op} allows us to write $\bar{a}(q, p) = a(\bar{q}, \bar{p})$. We can further verify, considering τ infinitesimal, that $\{\bar{q}, \bar{p}\} = 1$ and the transformation (5.4) is canonical. For a and b in $\mathfrak{S}$, we have (*55*) $\{\bar{a}, \bar{b}\}(\bar{q}, \bar{p}) = \{\bar{a}, \bar{b}\}\{q, p\} = \{a, b\}(q, p)$. For every $z \in \mathfrak{S}$ we have thus a set $\mathscr{T}_z(\tau)$

of canonical transformations parametrized by $\tau \in \mathfrak{R}$ which are, moreover, invertible and hence form a one-parameter group. As any two functions in $\mathfrak{S}$ which differ at most by an additive real constant generate the same transformation, we need only consider these equivalence classes of functions elements of $\mathfrak{S}/\mathfrak{R}$, and take them to be the Lie algebra of the group of classical canonical transformations כ (kaf) which can be cast in the form (5.4) and which act on the space $\mathfrak{S}/\mathfrak{R}$ itself. The Lie bracket of this algebra is the commutator between the operators (5.3) given by

$$[z_{1\mathrm{OD}}, z_{2\mathrm{OD}}] \qquad \{z_1, z_2\}_{\mathrm{OP}},$$

which is, again, an element of $\mathfrak{S}/\mathfrak{R}$.

Since the operators (5.4) in כ are invertible and continuous in τ, they must be one-to-one mappings of the full phase space onto the full phase space, and map compact regions into compact regions. We can define another correspondence between $\mathfrak{S}/\mathfrak{R}$ and the elements of כ by the use of the generating function (*55, 87*). This "parametrization" of כ is not convenient for our purposes, however, since we have no true way of building one-parameter subgroups. The correspondence between both approaches has been studied by Testa (*91–93*) and others (*94, 95*).

3. *The Inhomogeneous Linear Subgroup*

One subgroup of כ is the set of inhomogeneous linear transformations

$$\mathscr{F} : \begin{cases} q \to \bar{q} = aq + bp + e & (5.6a) \\ p \to \bar{p} = cq + dp + f & (5.6b) \end{cases}$$

($a, b, \ldots, f \in \mathfrak{R}$) which is canonical if $ad - bc = 1$, and can be identified with a group $ISp(2, R)$. The subset of $\mathfrak{S}/\mathfrak{R}$ generating this group through (5.4) can be seen to be the class $P_{\leq 2}$ of up-to-second-order polynomials in q and p, of which we can choose the linearly independent generators

$$l_1 = \tfrac{1}{4}(p^2 \quad q^2), \qquad l_2 = \tfrac{1}{2}pq, \qquad l_3 = \tfrac{1}{4}(p^2 + q^2), \qquad q, \qquad p.$$

4. *Point Transformations and Their Generators*

The set of transformations of the form

$$\mathscr{F} : \begin{cases} q \to \bar{q} = \varphi(q) & (5.8a) \\ p \to \bar{p} = \psi(q, p), & (5.8b) \end{cases}$$

i.e., where the new observable $\bar{q}$ is a function φ only of q, clearly forms a subgroup of ב. We shall call it the group פ (pey) of canonical *point* transformations in configuration space.

We can generate a one-dimensional subgroup $\mathscr{F}_z(\tau)$ of ב depending on the parameter τ and entirely in פ if we propose a generator function of the class P_a, i.e.,

$$z(q, p) = pf(q) \quad g(q), \tag{5.9}$$

so that, from (5.4), the corresponding τ-dependent function φ in (5.8a) is

$$\varphi_\tau(q) = q \quad \tau f(q) + \frac{\tau^2}{2!} f(q) \frac{d}{dq} f(q) \quad \frac{\tau^3}{3!} f(q) \frac{d}{dq} f(q) \frac{d}{dq} f(q) + \cdots \tag{5.10}$$

which satisfies the differential equation

$$\frac{\partial \varphi_\tau(q)}{\partial \tau} \quad z_{\mathrm{op}} \varphi_\tau(q) = \quad f(q) \frac{\partial \varphi_\tau(q)}{\partial q} \tag{5.11}$$

which, given $f(q)$, can be used to determine $\varphi_\tau(q)$ under the change of variable

$$\chi(q) \equiv \int dq [f(q)]^{-1}, \qquad q = \chi^{-1}(r), \tag{5.12a}$$

$$\varphi_\tau(q) \quad \exp[\tau(pf + g)_{\mathrm{op}}] q = \chi^{-1}(\chi(q) \quad \tau). \tag{5.12b}$$

It can be checked to satisfy (5.11) with the correct initial condition $\varphi_0(q) \quad q$.

Similarly, the function ψ_τ in (5.8b) will satisfy

$$\frac{\partial \psi_\tau(q, p)}{\partial \tau} = z_{\mathrm{op}} \psi_\tau(q, p) = \left[(p\dot{f} + \dot{g}) \frac{\partial}{\partial p} \quad f \frac{\partial}{\partial q} \right] \psi_\tau(q, p). \tag{5.13}$$

where the overdot stands for differentiation of a function with respect to its argument q. In order to conserve the canonical Poisson bracket relation, we can write

$$\psi_\tau(q, p) = p[\dot{\varphi}_\tau(q)]^{-1} + \gamma_\tau(q) \tag{5.14}$$

and, to determine $\gamma_\tau(q)$, we replace it in (5.13). Using (5.12) we obtain

$$\frac{\partial \gamma_\tau(q)}{\partial \tau} = [\dot{\varphi}_\tau(q)]^{-1} \dot{g}(q) \quad f(q) \frac{\partial \gamma_\tau(q)}{\partial q} \tag{5.15}$$

This differential equation is similar to (5.11) in the sense that the solution can be written as $\gamma_\tau(q) = c_1 \varphi_\tau(q) + \delta_\tau(q) + c_2$ where c_1 and c_2 are constants and $\delta_\tau(q)$ satisfies (5.15) again. The initial condition $\gamma_0(q) = 0$, however, requires that $c_1 = 0 = c_2$. For the trivial case $g \equiv 0$ we have $\gamma_\tau \equiv 0$. For the more general case $g \not\equiv 0$, it can be remarked that a point transformation can be effected in two steps: first, with the generator function (5.9) being $z(q, p) = pf(q)$ so that $\gamma_\tau \equiv 0$ in (5.14), and second, with one generated by functions of the kind $z(q, p) = pq + g(q)$, i.e., $f(q) = q$, hence $\chi = \ln$ and $\chi^{-1} = \exp$, so that (5.12) and (5.14) yield

$$\bar{q} = e^{-\tau} q \tag{5.16a}$$

$$\bar{p} = e^{\tau} p + \gamma_\tau(q). \tag{5.16b}$$

These transformations form by themselves a subgroup of $\mathfrak{D}$. A series expansion for $\gamma_\tau(q)$ in powers of τ can be found using (5.4) on p, i.e.,

$$\gamma_\tau(q) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} z_{\text{op}}^n p - e^{\tau} p = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \gamma^n(q), \tag{5.17a}$$

where $\gamma^0(q) \equiv 0$, $\gamma^1(q) = \dot{g}(q)$, and

$$\gamma^n(q) = \dot{g}(q) - q\dot{\gamma}^{n-1}(q). \tag{5.17b}$$

It is now a relatively simple matter to verify that the series (5.17) satisfies (5.15) for the transformation (5.16). We can determine directly that for $\tilde{g}(q) = q^m$ we obtain $\tilde{\gamma}^n(q) = [1 - (1 - m)^n]q^{m-1}$ and hence $\tilde{\gamma}_\tau(q) = e^{\tau}(1 - e^{-m\tau})q^{m-1}$. Since a linear combination of $\tilde{g}$'s will produce a linear combination of $\tilde{\gamma}_\tau$'s, the group multiplication of two transformations (5.16) being the addition of the functions, the Taylor expansion of a general $\tilde{g}(q)$ will yield

$$\tilde{\gamma}_\tau(q) = e^{\tau} q^{-1}[g(q) - g(e^{-\tau} q)],$$

which can be checked to have the correct boundary conditions and satisfy (5.15). Joining the two steps, the general point transformation in $\mathfrak{D}$ generated by (2.10) is

$$\mathscr{F}_z(\tau) \begin{cases} q \to \bar{q} = \varphi_\tau(q) = \chi^{-1}(\chi(q) - \tau) & (5.19a) \\ p \to \bar{p} = \psi_\tau(q, p) = p[\dot{\varphi}_\tau(q)]^{-1} + \gamma_\tau(q) & (5.19b) \end{cases}$$

with the definitions in (5.12a) and

$$\gamma_\tau(q) = [f(q)\dot{\varphi}_\tau(q)]^{-1}[g(q) - g(\varphi_\tau(q))].$$

Entirely similar considerations hold for generator functions of the class P_p which generate point transformations in p. Notice that a point transformation in q followed by a linear one composes to a new transformation of the type $\bar{q} = af(q) + bpg(q)$, $\bar{p} = ch(q) + dpg(q)$; i.e., the new $\bar{q}$ and $\bar{p}$ are elements of the class P_q. A point transformation in p followed by a linear one has an analogous form and $\bar{q}$ and $\bar{p}$ are elements of P_p.

B. Quantum Canonical Transformations

1. Definition

We now build a proper definition for a canonical transformation in quantum mechanics as a mapping of the Heisenberg–Weyl enveloping algebra $\mathscr{W}$ on itself through

$$\mathscr{F} \begin{cases} \mathscr{Q} \to \bar{\mathscr{Q}} = \Phi(\mathscr{Q}, \mathscr{P}, \mathscr{H}) & \text{(5.21a)} \\ \mathscr{P} \to \bar{\mathscr{P}} = \Psi(\mathscr{Q}, \mathscr{P}, \mathscr{H}) & \text{(5.21b)} \\ \mathscr{H} \to \bar{\mathscr{H}} = \Omega(\mathscr{H}) & \text{(5.21c)} \end{cases}$$

such that the entire functions Φ and Ψ include the specification of the order of the arguments (as, e.g., the standard form), and the commutation relations of the algebra $\mathscr{W}$ be preserved, i.e.,

$$[\Phi(\mathscr{Q}, \mathscr{P}, \mathscr{H}), \Psi(\mathscr{Q}, \mathscr{P}, \mathscr{H})] = i\Omega(\mathscr{H}) \tag{5.21d}$$

$$[\Phi, \Omega] = 0, \qquad [\Psi, \Omega] = 0, \tag{5.21e}$$

and such that the domains of $\bar{\mathscr{Q}}$, $\bar{\mathscr{P}}$, and $\bar{\mathscr{H}}$ be the same as those of $\mathscr{Q}$, $\mathscr{P}$ and $\mathscr{H}$. As $\bar{\mathscr{H}}$ is still in the center of $\mathscr{W}$, it cannot be but a function of $\mathscr{H}$ only [see (*45*, Theorem 4.5)].

We can realize a wide class of canonical transformations (5.21) as similarity and scale transformations in

$$\bar{\mathscr{Q}} = c_q \mathscr{S} \mathscr{Q} \mathscr{S}^{-1}, \tag{5.22a}$$

$$\bar{\mathscr{P}} = c_p \mathscr{S} \mathscr{P} \mathscr{S}^{-1}, \tag{5.22b}$$

$$\bar{\mathscr{H}} = c_n \mathscr{H}, \tag{5.22c}$$

where we assume the existence of a left inverse $\mathscr{S}^{-1}$ for every element $\mathscr{S}$ in $\mathscr{W}$ considered, which evidently satisfies (5.21) when the numbers c_q, c_p, and c_n are related as $c_q c_p = c_n$. We have then for the whole of $\mathscr{W}$, for $c_q = 1 = c_p$,

$$\begin{aligned} \mathscr{F} \quad \mathscr{A}(\mathscr{Q}, \mathscr{P}, \mathscr{H}) \to \bar{\mathscr{A}}(\mathscr{Q}, \mathscr{P}, \mathscr{H}) &= \mathscr{A}(\mathscr{S}\mathscr{Q}\mathscr{S}^{-1}, \mathscr{S}\mathscr{P}\mathscr{S}^{-1}, \mathscr{H}) \\ &= \mathscr{S}\mathscr{A}(\mathscr{Q}, \mathscr{P}, \mathscr{H})\mathscr{S}^{-1}. \end{aligned} \tag{5.23}$$

Furthermore, if the set of transformations (5.22) is to form a group, every $\mathscr{S}$ considered must have a two-sided inverse.

2. *Unitary Quantum Canonical Transformations*

The realization (5.22) is too general for our purposes, since it will not preserve the Hermiticity of the elements of $\mathscr{W}$ unless $\mathscr{S}$ is unitary with respect to the scalar product (2.35). This leads us to a more restricted group given by (5.22) with a unitary $\mathscr{S}$ which can be written as $\exp(i\tau'\mathscr{X})$ with $\mathscr{X}$ Hermitian (*22*). In this way we define the group of unitary canonical transformations ק (quf) as the set of all transformations on $\mathscr{W}$ given by

$$\begin{aligned} \mathscr{T}_{\mathscr{X}}(\tau') : \mathscr{A} \to \bar{\mathscr{A}} &= \exp(i\tau'\mathscr{X})\mathscr{A}\exp(-i\tau'\mathscr{X}) \\ &= \mathscr{A} + i\tau'[\mathscr{X},\mathscr{A}] + \frac{(i\tau')^2}{2!}[\mathscr{X},[\mathscr{X},\mathscr{A}]] + \\ &= \exp(i\tau'\mathscr{X}_{\text{com}})\mathscr{A}, \end{aligned}$$

where the Hermitian element $\mathscr{X}(\mathscr{Q},\mathscr{P},\mathscr{H}) \in \mathscr{W}$ generates the ק transformation on $\mathscr{W}$. We have defined the operator $\mathscr{X}_{\text{com}}$ associated to $\mathscr{X}$ through

$$\mathscr{X}_{\text{com}}\mathscr{A} = [\mathscr{X}, \mathscr{A}]$$

[the notation Ad$\mathscr{X}$ is also used in the literature, see (*30*)]. Note that elements of $\mathscr{W}$ which differ by an additive term involving $\mathscr{H}$ alone (members of the same equivalence class in $\mathscr{W}|\mathscr{H}$) generate the same transformation (5.24). Hence $\mathscr{W}|\mathscr{H}$ can be identified as the Lie algebra generating ק with the commutator bracket

$$[\mathscr{X}_{1\text{com}}, \mathscr{X}_{2\text{com}}] = [\mathscr{X}_1, \mathscr{X}_2]_{\text{com}}.$$

The space $\mathscr{W}|\mathscr{H}$ is also the space on which ק acts effectively, since additive terms in $\mathscr{H}$ remain unchanged under (5.14).

C. Correspondence Preserved and Broken

1. *The Question of Isomorphism*

What is the relation between ב and ק? Through the quantization procedure, for every generator function $z(q,p) \in \mathfrak{S}/\mathfrak{R}$ outside the classes $P_{\leq 2}$, P_q, and P_p, there will correspond an infinity of Hermitian operators in $\mathscr{W}|\mathscr{H}$. Hence, to every one-parameter subgroup in ב there will correspond one or *more* one-parameter subgroups in ק. There is thus a many-to-one correspondence between the elements of ב and those of ק. Assume, however, that we have chosen a definite quantization scheme so that out of ב we can build a unique subset of ק, and that to every one-parameter subgroup of ב generated by a function z there corresponds a one-parameter subgroup of ק generated by the unique operator $\mathscr{Z}$. In this way we can construct a one-to-one mapping between the elements of the generating algebras of ב and those of a subset of ק which may be extended, at least locally, to the group. This, however, is still not an *isomorphism* between the algebras or the groups, since we must still demand that under the Lie bracket operation the correspondence be preserved, i.e., that the Poisson bracket (5.5) for ב keep the correspondence with the commutator (5.26) for ק. This clearly does not hold in general. We conclude that ב is neither locally nor globally isomorphic with ק, not even within a definite quantization scheme subset.

We can restrict ourselves, however, to those subgroups of ב and ק whose generators *do* maintain the correspondence between Poisson brackets and commutators. These subgroups are precisely those of linear and point transformations. The classical and quantum versions of these subgroups will be isomorphic.

2. Extended Quantum Linear Transformations

The group of linear automorphisms of the Heisenberg–Weyl algebra $\mathscr{W}$ (2.1) is given by

$$\mathscr{F}: \begin{pmatrix} \mathscr{Q} \\ \mathscr{P} \\ \mathscr{H} \end{pmatrix} \to \begin{pmatrix} \bar{\mathscr{Q}} \\ \bar{\mathscr{P}} \\ \bar{\mathscr{H}} \end{pmatrix} \quad \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} \mathscr{Q} \\ \mathscr{P} \\ \mathscr{H} \end{pmatrix} \tag{5.27a}$$

such that $(ad - bc)g = 1$. The upper left 2×2 submatrix is, for $g = 1$, the group of real symplectic transformations $Sp(2, R)$ considered in (*96–99*). This is multiplied in direct product by the subgroup $T(1)$ of dilatations $\bar{\mathscr{H}} = g\mathscr{H}$ with $g > 0$, and multiplied in semidirect product by the subgroup $T(2)$ of "translations" $\mathscr{Q} \to \mathscr{Q} + e\mathscr{H}$, $\mathscr{P} \to \mathscr{P} + f\mathscr{H}$. This is further multiplied in semidirect product by the representatives of the two disconnected pieces of the dilatation group $g > 0$ and $g < 0$, a $C(2)$ group of two elements. The group of linear automorphisms of $\mathscr{W}$ is thus

$$C(2) \wedge [T(2) \wedge (Sp(2, R) \times T(1))]. \tag{5.27b}$$

This is (*100*), in fact, the same group of automorphisms of the Schrödinger equation of the free particle (*101*) and harmonic oscillator (*102*).

There is clearly an isomorphism between the $ISp(2, R) \cong T(2) \wedge Sp(2, R)$ subgroup of (5.27) and the group (5.7) of classical inhomogeneous linear transformations. Under these restricted canonical transformations, classical and quantum observables and operators of the class $P_{\leq 2}$ are mapped among themselves and hence, for these transformations on these observables, classical and quantum mechanics follow each other.

3. *Quantum Point Transformations*

The subset of $\mathfrak{P}$ of elements of the form

$$\mathscr{T} \begin{cases} \mathscr{Q} \to \bar{\mathscr{Q}} = \Phi(\mathscr{Q}) & (5.28a) \\ \mathscr{P} \to \bar{\mathscr{P}} = \Psi(\mathscr{Q}, \mathscr{P}, \mathscr{H}) & (5.28b) \\ \mathscr{H} \to \bar{\mathscr{H}} = \Omega(\mathscr{H}), & (5.28c) \end{cases}$$

will constitute the group of quantum point transformations in configuration space, when it preserves the $\mathscr{W}$ algebra (2.1). We can translate the results of Section V,A,4 to the quantum case because both the classical generator function (5.10) and the functions transformed in (5.28) belong to the class P_q. Under quantization we associate the symmetrized operator (4.4b) and it is easy to verify that the Poisson bracket and commutator of two quantities—classical or quantum—in P_q are in P_q [see (*30*)]. The expansion series (5.24) for point transformations generated by $\mathscr{Z}$ acting on $\mathscr{Q}$ and $\mathscr{P}$ is then identical to (5.4) generated by z acting on q and p, with $\tau'\mathscr{H} = \tau 1$ and the formal derivatives of operator functions taken as in (2.3). The group of quantum point transformations (5.28) is therefore isomorphic to the group $\mathfrak{D}$ of classical point transformations, and will be denoted by the same letter. Without further computation we can thus state that $\mathscr{Z}$ in (5.24) generates a point transformation (5.28) in $\mathfrak{D}$ with

$$\bar{\mathscr{Q}} = \Phi_{\tau'}(\mathscr{Q}) = X^{-1}(X(\mathscr{Q}) - \tau'\mathscr{H}), \tag{5.29a}$$

$$\bar{\mathscr{P}} = \Psi_{\tau'}(\mathscr{Q}, \mathscr{P}) = \tfrac{1}{2}\{\mathscr{P}[\dot{\Phi}_{\tau'}(\mathscr{Q})]^{-1} + [\dot{\Phi}_{\tau'}(\mathscr{Q})]^{-1}\mathscr{P}\} \quad \Gamma_{\tau'}(\mathscr{Q}), \tag{5.29b}$$

$$\bar{\mathscr{H}} = \mathscr{H}, \tag{5.29c}$$

where the capital Greek functions are identified with their classical lower-case counterparts in (5.12) and (5.20). The transformation (5.16) in particular will generate ray representations of the $\mathscr{W}$ algebra. We emphasize that we have asked our point transformations to be one-to-one invertible, infinitely differentiable mappings of the whole phase space onto itself. If this is violated in, for instance, mapping configuration space $\mathfrak{R}$ into a circle, as in Fock's projective transformation (*103*), we would find the spectrum of the operators, say $\mathscr{P}$, changed from the whole real line $\mathfrak{R}$ to a set of equally spaced values proportional to the integers $\mathfrak{Z}$. This is clearly impossible since the transformation (5.24) cannot change the spectrum of an operator.

For observables and operators of the class P_q, therefore, classical and quantum mechanics follow each other under point transformations in configuration space. Similar considerations apply to the class P_p under point transformations in momentum space.

4. *Example of Correspondence Broken*

Even though we have proved that the classical and quantum versions of inhomogeneous linear and point transformations are isomorphic, the *composition* of one linear and one point transformation may lie outside both subgroups. Similarly, the correspondence will break down if the new $\bar{\mathscr{Q}}$ and $\bar{\mathscr{P}}$ are not both elements of $P_{\leq 2}$, P_q, or P_p, as it may happen if we apply point transformations to observables other than those of the class P_q or linear transformations to observables other than those of the classes $P_{\leq 2}$, P_q, or P_p. Consider two or more systems related through a canonical transformation on the classical level. It is a question of central interest to know whether this correspondence can be carried over into quantum mechanics through a single quantization scheme. The answer is, in general, no. In order to show this it is sufficient to give a counterexample.

Consider the observable $a = qp^2$ and its unique corresponding Hermitian operator $\mathscr{A} = \frac{1}{2}(\mathscr{Q}\mathscr{P}^2 + \mathscr{P}^2\mathscr{Q})$. Now consider a classical transformation (5.19) with $\gamma_\tau \equiv 0$ for simplicity, and a fixed τ,

$$\mathscr{F} \quad a \to \bar{a} \quad \varphi(q)[\dot{\varphi}(q)]^{-2}p^2 \equiv \theta(q)p^2 \tag{5.30a}$$

Its quantization in the symmetrization scheme is thus

$$\bar{a} \overset{Q}{\Longrightarrow} \bar{\mathscr{A}} = \tfrac{1}{2}[\Theta(\mathscr{Q})\mathscr{P}^2 + \mathscr{P}^2\Theta(\mathscr{Q})]. \tag{5.30b}$$

Now consider the Hermitian operator obtained from $\mathscr{A}$ through the corresponding quantum point transformation (5.29). This is

$$\begin{aligned}\mathscr{F} : \mathscr{A} &\to \bar{\mathscr{A}}' \\ &= \tfrac{1}{8}(\Phi[(\dot{\Phi})^{-1}\mathscr{P} + \mathscr{P}(\dot{\Phi})^{-1}]^2 + [(\dot{\Phi})^{-1}\mathscr{P} + \mathscr{P}(\dot{\Phi})^{-1}]^2\Phi) \\ &= \tfrac{1}{2}[\Theta(\mathscr{Q})\mathscr{P}^2 + \mathscr{P}^2\Theta(\mathscr{Q})] + \tfrac{1}{4}\mathscr{H}^2(4\dot{\Phi}\Upsilon\dot{\Upsilon} + 2\Phi\Upsilon\ddot{\Upsilon} + 3\Phi\dot{\Upsilon}^2)\end{aligned} \tag{5.30c}$$

where $\Phi \equiv \Phi(\mathscr{Q})$ and $\Upsilon \equiv (\dot{\Phi})^{-1}$. This is obviously an operator differen from (5.30a). The breaking of the correspondence comes from acting on an element a in the class P_p with a point transformation in q space, whereby $\bar{a}$ is no longer in P_p. The quantization is no longer unique, but, indeed, dependent on the point transformation chosen. They both have the same classical limit a_c, however.

D. Discussion

The importance of canonical transformations in quantum mechanics was recognized within a year of its original formulation (*104–107*). Point transformations have been extensively used as transformation groups (*108*), the more recent applications of which include many-body and scattering problems (*109–117*), while the role of linear canonical transformations has only recently been appreciated (*96–99, 118–120*). Indeed, this seems to be a very promising field of research. The results of this section, however, have to be kept in mind when translating results of canonical transformations from classical into quantum mechanics (*60, 89*). (i) The transformation of a given physical system to a mathematically simpler one is a common technique in classical mechanics (*55*), usually by taking one of the new canonically conjugate observables to be a constant of motion as the Hamiltonian or the angular momentum. The other will be time or angle. This, though tempting, is not correct in our framework [see, e.g., (*107, 121–124*)]. Such transformations are not one to one and would change the momentum spectrum. (ii) All classical systems possess higher dynamical Lie algebras of observables under Poisson brackets (*47–49*). These can be quantized only for some exceptional systems as the harmonic oscillator, hydrogen atom, and the free particle. For other systems, the commutator bracket does not follow the Poisson bracket. (iii) Even if two or more classical systems have the same Lie algebra of quantizable observables [this is the case of the one-dimensional harmonic oscillator, hydrogen atom, and point rotor examined in (*125*), which share the $\mathscr{SO}(2,1)$ algebra, see (*126, 127*)], their quantum counterparts may have different spectra or, the Casimir operator being outside the classes $P_{\leq 2}$, P_q, or P_p, may indicate that the systems states belong to different irreducible representations of the algebra.

This is not to say that canonical transformations outside the classes mentioned above are necessarily incorrect. Indeed, we can have linear transformations in a higher dimensional space becoming nonlinear when the radial part is isolated in a differential operator realization (*128*) or when the requirement of conservation of the commutator bracket (2.1) is demanded only between well-chosen states in a particular basis (*129*, *130*). Finally, we should remark that the treatment of function groups and their associated infinitesimal algebras is still very incomplete. The group ק has a nilpotent structure (*15*) which should make it amenable to a relatively easy study. In the mathematical physics literature we can point out the works of van Hove (*88*) and Limić (*131*).

VI. Quantum Mechanics on a Compact Space

A. The Mixed Group $W^\star$

1. Definition

We shall start with the definition of a Heisenberg–Weyl group given through its composition rule

$$g(x_1, y_1, z_1)g(x_2, y_2, z_2) = g(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \tfrac{1}{2}[y_1x_2 - x_1y_2]), \tag{6.1}$$

i.e., (2.6), and assume that one of its infinitesimal generators $\mathscr{P}$ exists and has a discrete spectrum consisting of an infinite set of discrete eigenvalues

$$\sum(\mathscr{P}) = \{p = n_p p_0, \quad n_p \in \mathfrak{Z}\}. \tag{6.2}$$

This operator generates a one-parameter compact subgroup

$$W_P{}^\star = \{g(0, y, 0) \in W, \quad y \equiv y \bmod L\} \tag{6.3a}$$

where L is some arbitrary, fixed length. The one-valued representations of $W_P{}^\star$ are phases e^{ipy}, so that $y \equiv y \bmod L$ implies that $p_0 = 2\pi/L$ is the distance between two adjoining eigenvalues of $\mathscr{P}$.

The group identity element is $e = g(0, 0, 0)$ which is by (6.3a) equivalent to $g(0, L, 0)$. We thus have $g(x, y, z)e = g(x, y, z)g(0, L, 0) = g(x, y + L, z - \tfrac{1}{2}xL)$ and $z \equiv z \bmod \tfrac{1}{2}xL$. If x were a real variable, it could be as small as we please and hence $z \equiv 0$ would vanish as a group parameter, leaving us with an Abelian two-parameter group. In order to have a proper Heisenberg–Weyl group structure we must thus assume that x can only take discrete values, say n_x/M, with $n_x \in \mathfrak{Z}$ so that we have a second subgroup

$$W_Q{}^\star = \{g(x, 0, 0) \in W, \quad x = n_x/M, \quad n_x \in \mathfrak{Z}\}. \tag{6.3b}$$

In this way the third one-parameter subgroup $g(0, 0, z)$ becomes

$$W_H^\star \quad \{g(0, 0, z) \in W, \quad z \equiv z \bmod L/2M\}.$$

We can now compose the group $W^\star \subset W$ as

$$W^\star = W_Q{}^\star W_P{}^\star W_H{}^\star,$$

and check that the restrictions (6.3) on the group elements are preserved under the multiplication.

The group manifold of $W^\star$ is an infinite collection of disjoint tori. It can be called neither discrete nor compact for the same reason that an interval $(x_0, x_1]$ is neither open nor closed. Rather, it has one discrete (x) and two continuous parameters (y, z) which range over compact domains. Following Weyl (*29*) we shall call it a *mixed* group. It is a Lie group, since discrete groups are trivial Lie groups. Only elements of $W_P{}^\star$ and $W_H{}^\star$, however, can be generated through infinitesimal operators $\mathscr{P}$ and $\mathscr{H}$ as $g(0, y, 0) = \exp(iy\mathscr{P})$ and $g(0, 0, z) = \exp(iz\mathscr{H})$. Those of $W_Q{}^\star$ cannot: As $W_Q{}^\star$ is an infinite discrete Abelian group, it has a *finite* generator element which we can denote by $\mathscr{E}^{iQ}$, where the iQ is to be regarded as an upper index (not as an exponent). The general element of $W_Q{}^\star$ is then

$$g(x, 0, 0) = (\mathscr{E}^{iQ})^{n_x} \quad \mathscr{E}^{ixQ}. \tag{6.5a}$$

where we adopt a suggestive notation. The general group element $g(x, y, z)$ of $W^\star$ can thus be written, following (2.5b) and (2.5c), as

$$\begin{aligned} g(x, y, z) &= \mathscr{E}^{ixQ} \exp(iy\mathscr{P}) \exp(i[z + \tfrac{1}{2}xy]\mathscr{H}) \\ &= \exp(iy\mathscr{P})\mathscr{E}^{ixQ} \exp(i[z - \tfrac{1}{2}xy]\mathscr{H}), \end{aligned}$$

while the expression (2.5a) has no proper meaning. The one-valued representations of the one-parameter groups $W_Q{}^\star$, $W_P{}^\star$, and $W_H{}^\star$ by exponentials give the result that their generators have the spectra

$$\begin{aligned} &\Sigma\,(\mathscr{E}^{iQ}) = \{e^{iq}, \quad q \in \mathfrak{R}, \quad q \equiv q \bmod 2\pi M\}, \\ &\Sigma\,(\mathscr{P}) = \{p = n_p p_0, \quad n_p \in \mathfrak{Z}, \quad p_0 = 2\pi/L\}, \\ &\Sigma\,(\mathscr{H}) = \{\lambda = n_\lambda \lambda_0, \quad n_\lambda \in \mathfrak{Z}, \quad \lambda_0 = 4\pi M/L = 2p_0 M\}, \end{aligned}$$

while an operator "$\mathscr{Q}$" in this context has no meaning. A commutator bracket can be built considering $g(x, 0, 0)g(0, y, 0)g(x, 0, 0)^{-1}g(0, y, 0)^{-1}$

$g(0, 0, -xy)$, for finite $x = n_x/M$ and infinitesimal y as

$$[\mathscr{E}^{ixQ}, \mathscr{P}] \qquad \cdot n_x \mathscr{H} \mathscr{E}^{ixQ}, \tag{6.7}$$

which is of the nature of the Heisenberg–Weyl commutator bracket, but is not a Lie bracket.

2. *Representations of* $W^\star$

As in Section II,D, a procedure to obtain the unitary representations of the group $W^\star$ is to consider the action of the group on itself and then, through a coset decomposition, introduce functions f_c on the coset space $W^\star/W_Q{}^\star$ and their transformation (2.19) under the actions of the group. We now decompose f_c as in (2.20) into a *sum* of eigenfunctions of $\mathscr{H}$ labeled by a *discrete* set of eigenvalues λ given by (6.6c) which will label the representations of $W^\star$. We thus have a multiplier representation of $W^\star$ on the coset space $W^\star/W_Q{}^\star W_H{}^\star$ which is identical with (2.21) except for the ranges of the parameters. By introducing a complete and orthogonal set of functions $\{\psi_n{}^\lambda(y)\}_{n \in \mathfrak{J}}$ ($\mathfrak{J}$ some index set) on the circle with perimeter L, with the usual measure dy, we can find the unitary representation matrices of $W^\star$ as

$$\begin{aligned} D^\lambda_{nn'}(g') &= (\psi_n{}^\lambda, \mathscr{T}(g')\psi^\lambda_{n'}) \\ &= \int_{-L/2}^{L/2} dy\, \psi_n{}^\lambda(y)^* \exp(i\lambda[z \qquad x'y' - x'y])\psi^\lambda_{n'}(y \qquad y'). \end{aligned} \tag{6.8}$$

One such basis is

$$\chi^\lambda_{n_p}(y \qquad L^{-1/2} \exp(-2\pi i n_p y/L), \qquad n_p \in \mathfrak{Z}, \tag{6.9}$$

which are eigenfunctions of $\mathscr{P}$ with eigenvalue $p = n_p p_0$. Performing (6.8) we obtain, for $\lambda = n_\lambda \lambda_0$ and $x = n_x/M$,

$$D^\lambda_{n_p n_p'}(g(x, y, z)) \qquad \delta_{2n_x n_\lambda, n_p - n_p'} \exp i\left(\lambda z + \frac{\pi}{L} y[n_p + n_p']\right) \tag{6.10}$$

[compare with (2.24)], which can be verified to follow the group multiplication law. It is to be noticed, however, that this representation is not irreducible since, for fixed n_λ, (6.10) will have only nonzero matrix elements between rows n_p and columns n_p' such that $n_p - n_p'$ is a multiple of $2n_\lambda$. This means that (6.10) can be broken up into $2n_\lambda$ submatrices D^r such that $n_p = 2mn_\lambda + r$ ($m \in \mathfrak{Z}$, $r = 0, 1, \ldots, 2n_\lambda - 1$). They all belong to the same eigenvalue λ of $\mathscr{H}$, but differ in the range of the eigenvalues of $\mathscr{P}$: $p = m\lambda/M + rp_0$, $m \in \mathfrak{Z}$. If M is identified with the radius of the space, only the set D^0 provides us with physical angular momentum eigenvalues.

It will prove convenient to introduce a summation symbol which will deal only with one set D^r and will go over to an integral in the $\lambda \to 0$ limit. Noticing that the difference between two adjoining values of p in D^r is $\Delta p = \lambda/M$, we define

$$\mathop{\mathsf{S}}_{p} f(p) \equiv \sum_{\substack{p \\ (r \text{ fixed})}} \Delta p f(p) = (\lambda/M) \sum_{\substack{m \in \mathfrak{Z} \\ (n_p = 2mn_\lambda + r)}} f(n_p p_0). \tag{6.11}$$

In this way we can define from (6.10), for $p = m\lambda/M + rp_0$, $m \in \mathfrak{Z}$,

$$\begin{aligned} D^{\lambda,r}_{pp'}(g(x, y, z)) &= (M/\lambda) D^{\lambda}_{n_p n_p'}(g(x, y, z)) \\ &= \delta_P(\lambda x - [p - p']) \exp i(\lambda z + \tfrac{1}{2}[p + p']y) \\ &= \exp(irp_0 y) D^{\lambda,0}_{p,p'}(g(x, y, z)), \end{aligned}$$

where

$$\delta_P(\lambda x - [p - p']) = (M/\lambda)\, \delta_{2n_\lambda n_x, n_p - n_{p'}} \tag{6.12b}$$

is the reproducing kernel in (6.11). The analogy with (2.24) is now complete, the product of representation matrices follows (2.25) closely with the "integral" (6.11), and we can check orthogonality and the completeness relation. Integration over the group manifold will be indicated using the notation in (*132*)

$$\mathop{\mathsf{S}}_{g} \equiv \sum_{n_x} (1/M) \int_{-L/2}^{L/2} dy \int_{-L/4M}^{L/4M} dz.$$

since the difference between two adjoining values of x is $\Delta x = 1/M$. We can now write the orthogonality relation on the group as

$$\mathop{\mathsf{S}}_{g} D^{\lambda_1, r_1}_{p_1, p_1'}(g)^* D^{\lambda_2, r_2}_{p_2, p_2'}(g) = \delta_{\hat{W}^\star}(\lambda_1, r_1; \lambda_2, r_2)\, \delta_P(p_1 - p_2)\, \delta_P(p_1' - p_2') \tag{6.14}$$

[compare with (2.32)] where

$$\delta_{\hat{W}^\star}(\lambda_1, r_1; \lambda_2, r_2) = (L^2/2M^2)\, \delta_{n_{\lambda_1}, n_{\lambda_2}}\, \delta_{r_1, r_2} \tag{6.15}$$

should be the reproducing kernel under the Plancherel sum over $\hat{W}^\star$, the representation space of $W^\star$. Indeed, we can define an "integration" over representation space (*132*) as

$$\mathop{\mathsf{S}}_{\lambda, r} \equiv \sum_{n_\lambda \in \mathfrak{Z}} (2M^2/L^2) \sum_{r=0}^{2n_\lambda - 1}, \tag{6.16}$$

whereupon the completeness relation reads

$$\mathop{\mathbf{S}}_{\lambda,r} \mathop{\mathbf{S}}_{p} \mathop{\mathbf{S}}_{p'} D^{\lambda,r}_{pp'}(g_1)^* D^{\lambda,r}_{pp'}(g_2) = \delta_{W^\star}(g_1, g_2) \tag{6.17}$$

[compare with (2.34)] where the reproducing kernel under group integration (6.13) is

$$\delta_{W^\star}(g_1, g_2) = M\, \delta_{n_{x_1}, n_{x_2}}\, \delta(y_1 \quad y_2)\, \delta(z_1 \quad z_2). \tag{6.18}$$

Notice that as the dependence of $D^{\lambda,r}$ on r is only through a phase, the summation in (6.16) becomes

$$\sum_{n_\lambda \in \mathfrak{Z}} \frac{4M^2}{L^2} n_\lambda = \sum_{n_\lambda \in \mathfrak{Z}} \frac{|\lambda|}{4\pi^2} \Delta\lambda, \tag{6.19}$$

which can be directly compared with the Plancherel measure for W in (2.34).

A second basis for the circle can be constructed considering eigenfunctions of $\mathscr{E}^{iQ}$, orthonormal in the sense of Dirac:

$$\tilde{\chi}_q{}^\lambda(y) = |\lambda|^{1/2}\, \delta(q + \lambda y) \tag{6.20}$$

where the range of q will be examined below, which correspond to the eigenvalue e^{ixq}. Introduced in (6.8) they yield

$$D^\lambda_{qq'}(g(x, y, z)) = \delta(\lambda y - [q' - q]) \exp i(\lambda z + \tfrac{1}{2}[q + q']x) \tag{6.21}$$

[compare with (2.27)], which follow the group multiplication law, are orthogonal over $W^\star$:

$$\mathop{\mathbf{S}}_{g} D^{\lambda_1}_{q_1q_1'}(g)^* D^{\lambda_2}_{q_2q_2'}(g) = \sum_r \delta_{W^\star}(\lambda_1, r; \lambda_2, r)\, \delta(q_1 - q_1')\, \delta(q_2 \quad q_2'), \tag{6.22}$$

and complete. The relation between the two representations (6.11) and (6.22) can be given by the transformation bracket

$$(M/\lambda)^{1/2}(\tilde{\chi}_q{}^\lambda, \chi^\lambda_{n_p}) = (4\pi n_\lambda |\lambda|)^{-1/2} \exp(ipq/\lambda). \tag{6.23}$$

Whereas the required properties of (6.21) and (6.22) do not depend on the range of q, due to the expression $\lambda y - q$ in the Dirac δ in (6.21) for general $p = m\lambda/M + rp_0$ ($m \in \mathfrak{Z}$), the kernel (6.23) is a multivalued function of q in that range. Only for the set D^r of representations with $r = 0$ is the kernel periodic with period $2\pi M$, as it should have to be if q is to represent the coordinate on a circle with radius M. It corresponds to representations where the allowed spectrum of $\mathscr{P}$ is $p = m\lambda/M$, $m \in \mathfrak{Z}$, and should be the only one with a physical interpretation. For $r = n_\lambda$ we have $p = (m+\frac{1}{2})\lambda/M$ which resembles the angular momentum of objects of half-integer spin on the circle, in that we have a two-valued representation of the group $W^\star$ on q space.

Finally we shall come to use

$$\begin{aligned} D^\lambda_{qp}(g) &= (M/\lambda)^{1/2}(\tilde{\chi}_q{}^\lambda, \mathscr{E}(g)\chi^\lambda_{n_p}) \\ &= (4\pi n_\lambda \mid \lambda \mid)^{-1/2} \exp i(\lambda[z + \tfrac{1}{2}xy] + xq + yp + pq/\lambda), \quad (6.24a) \end{aligned}$$

$$\begin{aligned} D^\lambda_{pq}(g) &= (M/\lambda)^{1/2}(\chi^\lambda_{n_p}, \mathscr{E}(g)\tilde{\chi}_q{}^\lambda) \\ &= (4\pi n_\lambda \mid \lambda \mid)^{-1/2} \exp i(\lambda[z - \tfrac{1}{2}xy] + xq + yp - pq/\lambda). \quad (6.24b) \end{aligned}$$

B. The Ring $\mathfrak{W}^\star$

We shall now proceed to construct a ring structure over $W^\star$ in a convenient basis. The elements of the ring $\mathfrak{W}^\star$ are defined as in (3.1) through

$$\mathscr{A} = \mathop{\mathbf{S}}_{g} A(g)\mathscr{E}(g) \tag{6.25}$$

allowing us to express any ring element in terms of a function over the group $A(g)$. As before, the generators $\mathscr{E}^{iQ}$, $\mathscr{P}$, and $\mathscr{H}$ are in $\mathfrak{W}^\star$, and so are any linear combinations and products. These operations can be expressed, as in Section II, through operations of the group functions. Convenient bases can be constructed through the $W^\star$ harmonic transform

$$A^\lambda(r, r') = \mathop{\mathbf{S}}_{g} A(g)D^\lambda_{rr'}(g) \tag{6.26a}$$

$$A(g) = \mathop{\mathbf{S}}_{\lambda,r} \operatorname{tr}[A^\lambda D^\lambda(g^{-1})] \tag{6.26b}$$

[compare with (3.9)–(3.10)], where the D's are the representation matrices in any of the chains considered. (See Table II.) As in (2.29) we can set up a "coordinate basis" of functions $\psi^\lambda(q) = (\tilde{\chi}_q{}^\lambda, \phi^\lambda)$, $q \in (-\pi M, \pi M]$, and act with the ring elements as

$$g : \psi^\lambda(q) \to [\mathscr{E}(g)\psi^\lambda](q) = \int dq'\, D^\lambda_{qq'}(g)\psi^\lambda(q') \tag{6.27a}$$

$$\mathscr{A} : \psi^\lambda(q) \to [\mathscr{A}\psi^\lambda](q) = \int dq'\, A^\lambda(q, q')\psi^\lambda(q'). \tag{6.27b}$$

For the elements of the "Lie algebra" of W^* we have, as expected, the Schrödinger representation

$$\mathscr{E}^{iQ}\psi^\lambda(q) = e^{iq}\psi^\lambda(q), \tag{6.28a}$$

$$\mathscr{P}\psi^\lambda(q) = -i\lambda \frac{d}{dq}\,\psi^\lambda(q), \tag{6.28b}$$

$$\mathscr{H}\psi^\lambda(q) = \lambda\psi^\lambda(q), \tag{6.28c}$$

which are Hermitian with respect to the scalar product

$$(\psi_1, \psi_2) = \int_{-\pi M}^{\pi M} dq\, \psi_1(q)^*\psi_2(q), \tag{6.29}$$

and allows us to construct quantum mechanical states as the one-valued function on the circle, elements of the Hilbert space $\mathfrak{L}^2(-\pi M, \pi M]$. Hermiticity of a ring element is defined in terms of (6.29) or, equivalently, in terms of the unitarity of (6.28a) and the Hermiticity of (6.28b) and (6.28c).

In order to establish a connection with physical observables on the circle, we are led as before to consider the phase space representatives for the standard form

$$a_s^\lambda(q, p) = (4\pi n_\lambda\, |\, \lambda\, |)^{1/2} \exp(-ipq/\lambda) \underset{g}{\mathbf{S}}\, A(g)D^\lambda_{qp}(g), \tag{6.30a}$$

and the antistandard form

$$a_a^\lambda(q, p) = (4\pi n_\lambda\, |\, \lambda\, |)^{1/2} \exp(ipq/\lambda) \underset{g}{\mathbf{S}}\, A(g)D^\lambda_{pq}(g). \tag{6.30b}$$

In Table II we give two typical ring elements with their corresponding representatives. It must be remembered, though, that p takes values at all discrete points $m\lambda/M$, $m \in \mathfrak{Z}$, zero at all other values, and $q \equiv q \bmod 2\pi M$. Glancing through Table II we see that it looks very much like Table I. Indeed, if in the latter we build $\exp(ix\mathscr{Q})$ as $\sum (ix\mathscr{Q})^n/n!$, we arrive at exactly the entries of the former. Conversely, if out of $\mathscr{E}^{iQ}$ we build a "position operator" for the circle, which we enclose in quotation marks

$$\text{“}\mathscr{Q}\text{”} \quad i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (1/n) \cos n\pi\, \mathscr{E}^{inQ/M}; \tag{6.31a}$$

it will exhibit with $\mathscr{P}$ the "non-Heisenberg" commutation relations

$$[\text{“}\mathscr{Q}\text{”}, \mathscr{P}] \quad i\mathscr{H}(1 - 2\pi\, \delta(\text{“}\mathscr{Q}\text{”} \quad \pi)). \tag{6.31b}$$

TABLE II

SOME REPRESENTATIVE FUNCTIONS FOR TWO TYPICAL ELEMENTS OF THE HEISENBERG–WEYL RING $\mathfrak{W}^*$

Representative	$\mathscr{H}^l \mathscr{E}^{i\bar{x}Q} \mathscr{P}^n$	$\mathscr{H}^l \mathscr{P}^n \mathscr{E}^{i\bar{x}Q}$
$A(g(x, y, z))$	$M\, \delta_{n_x \bar{n}_x} i^{n+l}\, \delta^{(n)}(y)\, \delta^{(l)}(z + \frac{1}{2}xy)$	$M\, \delta_{n_x \bar{n}_x} i^{n+l}\, \delta^{(n)}(y)\, \delta^{(l)}(z - \frac{1}{2}xy)$
$A^\lambda(q, q')$	$\lambda^l \exp(i\bar{x}q)(-i\lambda)^n\, \delta^{(n)}(q - q')$	$\lambda^l \exp(i\bar{x}q')(-i\lambda)^n\, \delta^{(n)}(q - q')$
$A^\lambda(p, p')$	$\lambda^l\, \delta_P(\lambda\bar{x} - [p - p'])p'^n$	$\lambda^l\, \delta_P(\lambda\bar{x} - [p - p'])p^n$
$a_s^\lambda(q, p)$	$\lambda^l \exp(i\bar{x}q)p^n$	$\lambda^l \exp(i\bar{x}q)(p + \lambda\bar{x})^n$
$a_a^\lambda(q, p)$	$\lambda^l \exp(i\bar{x}q)(p - \lambda\bar{x})^n$	$\lambda^l \exp(i\bar{x}q)p^n$

Regarding the correspondence between the standard phase space representatives of two ring elements $\mathscr{A}$ and $\mathscr{B}$, and that of their product $\mathscr{A}\mathscr{B} = \mathscr{C}$, we follow (3.18) in writing

$$c(q,p) = (2\pi \mid \lambda \mid)^{-1/2} \int_{-\pi M}^{\pi M} dq' \; \mathop{\mathbf{S}}_{p'} a(q,p')b(q',p) \times \exp(-i[q-q'][p-p']/\lambda). \tag{6.32}$$

While we can write the Taylor expansion for b as a function of q, for a as a function of p we can only expand its Fourier transform $\tilde{a}$ as

$$a(q, p+\sqrt{\lambda}\,v) = (2\pi)^{-1/2} \int_{-\pi/\lambda}^{\pi/\lambda} d\zeta\, \tilde{a}(q,\zeta) \exp(-i[p+\sqrt{\lambda}\,v]\zeta) \sum_{s=0}^{\infty} \frac{(\sqrt{\lambda}\,v)^s}{s!} \frac{\Delta^s}{\Delta p^s} a(q,p) \tag{6.33a}$$

where we have defined

$$\frac{\Delta^s}{\Delta p^s} a(q,p) \equiv \mathop{\mathbf{S}}_{p'} a(q,p')Z^s(p \quad p'), \tag{6.33b}$$

i.e., the convolution between a and

$$Z^s(p) = (2\pi)^{-1} \int_{-\pi/\lambda}^{\pi/\lambda} d\zeta(\quad i\zeta)^s \exp(-ip\zeta). \tag{6.33c}$$

Both in the infinite-radius and in the classical limit, as $\lambda \to 0$, $Z^s(p)$ becomes $\delta^{(s)}(p)$. Since p is now dense in the real line, (6.33b) becomes $d^s a(q,p)/dp^s$. Moreover, for the special cases $a(q,p) = p^n$, $\Delta^s a/\Delta p^s = n!p^{n-s}/(n-s)!$, so that the symbol $\Delta^s/\Delta p^s$ stands very intuitively for a derivative. If now $\mathscr{D} = [\mathscr{A}, \mathscr{B}]$, we have

$$d_s^{\lambda}(q,p) = \sum_{k=1}^{\infty} \frac{-i\lambda)^k}{k!} \left[\frac{\partial^k a_s}{\partial q^k} \frac{\Delta^k b_s}{\Delta p^k} - \frac{\Delta^k a_s}{\Delta p^k} \frac{\partial^k b_s}{\partial q^k} \right] \tag{6.34}$$

where the first term is the analog of the classical Poisson bracket times $i\lambda$.

In Section III,D we reached the conclusion that Poisson brackets and commutators followed each other when all terms beyond the first one in (6.34) vanished. This happened there for the classes $P_{\leq 2}$, P_q, and P_p. The classes $P_{\leq 2}$ and P_p are not allowed in $\mathfrak{W}^*$, since "nonperiodic" functions of q lead to non-Heisenberg commutation relations (6.31b) which differ from the Poisson bracket by a Dirac δ. The correspondence in $\mathfrak{W}^*$ thus holds only for the class P_q with functions which are periodic in q with period $2\pi M$.

C. Characteristics of "Compact" Quantum Mechanics

1. *Infinite-Radius Limit*

If we let $M \to \infty$, the range of q will increase indefinitely. In order to avoid the collapse of all values of z to zero, we must let $L \to \infty$ simultaneously, setting $L = M^2$. With this, the group integration (6.13) will become one over a three-dimensional Euclidean space with the Haar measure (2.7) of W. The set of values of p becomes dense in the real line with the sum (6.11) becoming an ordinary integral which has to be taken $2n_\lambda$ times, one for each D^r ($r = 0, 1, \ldots, 2n_\lambda - 1$), which in the limit $p_0 \to 0$ become equivalent. Likewise q becomes extended over the full real line. As λ_0 vanishes, the summation over the $\hat{W}^\star$ representation space (6.16) is the integral with the correct Plancherel measure (2.33) of W. In doing so, n_λ has to take larger and larger values, otherwise λ itself would vanish. Finally, for M growing without bound, wave functions over q space (6.27) to (6.29), if they are to remain square integrable and not move to infinity, must become vanishingly small at $q = \pm\pi M$. The commutator (6.2) then has the matrix elements of (2.1).

2. *The Classical Limit*

The classical limit is obtained letting $L \to \infty$ and keeping M fixed. For finite n_λ, all λ's collapse to zero and all operators commute since $\mathscr{H}$ is represented by zero. As $\Delta p \to 0$, the values of p become dense on the real line while q stays bounded in $(-\pi M, \pi M]$. Table II shows that $a_s{}^\lambda(q, p)$ and $a_a{}^\lambda(q, p)$ become equal, and this is the classical phase space function.

3. *The Quantization Process*

As in Section IV, the inverse process to the classical limit is quantization: We start with a classical phase space function $a_c(q, p)$ which is here assumed to be periodic in q with period $2\pi M$. We want to arrive at some Hermitian element $\mathscr{A}$ of the ring for which $a_s{}^\lambda(q, p) = a_a{}^\lambda(q, p)^*$. The quantization schemes presented in Section IV which are defined in terms of the quantization of monomials of the type $\mathscr{Q}^m\mathscr{P}^n$ are invalid here. Nevertheless, Dirac's construction as well as Cohen's general scheme are formulated so that they can be used when we have general periodic functions $f(e^{iq})$. The conclusions then follow those presented previously in stating that phase space functions of the class P_q [i.e., of the type $pf(e^{iq}) + g(e^{iq})$] are quantized as

$$\tfrac{1}{2}[f(\mathscr{E}^{iQ})\mathscr{P} + \mathscr{P}f(\mathscr{E}^{iQ})] + g(\mathscr{E}^{iQ})],$$

and that this is scheme independent.

4. *Canonical Transformations*

The definition of classical canonical transformations on a compact space can be given in the same terms as for the infinite space of Section V,A. This is not so in quantum mechanics, where the commutator we want to be preserved is (6.7), in addition to preserving the Hermiticity of the operators involved. The first question which comes to mind is to inquire whether a linear transformation between $\mathscr{E}^{iQ}$ and $\mathscr{P}$ is canonical. Direct replacement of such a transformation in (6.7) shows that it is not. Similarly, a linear transformation between "$\mathscr{Q}$" in (6.31) and $\mathscr{P}$ fails to preserve the bracket (6.31b).

The other class of canonical transformations we are interested in is point transformations in q space, where the periodicity interval $2\pi M$ is mapped onto itself through a one-to-one, invertible, differentiable function. Here, classical and quantum mechanics will follow each other as Poisson brackets and commutators agree. The relevant characteristics of such transformations were already developed in Section V,C. We would only like to point out that a particularly important case of point transformations on the circle is made up of those generated by the quantization of

$$j_1 = p\cos q + l\sin q, \qquad j_2 = p\sin q - l\cos q, \qquad j_3 = p, \tag{6.35}$$

which close onto an $\mathscr{SO}(2, 1)$ algebra. Indeed, they can be shown to be the most general $\mathscr{SO}(2, 1)$ algebra of functions in the class P_q with a constant Casimir operator. The integrated action of (6.35) can be found, using (5.12) through (5.19). The operator $\exp(\tau j_{3\text{op}})$ generates rotations in q by the angle τ while the transformation generated by j_2 is

$$\exp(\tau j_{2\text{op}}) \begin{cases} q \to \bar{q} = 2\arctan(e^{\tau}\tan(q/2)) \\ p \to \bar{p} = p(\cosh\tau - \sinh\tau\cos q). \end{cases} \tag{6.36}$$

These, as constructed by Bargmann (*133*), are transformations of the $SO(2, 1)$ coset [by $O(2)$] manifold which provide the principal series of representations of the $SO(2, 1)$ group for $l = -\frac{1}{2} + i\varrho$, $\varrho \in \mathfrak{R}$, and the value $l(l + 1)$ for the Casimir operator.

D. Discussion

In this section we have carried out a program parallel to that of the former sections but dealing with quantum mechanics on a compact one-dimensional space. The system was originally defined as a quantum system whose momentum operator has a discrete, infinite spectrum of equally spaced eigenvalues. Although it was shown that an appropriate Hilbert space of wave functions is $\mathfrak{L}^2(-L/2, L/2]$, it would be wrong to state that we can just replace this space into the domain of (2.1); indeed, we have avoided the use of (2.1) altogether since the assumption that a position operator exists with a continuous, compact eigenvalue spectrum leads to severe difficulties. These were recognized as early as 1927 by Jordan (*134*, Postulate D, p. 812 and statement on p. 816); see, however, (*135*, p. 2); other relevant earlier references are (*105*, *107*, *136*, *137*). Jordan showed that an operator $\mathscr{P}$ with a discrete spectrum *cannot* have a conjugate operator "$\mathscr{Q}$" such that (2.1) holds. It can also *not* be satisfied by any two bounded operators (*35*, Sect. 6.1.1). The simplest argument can be given (*138*, Sect. IV) considering the matrix elements of (2.1) assumed to hold between eigenstates ψ_m of $\mathscr{P}$ with eigenvalue m:

$$i\hbar\, \delta_{m.m} \qquad (\psi_m \qquad\qquad \mathscr{P}]\psi_m \qquad (m' - m)(\psi_m, \qquad '\psi_{m'}), \qquad (6.37)$$

which is impossible for $m = m'$. The problem basically stems from the definition of an angle observable q which must exhibit a discontinuity of L somewhere along the circle. Classical mechanics, being basically a local theory, can write the Poisson bracket $\{q, p\} = 1$ and ignore the discontinuity of q, shifting if necessary the origin of coordinates. In its quantum mechanical translation, however, the discontinuity cannot be ignored. We thus have to reject the Heisenberg commutation relations (2.1) and, if we still want to keep a position operator "$\mathscr{Q}$" with some of its characteristics, we have to replace (2.1) by the non-Heisenberg commutation relation (6.31b) (*138–141*). The well-known uncertainty relation $\Delta q \cdot \Delta p \gtrsim \frac{1}{2}\hbar$ must also be revised: When q runs over a compact domain, Δq cannot exceed a finite number, while it is not difficult to prepare samples with a Δp as small as we please, thus violating the inequality. This has led Judge and Lewis (*139*) to postulate a redefinition of Δq so as to make it insensitive to the position of the discontinuity (*142–147*).

The alternative which we have essentially followed skips the problem of using an ill-defined "$\mathscr{Q}$" when we consider only *periodic* functions in q; the discontinuity in the coordinate definition is washed out and we make use of the Weyl commutation relation (2.5b), (2.5c) instead of (2.1) (*35*, Chapter 6; *105*; *107*; *136–138*; *141*; *148*). We have further shown that we can arrive at a consistent and orderly mathematical framework through the use of the Heisenberg–Weyl group defined by (2.5). This matches the "safe" use of the coordinate and momentum formalism in (*138–148*). We should here mention a similar work by Boon (*21*) on the variant of the Heisenberg–Weyl group presented in Section II,C,2.

[In order to avoid a possible misunderstanding, we would like to stress that we are concerned with momentum operators which exhibit an infinite discrete spectrum as does angular momentum, *not* a half-infinite Hamiltonian-type spectrum with a lower bound. Although analogies (*138*, *141*) and connections (*148*) between these two exist, the problems connected with a "time operator" (*141*, *149–153*) have a quite different origin.]

The problem has also been treated in two or more dimensions and then "pseudoquantized" through a coordinate transformation (*148*), or treated through a larger algebra of well-defined periodic operators as (6.35), thus avoiding the use of "compact" quantum mechanics completely (*7*, *154*).

As stated in the Introduction, however, we have followed a construction which puts stress on the use of the Heisenberg–Weyl algebra, group, and ring as a scheme for quantum mechanics. The material presented here has been developed in (*155–159*).

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