Appendix A

The Gamma Function

The *factorial* of a natural number n is defined as

$$n! := n(n-1)(n-2)\cdots 3\cdot 2\cdot 1. \tag{A.1}$$

Recursively, it can be characterized by

$$n! := n(n-1)!, \qquad 0! := 1. \tag{A.2}$$

A function which generalizes the factorial for complex numbers is the *gamma* function, defined by the *Euler integral*

$$\begin{aligned}\Gamma(z) &:= \int_0^\infty dt\, t^{z-1}\exp(-t)\\ &= 2\int_0^\infty du\, u^{2z-1}\exp(-u^2), \qquad \operatorname{Re} z > 0.\end{aligned} \tag{A.3}$$

From this form it follows by integration by parts that

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(1) = 1, \tag{A.4}$$

and hence

$$\Gamma(n) = (n-1)!. \tag{A.5}$$

We can define $\Gamma(z)$ for $\operatorname{Re} z \leqslant 0$ by $\Gamma(z) = \Gamma(z+1)/z$ repeated the number of times necessary for the argument to reach positive values for the real part. A special value is

$$\Gamma(\tfrac{1}{2}) = \pi^{1/2} = 1.7724538509\ldots, \tag{A.6}$$

found from the last expression in (A.3), which is just the integral (7.21). A plot of the gamma function appears in Fig. A.1. It is an analytic function with simple poles at zero and the negative integers, since

$$\begin{aligned}\Gamma(x-n) &= (x-n)^{-1}\Gamma(x-n+1) = \cdots\\ &= (-1)^n[(n-x)(n-1-x)\cdots(1-x)x]^{-1}\Gamma(1+x).\end{aligned} \tag{A.7a}$$

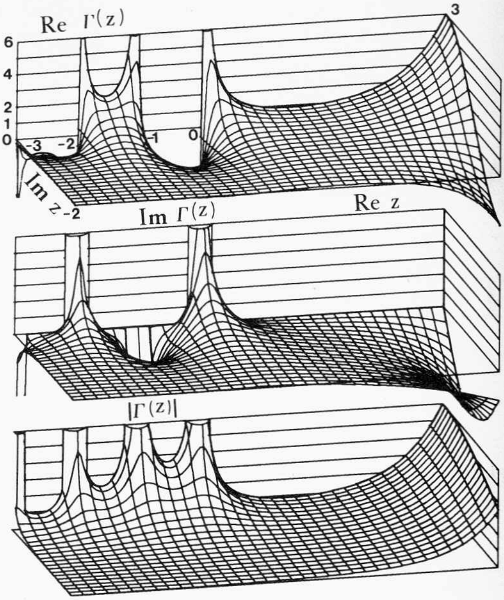

Fig. A.1. Real, imaginary, and absolute values of the gamma function $\Gamma(z)$ in the complex plane of the argument. $\mathrm{Re}\, z \in (-3, 3)$, $\mathrm{Im}\, z \in (-2, 0)$. Simple poles are present at $z = 0, -1, -2, \ldots$.

Thus for $x \to 0$ the residues of the poles are

$$\mathrm{Res}\, \Gamma(z)|_{z=-n} = (-1)^n/n!. \tag{A.7b}$$

Two other expressions for the gamma function, the Euler infinite limit and the Weierstrass infinite product, can be seen, for instance, in Whittaker and Watson (1903, Chapter 12). In the former we can find the proofs of the following useful relations: the Gauss multiplication formula

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nx-1/2} \prod_{k=0}^{n-1} \Gamma(z + k/n) \tag{A.8}$$

and the reflection formulas

$$\Gamma(z)\Gamma(1 - z) = \pi \csc(\pi z), \tag{A.9a}$$

$$\Gamma(\tfrac{1}{2} + z)\Gamma(\tfrac{1}{2} - z) = \pi \sec(\pi z). \tag{A.9b}$$

The numerical computation of the gamma function is usually performed by approximating it by the polynomial

$$\begin{aligned}\Gamma(z + 1) = 1 &- 0.577191652z + 0.988205891z^2 - 0.897056936z^3 \\ &+ 0.918206857z^4 - 0.756704078z^5 + 0.482199394z^6 \\ &- 0.193527818z^7 + 0.035868343z^8 + \varepsilon(z),\end{aligned} \tag{A.10}$$

which for $0 \leqslant z \leqslant 1$ is valid with $|\varepsilon(z)| \leqslant 3 \times 10^{-7}$. The argument is moved to the strip $0 \leq \operatorname{Re} z \leq 1$ by repeated use of (A.4) or (A.7a); then, if need be, the absolute value of the imaginary part is contracted to less than $\frac{1}{2}$ with the use of the Gauss formula (A.8). For this and several other computer algorithms and approximations, see Hastings (1955) and the periodically updated communications of the American Computer Society (ACS). The analytic aspects of the gamma function have been elegantly developed by Artin (1964) and Lösch and Schoblik (1951). A summary of properties, tables, and references can be found in Abramowitz and Stegun (1964, Chapter 6).