Groups of Integral Transforms Generated by Lie Algebras of Second- and Higher-Order Differential Operators.

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Summary. — We study the construction and action of certain Lie algebras of second- and higher-order differential operators on spaces of solutions of well-known parabolic, hyperbolic and elliptic linear differential equations. The latter include the N-dimensional quadratic quantum Hamiltonian Schrödinger equations, the one-dimensional heat and wave equations and the two-dimensional Helmholtz equation. In one approach, the usual similarity first-order differential operator algebra of the equation is embedded in the larger one, which appears as a quantum-mechanical dynamic algebra. In a second approach, the new algebra is built as the time evolution of a finite-transformation algebra on the initial conditions. In a third approach, the inhomogeneous similarity algebra is deformed to a noncompact classical one. In every case, we can integrate the algebra to a Lie group of integral transforms acting effectively on the solution space of the differential equation.

1. – Introduction.

The last fifteen years have seen a renewed interest in the Lie theoretical treatment of partial differential equations, both linear and nonlinear (1-3). From the method of construction of the similarity algebra, however, it is clear

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that only algebras of first-order differential operators in the independent and dependent variables will appear. During the same period, theoretical physicists have been working with « hidden » symmetry and dynamical algebras for Schrödinger systems. These usually turn out to be differential operators of order higher than the first. The quadrupole operators in the harmonic-oscillator $\mathscr{SU}_3$ algebra [4] and the step operators of the $\mathscr{Sp}_{6,R}$ dynamical algebra ([5], Chapter 20) are of second order in the space derivatives; for the Kepler system, the Runge-Lenz vector components in $\mathscr{SO}_4$ are of second order [6,7], while the generators which are noncompact in the $\mathscr{SO}_{4,2}$ dynamical algebra —conformal transformations of the Fock sphere in momentum space—are integral operators in configuration space ([8,9], [5] Chapter 21). (See, however, [10].) Other cases of symmetry algebras also involve operators of infinite order expressible as integral transforms [11,12]. These algebras have been used in order to find transition operator matrix elements and expectation values in nuclear shell theory ([4], [5] Chapter 20) and hydrogenic physics ([13,14], [15] Chapter 21), extending the techniques of the angular-momentum Racah algebra to these fields.

Both in differential equations and in dynamical-algebra physics, the study of the associated Lie group seems to have been of secondary importance, exception taken of the rotation, Galilean, Lorentz and Poincaré transformations. The work of Moshinsky and collaborators [16-20] has shown, however, that,

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BLUMAN).

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at least regarding the harmonic-oscillator $Sp_{2N,C}$ group, finite transformations result in a semi-group of integral transforms which represent canonical transformations for a quantum system, and thus suggest a host of interesting problems both in quantum mechanics and in group theory. This semi-group and associated ones include, as particular cases of finite group elements, the integral transforms of Fourier, Bergmann, Mellin, bilateral Laplace, Gauss-Weierstrass, Hankel and Barut-Girardello (([21,22]), ([23]) Part. 4) and have led to what seems to be a significant integration of various developments in group theory, special functions and partial differential equations of parabolic type (([24-26]), ([23]) Chapter 10). In the mathematical literature, it has been shown that every N-dimensional real Lie algebra has a realization in second-order differential operators in N variables, with no first-order part ([27]).

Consider n-th order differential operators in one independent variable written as

$$(1.1)\qquad \mathbf{P}^{(n)} = \sum_{k=0}^{n} P_k(q)\,\frac{d^k}{dq^k}\,,$$

where $d^0/dq^0 = 1$. We are interested in the exponentiation of (1.1), $\exp[i\alpha \mathbf{P}^{(n)}]$, and that of analogous versions in more than one independent variable. The original work of Lie ([28]) dealt with the cases (1.1) for $n = 1$ and $P_0(q) = 0$. These can be called *point* transformations, mapping a function $f(q)$ on $f(q'(q, \alpha))$. The present work with similarity groups in differential equations entails considering the solution f as an independent variable. If the equation is linear, all similarity operators involving f appear as $f\partial/\partial f$, which is equivalent to 1 acting on $f(q)$. This remark brings in operators (1.1) with a $P_0(q)$ term whose exponentiated action is

$$(1.2)\qquad f_\alpha(q) = \exp\left[i\alpha\left[P_1(q)\,\frac{d}{dq} + P_0(q)\right]\right] f(q) = \mu(q, \alpha) f(q'(q, \alpha))\,,$$

where a multiplier function $\mu(q, \alpha)$ thus appears. Groups generated out of operators of this kind seem to have been first applied by BARGMANN ([29]). The

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action (1.2) will be called *geometric* as the value of $f_\alpha(q)$ at some q depends on the value of $f(q')$ at a single point q'. Of course, when $P_0(q)=0$, $\mu(q,\alpha)=1$. The next step, $n=2$, has been solved completely for $P_2(q)=c_{20}$, $P_1(q)=c_{11}q+c_{10}$ and $P_0(q)=c_{02}q^2+c_{01}q+c_{00}$ with c_{mn} constant [21,25] and when $P_2(q)=c_{20}$, $P_1(q)=c_{11}q-c_{1,-1}q^{-1}$, $P_0(q)=c_{02}q^2+c_{0,-2}q^{-2}$ [22]. In these cases, the group action is that of an *integral transform*

$$f_\alpha(q)=\exp[i\alpha P^{(2)}]f(q)=\int_I dq'\,f(q')K(q,q';\alpha) \tag{1.3}$$

over an interval I which is R and R^+, respectively. The kernel $K(q,q';\alpha)$ is a Gaussian, exponential and/or Bessel function. As $\alpha\to 0$ (or as $c_{20}\to 0$) this kernel becomes a Dirac δ.

The present article is devoted to explore some groups generated by operators of the kind (1.1) in more than one independent variable, acting on spaces of functions which satisfy a linear partial differential equation. In doing so, we must emphasize several points. First, that for one-parameter groups, the kernel in (1.3) involves only the Green's function of the equation, so it is actually the case of finite-dimensional groups we are interested in. Second, if $f(q,t)$ belongs to the space of solutions of a given partial differential equation

$$(\boldsymbol{H}-\partial_t)f(q,t)=0\,, \tag{1.4}$$

and the group action is required to map this space onto itself, the algebra of operators (1.1) must satisfy the well-known condition

$$[(\boldsymbol{H}-\partial_t),\boldsymbol{P}_k^{(n)}]=\boldsymbol{R}_k(\boldsymbol{H}-\partial_t)\,, \tag{1.5}$$

where the operator $\boldsymbol{R}_k$ is determined by $\boldsymbol{P}_k^{(n)}$. Now, if $\boldsymbol{X}$ is an operator independent of t, then it is easy to verify that

$$\boldsymbol{X}^{(t)}=\exp[t\boldsymbol{H}]\boldsymbol{X}\exp[-t\boldsymbol{H}] \tag{1.6}$$

satisfies (1.5) with $\boldsymbol{R}=0$. This only means that any transformation of the initial conditions will produce a transformation in the solution space of (1.4). Although this remark would appear to trivialize the search for symmetry transformations, it rather directs us, as for the dynamical groups of quantum mechanics, to search for finite algebras of operators which include $\boldsymbol{H}$ or under which $\boldsymbol{H}$ transforms in a simple way. Thirdly, for hyperbolic or elliptic equations, the form (1.4) still applies, if we consider $\boldsymbol{H}$ to be a matrix with operator entries and f as a column vector composed out of a function and its t-derivative. The group of transforms (1.3) will require likewise a function matrix kernel on which the group identity $\alpha = 0$ in (1.3) only requires that it reduces to the reproducing kernel of the space of solutions—not necessarily a Dirac δ.

Rather than embark here on a general theory of hyperdifferential operators (for these, see [30]), it seems to us that it is relevant at present to give various specific examples of partial differential equations of parabolic, hyperbolic and elliptic type: the free-particle Schrödinger, heat, wave and Helmholtz equations, in which algebras of second- and higher-order differential operators appear, and where the remarks listed above apply one at a time.

In sect. **2** we follow the « dynamical algebra » construct of quadratic quantum Hamiltonians in N space dimensions: « noninvariance » algebras, properly containing the similarity algebra of the differential equation. This algebra is exponentiated to a group $W_N Sp_{2N,R}$. The Schrödinger similarity algebra [31,32], $\mathscr{S}ch_N = \mathscr{W}_N \wedge [\mathscr{S}p_{2,R} \oplus \mathscr{O}_N]$ contains all and only first-order operators in space and time, while $\mathscr{W}_N \mathscr{S}p_{2N,R}$ contains all up-to-second-order ones. The integral transform kernels are given explicitly and related to the algebra generators. As the time evolution operator need not be in a given group when the latter maps solutions of an equation into solutions of the same, in sect. **3** we develop one such case: conformal symmetry operators acting on the initial conditions of a (Schrödinger-invariant) parabolic differential equation. For arbitrary time we find that this group evolves into an isomorphic group of integral transforms. In sect. **4** we produce a complementary Schrödinger symmetry group for the (conformal invariant) wave equation. Lastly, in sect. **5**, we deform [33-35] the symmetry algebra of the Helmholtz equation $[\mathscr{I}\mathscr{O}_2 \Rightarrow \mathscr{S}\mathscr{O}_{2,1}]$ and examine the resulting integrated process in terms of the Green's function and boundary data.

Each of the three approaches listed here, dynamical algebras (sect. **2**), transformation of initial conditions (sect. **3** and **4**) and algebra deformation (sect. **5**), seem extensible to wide, overlapping classes of partial differential equations, linear as well as nonlinear. Indeed, as will be clear, infinite-dimensional algebras can be produced in every case and the problem is rather to be able to reduce oneself to finite-dimensional ones. In the concluding sect. **6** we offer some comments on this situation.

2. – Symplectic symmetry in a class of parabolic equations.

In this section we give a detailed account of the Lie algebra $\mathscr{W}_N \mathscr{S}p_{2N,R}$, the corresponding Lie group and its action on a space of functions on R^N; this

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is with the purpose of fixing the notation for subsequent sections, present results which have not appeared explicitly before, and show how this algebra appears as a symmetry algebra for the free-particle Schrödinger equation in particular.

Consider the operators defined through their action on a suitable space of functions on R^N:

$$Q_m f(q) = q_m f(q)\,, \qquad q \in R^N\,, \tag{2.1a}$$

$$P_n f(q) = -i\partial f(q)/\partial q_n\,, \qquad m, n = 1, 2, \ldots, N\,, \tag{2.1b}$$

and the unit operator 1. These $2N+1$ operators form a basis for ([36]) a realization of the Heisenberg-Weyl algebra $\mathscr{W}_N$ with commutation relations

$$[Q_m, P_n] = i\delta_{mn}1\,, \quad [Q_m, 1] = 0\,, \quad [P_n, 1] = 0\,. \tag{2.2}$$

We shall consider q to be a row vector and q^T, its transpose, a column vector. The Lie algebra exponentiates to the Heisenberg-Weyl group W_N whose elements $W(x, y, z)$, parametrized through row vectors $x, y \in R^N$ and $z \in R$ act on suitable functions of q as

$$W(x, y, z) f(q) = \exp\left[i\left(\sum_m x_m Q_m + \sum_n y_n P_n + z1\right)\right] f(q) = \\ = \exp[i(xq^T + xy^T/2 + z)] f(q + y)\,. \tag{2.3}$$

We shall reserve Roman lower- and upper-case letters for N-dimensional row vectors and matrices, and corresponding Greek letters for $2N$-dimensional ones. Let thus $\xi = (x, y)$. The group composition rule for W_N can be found from (2.3) or directly from (2.2) to be

$$W(\xi_1, z_1)\, W(\xi_2, z_2) = W(\xi_1 + \xi_2, z_1 + z_2 + \tfrac{1}{2}\xi_1 \Omega \xi_2^T)\,, \tag{2.4a}$$

where

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.4b}$$

1 being the $N \times N$ unit matrix.

Out of the universal enveloping algebra of $\mathscr{W}_N$ we can now build the second-

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order operators and their associated $2N \times 2N$ matrices

(2.5a) $$\mathsf{C}_{mn} = \mathsf{Q}_m \mathsf{Q}_n \leftrightarrow \Gamma(\mathsf{C}_{mn}) = -i \begin{pmatrix} 0 & 0 \\ S^{mn} & 0 \end{pmatrix},$$

(2.5b) $$\mathsf{B}_{mn} = \mathsf{P}_m \mathsf{P}_n \leftrightarrow \Gamma(\mathsf{B}_{mn}) = i \begin{pmatrix} 0 & S^{mn} \\ 0 & 0 \end{pmatrix},$$

(2.5c) $$\mathsf{A}_{mn} = \frac{1}{2}(\mathsf{Q}_m \mathsf{P}_n + \mathsf{P}_n \mathsf{Q}_m) \leftrightarrow \Gamma(\mathsf{A}_{mn}) = i \begin{pmatrix} E^{nm} & 0 \\ 0 & -E^{mn} \end{pmatrix},$$

where we have used the $N \times N$ matrices

(2.5d) $$E^{mn} = \| E^{mn}_{jk} \|, \qquad E^{mn}_{jk} = \delta_{mj}\delta_{nk}, \qquad S^{mn} = E^{mn} + E^{nm} = S^{nm},$$

which have, respectively, one and two nonzero elements. There are $N(2N+1)$ of these operators. They can be shown, moreover, to satisfy the commutation relation of the $2N$-dimensional symplectic real Lie algebra $\mathscr{S}p_{2N,R}$. Similarly, one can verify that the commutation relations of the operators and their associated matrices are identical. These matrices —and linear combinations thereof— satisfy $\Gamma\Omega + \Omega\Gamma^T = 0$. Under exponentiation, $\Sigma = \exp[i\alpha\Gamma]$ will satisfy $\Sigma\Omega\Sigma^T = \Omega$. This property defines the $2N \times 2N$ real symplectic group. We write Σ in four $N \times N$ blocks:

(2.6) $$\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = 1.$$

The exponentiation of the individual matrices in (2.5) is easy as all but $\Gamma(\mathsf{A}_{mn})$ are nilpotent. The exponentiation of linear combinations is relatively more complicated, but can be handled by using various subgroup decomposition. In particular, the A-operators have a sublagebra $\mathscr{S}C_N$ generated by

(2.7) $$\mathsf{M}_{mn} = \mathsf{A}_{mn} - \mathsf{A}_{nm},$$

for which the representation matrices exponentiate to orthogonal ones.

The adjoint action of $Sp_{2N,R}$ on the generators of $\mathscr{W}_N$ can be written in $2N$-vector form as

(2.8) $$\mathsf{S}_\Sigma \begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix} \mathsf{S}_\Sigma^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix},$$

the composition law holding accordingly as

(2.9) $$\mathsf{S}_{\Sigma_1} \mathsf{S}_{\Sigma_2} = \varphi \mathsf{S}_{\Sigma_1\Sigma_2},$$

and $S_1 = 1$, where φ is a possible sign factor (see below). From (2.8), in a manner entirely analogous to (([23]) sect. 9'1), one can find the action of this $Sp_{2N,R}$ group of operators on a suitable space of functions of $q \in R^N$ as an (in general) integral transform

$$(2.10) \qquad f(q) \xrightarrow{S_\Sigma} S_\Sigma f(q) = \int_{R^N} d^N q' \, f(q') \, C_\Sigma(q, q')$$

with kernel

$$(2.11) \qquad C_\Sigma(q, q') = \exp[-iN\pi/4]([2\pi]^N \det B)^{-\frac{1}{2}} \cdot \\ \cdot \exp[i(\tfrac{1}{2} q' B^{-1} A q'^T - q' B^{-1} q^T + \tfrac{1}{2} q D B^{-1} q^T)] \, .$$

This class of integral transforms has been termed *canonical*. The sign factor φ in (2.9) turns out to be given by sign[$\det(B_1 B_2)/\det(B_1)\det(B_2)$] so that the adjoint action (2.8) is faithful, but (2.10) yields a 2:1 ray representation of $Sp_{2N,R}$. When the group parameters are allowed to go complex, (2.11) still holds for a subsemi-group and φ is still only a sign. The analysis follows that of ref. (([21]) appendix B) for the $N = 1$ case, with $\det B$ in place of b. The case $\det B = 0$ will be analyzed next.

Since any $Sp_{2N,R}$ matrix with $\det A \neq 0$ can be decomposed into elements as

$$(2.12) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{T-1} \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix} ,$$

we can look separately into each of the subgroups in this product. The action of the first two can be found independently of (2.10)-(2.11) and yields

a) Gaussian multiplication

$$(2.13) \qquad S\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} f(q) = \exp\left[\frac{1}{2} iqCq^T\right] f(q) \, ,$$

b) linear transformation

$$(2.14) \qquad S\begin{pmatrix} A & 0 \\ 0 & A^{T-1} \end{pmatrix} f(q) = (\det A)^{-\frac{1}{2}} f(qA^{T-1}) \, ,$$

c) pure integral transformation $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$.

These particularize the transform (2.11). Their behaviour as $\det B \to 0$ can be analyzed noting that, from (2.7), B is a symmetric matrix in this case, which can be brought to diagonal form through an orthogonal matrix R.

When $B = \|b_m \delta_{mn}\|$, the kernel (2.11) is a product of N Gaussians of the form

$$(2\pi b_m)^{-\frac{1}{2}} \exp[i(q_m - q'_m)^2/2b_m]\,, \qquad m = 1, 2, \ldots, N\,,$$

for which the $b_m \to 0$ limit can be ascertained as in ([23] sect. 9'1) where the one-dimensional case is shown to become a Dirac $\delta(q_m - q'_m)$. It follows that when B is a matrix of rank r and R is such that $B = RB'R^T$ with B' diagonal $(b_1 b_2 \ldots b_r 0 \ldots 0)$, then the kernel (2.11) is

$$(2.15) \qquad [(2\pi)^r b_1 b_2 \ldots b_r]^{-\frac{1}{2}} \exp[-ir\pi/4] \cdot \\ \cdot \exp\left[\frac{1}{2} i \left[\sum_{k=1}^{r} i b_k^{-1}([q - q']R)_k^2\right]\right] \prod_{n=r+1}^{N} \delta([(q - q')R]_n)\,,$$

and the transform (2.10) effectively only involves r integrations.

The W_N and $Sp_{2N,R}$ groups can be now composed as

$$(2.16) \qquad I\{\Sigma, \xi, z\} = S_\Sigma W(\xi, z)\,.$$

As the generators of W_N transform among themselves under the adjoint action of $Sp_{2N,R}$, the composite operators (2.16) will be elements of the semi-direct product $W_N \wedge Sp_{2N,R}$, with W_N normal. We call this the Weyl-symplectic group $W_N Sp_{2N,R}$. The group composition reads

$$(2.17) \qquad I\{\Sigma_1, \xi_1, z_1\} I\{\Sigma_2, \xi_2, z_2\} = I\{\Sigma_1\Sigma_2, \xi_1\Sigma_2 + \xi_2, z_1 + z_2 + \tfrac{1}{2}\xi_1\Sigma_2\Omega\xi_2^T\}\,,$$

as can be found from (2.16), (2.8) and (2.4). The identity element is $I\{1, 0, 0\}$ and $I\{\Sigma, \xi, z\}^{-1} = I\{\Sigma^{-1}, -\xi\Sigma^{-1}, -z\}$. (We note that $\xi\Omega\xi^T = 0$.)

A particularly important subgroup is that of purely geometric transformations

$$(2.18) \qquad I\left\{\begin{pmatrix} A & 0 \\ C & A^{T-1}\end{pmatrix}, x, y, z\right\} f(q) = \\ = (\det A)^{-\frac{1}{2}} \exp\left[i\left(\tfrac{1}{2} qCA^{-1}q^T + qA^{T-1}x^T + \tfrac{1}{2} yx^T + z\right)\right] f(qA^{T-1} + y)\,,$$

where recall $\xi = (x, y)$. Here the value of $If(q)$ depends only on the value of $f(q')$ at one point, and (2.18) is the subset of all transformations with this property.

Consider now the time evolution of a function $f(q)$ as dictated by the N-dimensional free-particle Schrödinger equation

$$(2.19a) \qquad \left(\tfrac{1}{2}\sum_n P_n P_n\right) f(q, t) = i\partial_t f(q, t)\,.$$

This is a canonical transform of the initial condition $f_0(q)$ given by

$$(2.19b)\qquad f(q,t) = \exp[t\partial_{t'}]f(q,t')|_{t'=0} = \\ = \exp\left[-\frac{1}{2}it\sum_n P_n P_n\right] f_0(q) = I\left\{\begin{pmatrix}1 & t1\\ 0 & 1\end{pmatrix}, 0, 0, 0\right\} f_0(q),$$

where the last transform is the evolution operator of the system. Now, if the initial condition $f_0(q)$ is subject to a $W_N Sp_{2N,R}$ transformation and becomes a new $g_0(q)$, $f(q,t)$ will change correspondingly as

$$(2.20a)\qquad g(q,t) = I\left\{\begin{pmatrix}1 & t1\\ 0 & 1\end{pmatrix}, 0, 0, 0\right\} g_0(q) = \\ = \left[I\left\{\begin{pmatrix}1 & t1\\ 0 & 1\end{pmatrix}, 0, 0, 0\right\} I\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}, x, y, z\right\}\right] f_0(q) = \\ = I\left\{\begin{pmatrix}A+tC & B+tD\\ C & D\end{pmatrix}, x, y, z\right\} f_0(q).$$

We now write the last group element as the product of one geometric transformation times an evolution operator for the free particle in a transformed time t', here a $N\times N$ matrix T', as

$$(2.20b)\qquad \left[I\left\{\begin{pmatrix}A+tC & 0\\ C & (A+tC)^{T-1}\end{pmatrix}, x, y - xT', z\right\} I\left\{\begin{pmatrix}1 & T'\\ 0 & 1\end{pmatrix}, 0, 0, 0\right\}\right] f_0(q),$$

where

$$(2.20c)\qquad T' = (A+tC)^{-1}(B+tD).$$

The most obvious difference with the one-dimensional case is that here in general T' will not be a scalar multiple of the unit matrix. Let us consider then the question: when is $T' = t'1$ for all t? Clearly, this happens when the four submatrices of Σ are multiples of the same $N\times N$ matrix. Conditions (2.6) then further require that this matrix be orthogonal—call it R—so that $A = aR$, $B = bR$, $C = cR$ and $D = dR$ with $ad - bc = 1$. The set of all $W_N Sp_{2N,R}$ elements of the form

$$(2.21a)\qquad I\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\otimes R, \xi, z\right\},\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in Sp_{2,R},\quad R\in O_N,$$

constitute the Schrödinger group

$$(2.21b)\qquad Sch_N = W_N \wedge [Sp_{2,R}\otimes O_N] \subset W_N Sp_{2N,R},$$

which is regarded [28,31] as the similarity group for the free-particle Schrödinger equation, as well as that of the associated harmonic and repulsive oscillator and the free-fall (linear potential) equations [31]. When Σ belongs to this subgroup, eqs. (2.20) can be further manipulated to yield the result

(2.22a) $$g(\boldsymbol{q}, t) = I\left\{\begin{pmatrix}(a+tc)R & 0\\ cR & (a+tc)^{-1}R\end{pmatrix}\boldsymbol{x}, \boldsymbol{y}+t'\boldsymbol{x}, z\right\}f(\boldsymbol{q}, t') =$$

$$= (a+tc)^{-N/2}\exp\left[i\left(\frac{1}{2}ca^{-1}\boldsymbol{q}^2+\left\{a^{-1}\boldsymbol{q}+\frac{1}{2}[\boldsymbol{y}-t'\boldsymbol{x}]\right\}\boldsymbol{x}^T+z\right)\right]f(\boldsymbol{q}', t'),$$

(2.22b) $$t' = (dt+b)/(a+ct),$$

(2.22c) $$\boldsymbol{q}' = (a+ct)^{-1}(\boldsymbol{q}R+[dt+b]\boldsymbol{x})+\boldsymbol{y}.$$

In this form, $f(\boldsymbol{q}, t)$ is shown to have undergone a geometric transformation in the $(\boldsymbol{q}, t)$-space, to a function $g(\boldsymbol{q}, t)$ which is a multiplier function times $f(\boldsymbol{q}', t')$, the original function in the transformed variables $(\boldsymbol{q}', t')$. Sch_N is generated by the Heisenberg-Weyl operators (2.1), the angular-momentum generators (2.7) and by the « isotropic » $SO_{2,1}$ generators of Gaussian multiplication, dilatation and time evolution:

(2.23a) $$\mathbf{H}_2 = \tfrac{1}{2}\sum_m \mathbf{Q}_m\mathbf{Q}_m = \tfrac{1}{2}\sum_m \mathbf{C}_{mm},$$

(2.23b) $$\mathbf{H}_0 = \tfrac{1}{2}\sum_m[\mathbf{Q}_m\mathbf{P}_m+\mathbf{P}_m\mathbf{Q}_m] = \tfrac{1}{2}\sum_m \mathbf{A}_{mm},$$

(2.23c) $$\mathbf{H}_{-2} = \tfrac{1}{2}\sum_m \mathbf{P}_m\mathbf{P}_m = \tfrac{1}{2}\sum_m \mathbf{B}_{mm}.$$

The time-t Schrödinger algebra is obtained from this through application of (2.8) on the time-0 Schrödinger algebra, $\mathbf{S}_\Sigma$ given by the time evolution operator in (2.19b):

(2.24a) $$\mathbf{Q}_m^{(t)} = \mathbf{Q}_m - t\mathbf{P}_m, \qquad \mathbf{P}_n^{(t)} = \mathbf{P}_n, \qquad \mathbf{M}_{mn}^{(t)} = \mathbf{M}_{mn}.$$

The three « dynamical » group generators (2.23) are, for time t,

(2.24b) $$\mathbf{H}_{-2}^{(t)} = \mathbf{H}_{-2} - t\mathbf{H}_0 - it^2\partial_t,$$

(2.24c) $$\mathbf{H}_0^{(t)} = \mathbf{H}_0 + 2it\partial_t,$$

(2.24d) $$\mathbf{H}_2^{(t)} = \mathbf{H}_2 = -i\partial_t,$$

where the second-order operator $\mathbf{H}_2$ on the initial conditions has been replaced by $-i\partial_t$ on the space of solutions of (2.19a). The generators (2.24) of Sch_N are those of Bluman and Cole [2] in the notation used by MILLER ([3] sect. 2.2, [37]).

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[37] E. G. KALNINS and W. MILLER jr.: *Journ. Math. Phys.*, **15**, 1728 (1974).

For $N = 1$ this is indeed the full symmetry algebra as $W_1 Sp_{2,R} \simeq Sch_1$. For $N > 1$ however, $W_N Sp_{2N,R}$ contains extra generators which remain of second order on the space of solutions of (2.19a) at time t. These can be written out explicitly through (2.5) and (2.24a). The operator $-i\partial_t$ is present as in (2.24b)-(2.24d), but not elsewhere. The assignment (2.24d), of course, holds only for the free-particle Schrödinger-equation case written out here. For the harmonic and repulsive oscillators, the free-fall system or, in fact, any system whose time evolution operator lies in the $\mathscr{W}_N \mathscr{S}p_{2N,R}$ algebra, this continues to be the full dynamical algebra, containing properly the similarity algebra of first-order differential operators ([31]). In fact, the similarity group is even smaller than Sch_N in the cases where the Hamiltonian rotational invariance is broken, as for the anisotropic oscillator ([31,38]), where $\mathscr{W}_N \mathscr{S}p_{2N,R}$ continues to be the dinamical algebra of the system. In every case the generators of $W_N Sp_{2N,R}$, which are not in the similarity group, generate an integral transform action on the time-t solutions of the corresponding equation.

3. – Conformal symmetry for parabolic equations.

A function $f(q, t)$ which is a solution to a linear evolution equation

$$Hf(q, t) = \partial_t f(q, t) \tag{3.1a}$$

with initial conditions

$$f(q, t)|_{t=0} = f_0(q) \tag{3.1b}$$

can be found from the latter through a formal evolution operator

$$f(q, t) = \exp(tH) f_0(q) = \int_{-\infty}^{\infty} dq' f_0(q') G(q, q'; t) . \tag{3.2}$$

For simplicity we shall work here with only one space dimension. The general case, although not identical, follows basically the same line of argument. The system characterized by H is clearly independent of the initial condition $f_0(q)$. As long as this gives rise to a unique $f(q, t)$, the latter can be traced back to its initial condition $f_0(q)$.

If we effect a transformation T on $f_0(q)$, a transformation $T^{(t)}$ will be induced at the time-t solution, given by

$$T^{(t)} = \exp[tH] T \exp[-tH] . \tag{3.3}$$

([38]) C. P. Boyer and K. B. Wolf: *Journ. Math. Phys.*, **16**, 2215 (1975).

We would like to particularize our considerations to geometric transformations generated by operators of the kind

$$\mathbf{P}^{(1)} = P_1(q)\,\partial_q + \sigma P_0(q)\,, \qquad \sigma \in C\,. \tag{3.4}$$

These form an infinite-dimensional algebra, as can be easily verified seeing that linear combinations and commutators of operators of this class lie again in the same class. Another operator relevant to the transformation of the initial conditions is $\mathbf{H}$ itself. The commutator of $\mathbf{H}$ and (3.4), when $\mathbf{H}$ contains second-derivative terms, will lie in general outside the class of geometric transformations. The one-parameter transformation group generated by (3.4) can be found through the standard techniques of Lie theory and lead to a multiplier action

$$f_\alpha(q) = \mathbf{T}_\alpha f_0(q) = \exp\left[\alpha\{P_1(q)\,\partial_q + \sigma P_0(q)\}\right] f_0(q) = [\mu(q,\alpha)]^\sigma f_0(\varphi_\alpha(q)), \tag{3.5a}$$

$$\varphi_\alpha(q) = F^{-1}(F(q)-\alpha), \qquad \mu(q,\alpha) = G(q)/G(\varphi_\alpha(q)) = 1/\mu(\varphi_\alpha(q), -\alpha), \tag{3.5b}$$

$$F(q) = -\int \frac{dq}{P_1(q)}, \qquad G(q) = \exp\left[\int dq\, \frac{P_0(q)}{P_1(q)}\right], \tag{3.5c}$$

where, in order that $\varphi_\alpha(q)$ be one-to-one, $F(q)$ must be strictly monotonic.

A Lie algebra of operators generated by (3.4) on the space of initial conditions will generate a corresponding Lie algebra on time-t evolved solutions of (3.1a)

$$\mathbf{P}^{(1,t)} = \exp[t\mathbf{H}]\,[P_1(q)\,\partial_q + \sigma P_0(q)]\exp[-t\mathbf{H}] \tag{3.6}$$

and the exponentiated operators will map solutions of this equation into solutions. As we assume $f(q,t)$ can be traced back to its $f_0(q)$, then

$$\begin{aligned} f_\alpha(q,t) &= \exp[\alpha\mathbf{P}^{(1,t)}]f(q,t) = \\ &= \exp[t\mathbf{H}]\exp[\alpha\mathbf{P}^{(1)}]\int_D d\nu(q')\,f(q',t)\,G(q,q';t)^* = \\ &= \exp[t\mathbf{H}]\int_D d\nu(q')\,f(q',t)\,\mu(q,\alpha)^\sigma G(\varphi_\alpha(q),q';t)^* = \\ &= \int_{-\infty}^{\infty} dq'' \int_D d\nu(q')\,f(q',t)\,\mu(q'',\alpha)^\sigma G(\varphi_\alpha(q),q';t)^* G(q,q'';t)\,. \end{aligned} \tag{3.7}$$

In the preceding formulae we have written the inverse time evolution as an integral with a measure $d\nu(q')$ over a region D in the complex plane, with the complex conjugate of the direct time evolution kernel. When the evolution operator is unitary on $\mathscr{L}^2(R)$, $d\nu(q') = dq'$ and D is the real line itself. When the process is diffusive as for the heat or « radial » heat equation, an integration over the q' complex plane C is necessary with an appropriate measure given below ((21,22), (23) section 9'2). The expression of $f_\alpha(q, t)$ in terms of $f(q, t)$ is thus that of a double-integral transform

$$f_\alpha(q, t) = \int_{-\infty}^{\infty} dq'' \int_D d\nu(q') f(q', t) K^{(2)}(q, q', q''; t; \alpha), \tag{3.8a}$$

$$K^{(2)}(q, q', q''; t; \alpha) = \mu(q'', \alpha)^\sigma G(\varphi_\alpha(q''), q'; t)^* G(q, q''; t). \tag{3.8b}$$

The order of integration in (3.8a) can be exchanged only when both integrals are finite and well defined. When this is possible, we can write

$$f_\alpha(q, t) = \int_D d\nu(q') f(q', t) K^{(1)}(q, q'; t; \alpha), \tag{3.9a}$$

$$K^{(1)}(q, q'; t; \alpha) = \int_{-\infty}^{\infty} dq'' K^{(2)}(q, q', q''; t; \alpha). \tag{3.9b}$$

We should point out clearly that the single-integral form (3.9) is not always possible. When H is a diffusion operator, (3.9) would imply that $K^{(1)}(q, q'; t; \alpha)$ itself is a solution of the evolution equation (3.1) in q and t corresponding to a Dirac δ-distribution. As this is not regressible in time at all, it cannot be the subject of the first step in (3.8). These steps can be applied, however, when $f(q, t) = G(q, c; t)$, corresponding to $f_0(q) = \delta(q - c)$. In this case $f_\alpha(q, 0)$ is a Dirac δ centered at $c' = \varphi_{-\alpha}(c)$, and with a modulation factor which can be obtained as

$$\begin{aligned} f_\alpha(q, 0) &= \mu(q, \alpha)^\sigma \delta(\varphi_\alpha(q) - c) = \mu(q, \alpha)^\sigma \delta(q - \varphi_{-\alpha}(c))/\varphi'_\alpha(\varphi_{-\alpha}(c)) = \\ &= \mu(\varphi_{-\alpha}(c), \alpha)^\sigma |P_1(\varphi_{-\alpha}(c))| \delta(q - \varphi_{-\alpha}(c))/|P_1(c)|, \end{aligned} \tag{3.10}$$

where we have assumed that $q' = \varphi_\alpha(q)$ has only one solution and used the identities

$$\delta(g(x) - x') = |g'(g^{-1}(x'))|^{-1} \delta(x - g^{-1}(x')), \tag{3.11a}$$

$$\varphi'_\alpha(y) = F'(y)/F'\big(F^{-1}(F(x) - \alpha)\big) = P_1(\varphi_\alpha(y))/P_1(y). \tag{3.11b}$$

It follows that under geometric transformations of the initial conditions, $G(q, c; t)$, the Green's function « centered » on c is transformed into

$$G_\alpha(q, c; t) = \mu(\varphi_{-\alpha}(c), \alpha)^\sigma |P_1(\varphi_{-\alpha}(c))| G(q, \varphi_{-\alpha}(c); t)/|P_1(c)|, \tag{3.12}$$

which is only a recentered and rescaled function.

We would like to give a more detailed account of the generating operators in (3.6) for the case when H is a Hamiltonian or diffusion operator which belongs to the class of $WSp_{2,R}$-similar systems. In this case, the transformations one can implement on $P^{(1)}$ are given by the Weyl-symplectic transformations of the last section:

$$(3.13)\qquad I\{\Sigma, \xi, z\}\begin{pmatrix} Q \\ P \end{pmatrix} I\{\Sigma, \xi, z\}^{-1} = \begin{pmatrix} dQ - bP + y1 \\ - cQ + aP - x1 \end{pmatrix}.$$

The operators which generate symmetry transformations at time t are then

$$(3.14)\qquad P^{(1,t)} = P_1(dq + ib\partial_q + y)(- icq + a\partial_q - ix) + \sigma P_0(dq + ib\partial_q + y)$$

with a, b, c, d, x and y given functions of t. Time derivatives will appear in (3.14) whenever H is reconstituted by q and ∂_q.

In order to be specific, let us consider the three-dimensional Lie algebra of conformal transformations of the real line generated by

$$(3.15a)\qquad C_{-1} = \partial_q , \qquad C_0 = q\partial_q , \qquad C_{+1} = q^2\partial_q ,$$

with commutation relations

$$(3.15b)\qquad [C_0, C_{\pm 1}] = \pm C_{\pm 1} , \qquad [C_{-1}, C_{+1}] = 2C_0 .$$

The generators (3.15) will correspond, respectively, to translations, dilatations and pure conformal transformations obtained from (3.5) and locally generate $Sp_{2,R} \simeq SO_{2,1}$ through the subgroups given in (2.12). When the time evolution of the system is governed by the heat equation (eq. (3.13) with $a = 1 = d$, $c = 0$, $b = - 2it$ and $x = y = z = 0$), then (3.14), as applied to (3.15a), reads

$$(3.16a)\qquad C^{(t)}_{-1} = \partial_q = C_{-1} ,$$

$$(3.16b)\qquad C^{(t)}_0 = (q + 2t\partial_q)\partial_q = C_0 + 2t\partial_q^2 \overset{H}{\simeq} q\partial_q + 2t\partial_t ,$$

$$(3.16c)\qquad C^{(t)}_{+1} = (q + 2t\partial_q)^2\partial_q = C_{+1} + 2t\partial_q + 4tq\partial_q^2 + 4t^2\partial_q^3 \overset{H}{\simeq}$$

$$\overset{H}{\simeq} (2t + q^2)\partial_q + 4qt\partial_t + 4t^2\partial_q\partial_t ,$$

the symbol $\overset{H}{\simeq}$ meaning that the two operators are equivalent when applied to the solution space of the heat equation. One may check easily that (1.5) holds for (3.16). The operator (3.16a) generates translations $q \to q + \alpha$, $t \to t$, (3.16b) generates dilatations $q \to \exp[\beta]q$, $t \to \exp[2\beta]t$, while the action of (3.16c), which for $t = 0$ is the conformal transformation $q \to q/(1 + \gamma q)$, for general t requires a double integral transform of the type (3.8). From (([21]), ([23]) sect. 9·2) we set

$$\mathrm{d}\nu(q) = (2\pi t)^{-\frac{1}{2}} \exp[-(\operatorname{Im} q)^2/2t]\,\mathrm{d}\operatorname{Re} q\,\mathrm{d}\operatorname{Im} q\,. \tag{3.17}$$

Correspondingly, we find the kernel

$$K^{(2)}(q, q', q''; t; \gamma) = (4\pi t)^{-1} \exp\left[-\left\{\left(\frac{q''}{1+\gamma q''} - q'\right)^{*2} - (q - q'')^2\right\}\Big/ 4t\right], \tag{3.18}$$

and D to be the complex plane. As expected, the first integral over q' can be performed only if $f(q', t)$ is an entire analytic function of growth $(2, 1/4t)$. The double-integral process (3.8)-(3.17)-(3.18) can be explicitly verified for Gaussian functions of appropriate growth. Finally, it is interesting in itself to note that the procedure followed here will yield a contininum of conformal $\mathscr{S}C_{2,1}$ algebras built out of the enveloping algebra of the original $\mathscr{S}O_{2,1}$ algebra (3.15a): the commutation relations (3.15b) continue to hold for the operators (3.16), whether one or both operators in the commutator assumes either of the forms separated by the $\overset{H}{\simeq}$ sign. As we can again easily verify, the algebra generators (3.16) constitute a particular case of a deformation induced by (3.13) on $\mathscr{S}C_{2,1}$. The general case is

$$\mathbf{D}_{-1} = a\mathbf{C}_{-1}\,, \qquad a \neq 0\,, \tag{3.19a}$$

$$\mathbf{D}_0 = \mathbf{C}_0 + ay\mathbf{C}_{-1} + \beta a\mathbf{C}_{-1}^2\,, \qquad y, \beta \in R\,, \tag{3.19b}$$

$$\mathbf{D}_{+1} = a^{-1}\mathbf{C}_{+1} + 2y\mathbf{C}_0 + (\beta + ay^2)\mathbf{C}_{-1} + 2\beta\mathbf{C}_0\mathbf{C}_{-1} + 2a\beta y\mathbf{C}_{-1}^2 + a\beta^2\mathbf{C}_{-1}^3\,. \tag{3.19c}$$

The case (3.16) corresponds to $a = 1$, $\beta = 2t$, $y = 0$.

These will be up-to-third-order differential operators in the underlying space co-ordinate (q in this case, but the circle co-ordinate, for instance, in Bargmann's realization ([29]) of $SO_{2,1}$). If we wish to reduce the action of this algebra to the solution space of a parabolic differential equation with a quadratic Hamiltonian operator $\mathbf{H}$, the $\mathbf{H}$ operators present in the extended algebra (3.19) will be replaced by ∂_t, yielding up-to-second-order differential operators in a two-dimensional space (q, t). All of these conformal algebras for the differential equation, will be symmetry algebras of operators of second order in the derivatives.

4. – Point and Schrödinger symmetries for the wave equation.

We now turn to the problem of finding symmetry operators for a class of equations of the form

$$\partial_t^2 f(q, t) + \varepsilon\partial_t f(q, t) + \mathbf{D}f(q, t) = 0\,, \tag{4.1}$$

which include the wave equation (for $\mathbf{D} = -\partial_q^2$, $\varepsilon = 0$), the damped-wave equation ($\varepsilon > 0$) and the Klein-Gordon equation ($\mathbf{D} = -\partial_q^2 + \mu^2$, $\varepsilon = 0$). Although the method of solution may apply also for elliptic equations, as the Helmholtz equation ($\mathbf{D} = +\partial_q^2 + k^2$, $\varepsilon = 0$), we reserve next section for an alternative treatment of the latter. In fact, we shall present the results explicitly only for the wave equation in one space dimension. Other cases and dimensions lead to less transparent formulae.

For the wave equation, (4.1) can be written as

$$\mathbf{W}\begin{pmatrix} f \\ f_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_q^2 & 0 \end{pmatrix}\begin{pmatrix} f \\ f_t \end{pmatrix} = \partial_t \begin{pmatrix} f \\ f_t \end{pmatrix}, \tag{4.2}$$

where the first row defines $f_t(q, t) = \partial f(q, t)/\partial t$ and the second reproduces (4.1). Now, whereas the similarity algebra of the Klein-Gordon equation is the $\mathscr{I}\mathscr{O}_{1,1}$ algebra of space and time translations (generated by ∂_q and ∂_t) and Lorentz rotations (generated by $t\partial_q + q\partial_t$) and the same holds for the damped cases when we replace ∂_t by $\partial_t + \varepsilon/2$, the wave equation has a larger symmetry group. To display this we can pass to the characteristic co-ordinates $\xi = q + t$, $\eta = q - t$, so that the equation now reads $\partial^2 f/\partial\xi\partial\eta = 0$. Any point transformation generated by (3.4) with $P_0 =$ constant in ξ or η will leave this equation invariant. If

$$\mathbf{X} = A(\xi)\partial_\xi + B(\eta)\partial_\eta + c, \tag{4.3}$$

its 2×2 matrix representative will be

$$\mathbf{X} = \begin{pmatrix} (A + B)\partial_q + c & A - B \\ \partial_q(A - B)\partial_q & \partial_q(A + B) + c \end{pmatrix} \tag{4.4}$$

with $A = A(q + t)$ and $B = B(q - t)$ and c constant. We can check that $[\mathbf{W} - \partial_t 1, \mathbf{X}] = 0$.

Since infinite-dimensional algebras are not very informative, one restricts $A(\xi)$ to be 1, ξ or ξ^2, and similarly for $B(\eta)$, following (3.15). The c element trivially multiplies the solutions by a constant. The similarity group of this equation can thus said to be the conformal algebra

$$\mathscr{A}_1 \oplus \mathscr{SL}_{2,R,\xi} \oplus \mathscr{SL}_{2,R,\eta} \simeq \mathscr{A}_1 \oplus \mathscr{SO}_{2,2}\,.$$

This algebra contains time translations ($A = \frac{1}{2} = -B$), space translations ($A = \frac{1}{2} = B$), hyperbolic rotations ($A = \xi/2$, $B = -\eta/2$), dilatations ($A = \xi/2$, $B = \eta/2$) and two pure conformal transformations ($A = \xi^2/2$, $B = \pm\eta^2/2$)

all with $c = 0$. In addition, as with any linear equation, we have scaling ($A = 0 = B$, $c \neq 0$). We can now apply an approach similar to that used in sect. 3, namely, the time evolution operator will translate the transformation of the initial conditions to a transformation of the ensuing solution. Here, the time evolution operator is the operator matrix

$$\exp[tW] = \begin{pmatrix} \dot{G} & G \\ \ddot{G} & \dot{G} \end{pmatrix}, \tag{4.5a}$$

$$G = \partial_q^{-1}\sinh(t\partial_q)\,, \qquad \dot{G} = \cosh(t\partial_q)\,, \qquad \ddot{G} = \partial_q \sinh(t\partial_q)\,, \tag{4.5b}$$

where we note that only nonnegative powers of ∂_q are involved. In fact, on a suitable space of functions,

$$\cosh(t\partial_q)h(q) = \tfrac{1}{2}[h(q+t) + h(q-t)]\,, \tag{4.6a}$$

$$\partial_q^{-1}\sinh(t\partial_q)h(q) = \tfrac{1}{2}\int_{q+t}^{q-t} dq'\, h(q')\,. \tag{4.6b}$$

Now, the most general transformation of the initial conditions is not only the most general operator X in (4.4) evaluated at $t = 0$, but includes all 2×2 matrices with operator entries. The diagonal elements of this matrix operate separately on the initial function and its time derivative, while the off-diagonal elements effect linear combinations between them. Again, in order to curtail the excessive freedom, we can resort to the one-dimensional $\mathscr{W}_1 \mathscr{S}p_{2,R} \simeq \mathscr{S}ch_1$ algebra of sect. 2, spanned by P^2, $\frac{1}{2}(PQ + QP)$, Q^2, P, Q and 1. The 2×2 matrices can be built considering the direct sum composition, whereby independent Schrödinger operators are placed in the diagonal entries and zeroes of the off-diagonal ones. A subalgebra of this is obtained when the two operators are equal. We shall call it $\mathscr{S}ch_{1,D}$. A second possibility is that of the tensor sum of the Schrödinger algebra with a $\mathscr{G}\mathscr{L}_{2,R}$ algebra of 2×2 matrices with constant coefficients. The latter isolates the linear combinations between the two initial conditions. Again, $\mathscr{S}ch_{1,D}$ is contained as a subalgebra when the $\mathscr{G}\mathscr{L}_{2,R}$ element is zero. In concentrating on the $\mathscr{S}ch_{1,D}$ algebra of diagonal matrix operators, we note that, out of the six generators of this algebra, $\frac{1}{2}(QP + PQ)$, P and 1 belong to the $\mathscr{S}C_{2,2}$ similarity algebra (4.4) and generate dilatations, space translations and scaling, respectively. The P^2 operator generates integral transformations of the diffusive or free-particle Schrödinger type (eq. (2.13b) with $r = 1 = N$). Since P^2 commutes with the time evolution operator W in (4.3), the action of this operator is the same for all time t. We are thus left to explore the action generated by Q and Q^2. On the $t = 0$ conditions this is just multiplication by an exponential or Gaussian factor. In order to study the time evolution of their action and the corresponding integration to the group, let us consider the time evolution of a function $F(\mathbf{Q})$, *i.e.*

$$(4.7)\qquad \begin{pmatrix} F(\mathbf{Q})^{(t)} & 0 \\ 0 & F(\mathbf{Q})^{(t)} \end{pmatrix} = \exp[t\mathbf{W}] \begin{pmatrix} F(\mathbf{Q}) & 0 \\ 0 & F(\mathbf{Q}) \end{pmatrix} \exp[-t\mathbf{W}] =$$

$$= \begin{pmatrix} \dot{\mathbf{G}}F(\mathbf{Q})\,\dot{\mathbf{G}} - \mathbf{G}F(\mathbf{G})\,\ddot{\mathbf{G}} & \mathbf{G}F(\mathbf{Q})\,\dot{\mathbf{G}} - \dot{\mathbf{G}}F(\mathbf{Q})\,\mathbf{G} \\ \ddot{\mathbf{G}}F(\mathbf{Q})\,\dot{\mathbf{G}} - \dot{\mathbf{G}}F(\mathbf{Q})\,\ddot{\mathbf{G}} & \dot{\mathbf{G}}F(\mathbf{Q})\,\dot{\mathbf{G}} - \ddot{\mathbf{G}}F(\mathbf{Q})\,\mathbf{G} \end{pmatrix}.$$

When we replace $\mathbf{G}$ and its derivatives by (4.5*b*) and use their action (4.6), we obtain, for the first row,

$$(4.8)\qquad F(\mathbf{Q})^{(t)}f(q,t) =$$

$$= [\cosh(t\partial_q)F(q)\cosh(t\partial_q) - \partial_q^{-1}\sinh(t\partial_q)F(q)\,\partial_q\sinh(t\partial_q)]f(q,t) +$$

$$+ [\partial_q^{-1}\sinh(t\partial_q)F(q)\cosh(t\partial_q) - \cosh(t\partial_q)F(q)\,\partial_q^{-1}\sinh(t\partial_q)]f_t(q,t) =$$

$$= \tfrac{1}{4}\{F(q+t)f(q+2t,t) + [F(q+t)+F(q-t)]f(q,t) + F(q-t)f(q-2t,t)\} +$$

$$+ \tfrac{1}{2}\int\limits_{q-t}^{q+t}\mathrm{d}q'\,F(q')[f(q'+t,t) + f(q'-t,t)] -$$

$$- \tfrac{1}{2}\Big\{F(q+t)\int\limits_{q}^{q+2t}\mathrm{d}q'\,f_t(q',t) + F(q-t)\int\limits_{q-2t}^{q}\mathrm{d}q'\,f_t(q',t)\Big\} - \int\limits_{q-t}^{q+t}\mathrm{d}q'\,F(q')\int\limits_{q'-t}^{q'+t}\mathrm{d}q''\,f_t(q'',t)\,.$$

The second row only involves differentiation of (4.8) with respect to t.

When $F(q) = q$ and q^2, (4.8) represents the action of the algebra operators which can be thought of as infinite-order differential operators. Together with the differential operators $\mathbf{P}$, $\mathbf{P}^2$, $q\partial_q + t\partial_t$ and $\mathbf{1}$, they constitute a Schrödinger symmetry algebra for time-t wave equation solutions. Although $\mathbf{Q}^{(t)}$ and $\mathbf{Q}^{(t)2}$ are thus integral operators, their exponentiation to the group is well defined by (4.8) itself for $F(q) = \exp[i\alpha q]$ and $\exp[i\beta q^2]$, respectively. These transformations, together with Schrödinger free-particle diffusion, translation, dilatation and scaling constitute the symmetry group for time t. Note that the first two transformations are time independent, whereas the other are not. Moreover, time translation is not an element of this Schrödinger $Sch_{N,D}$-group. If $\mathbf{W}$ were included in the algebra, we would immediately generate an infinite-dimensional one.

5. – Deformation of the symmetry algebra of the Helmholtz equation.

In this section we examine in some detail the deformation of the similarity algebra of the two-dimensional Helmholtz equation

$$(5.1)\qquad \mathbf{H}_k f(q) = (-\mathbf{P}_1^2 - \mathbf{P}_2^2 + k^2)f(q_1, q_2) = 0\,.$$

In this context, the deformation of an algebra is the following: out of the universal enveloping algebra of a given Lie algebra, we search for finite-dimensional subalgebras. This was done for $\mathscr{S}\mathscr{C}_{2,1}$ in (3.19) and yielded another $\mathscr{S}\mathscr{C}_{2,1}$ algebra. Here, the known two-dimensional Euclidean similarity algebra $\mathscr{I}\mathscr{C}_2$ for (5.1) will be subject to a similar process: $\mathscr{I}\mathscr{O}_2 \Rightarrow \mathscr{S}\mathscr{C}_{2,1}$. Although this deformation is representation dependent [33,35], it turns out that the solution space of (5.1) exactly fits the necessary requirements. In principle, the contents of sect. 2 and 3 can be seen also as the deformation of $\mathscr{W}_N$, and so will the results of this section in terms of $\mathscr{W}_2$. The method of deriving results, however, seems to indicate that here $\mathscr{I}\mathscr{C}_2$ is the natural starting point.

It is well known ([3] Chapter 1) that the similarity algebra of the two-dimensional equation (5.1) is a vector space $\mathscr{E}_1$ spanned by the four operators

$$P_1,\ P_2,\ M_3 = Q_1 P_2 - Q_2 P_1,\ 1 \tag{5.2}$$

satisfying $[H_k, X] = 0$. (See eqs. (2.1).) This algebra is $\mathscr{A}_1 \oplus \mathscr{I}\mathscr{O}_2$, direct sum of the one-dimensional algebra generated by 1 and the Euclidean algebra $\mathscr{I}\mathscr{O}_2$ generated by the rest of (5.2). As the former acts trivially—the subgroup generated by it multiplies the solutions of (5.1) by a constant, as is always permissible in a linear equation—we shall work with the factor algebra and space $\mathscr{S}_1 = \mathscr{E}_1/\mathscr{A}_1 = \mathscr{I}\mathscr{C}_2$. The $\mathscr{I}\mathscr{O}_2$ operators can be exponentiated to a Euclidean symmetry group IO_2 of geometric transformations. This manifest similarity group is actually the full symmetry group, in contradistinction to the free-particle Schrödinger equation of sect. 2. The Helmholtz equation has thus a rather poor symmetry.

We can construct further operators commuting with the operator H_k in (5.1) through considering the universal enveloping algebra of (5.2). The elements of this which are at most quadratic in (5.2) form a ten-dimensional space which we call $\mathscr{E}_2$, with a vector basis given by (5.2) plus P_1^2, P_2^2, P_1P_2, M_3^2, $\{M_3, P_1\}$ and $\{M_3, P_2\}$, where $\{A, B\} = AB + BA$ is the anticommutator of A and B. On the space of solutions, the operator $P_1^2 + P_2^2$ is equivalent to $k^2 1$, which suggests that we work with the eight-dimensional space $\mathscr{S}_2 = \mathscr{E}_2/(H_k, 1)$. The adjoint action of IO_2 on $\mathscr{S}_2$ then divides it into five orbit families, whose representatives we choose as

$$S_1 = P_1^2 + aM_3\,, \tag{5.3a}$$

$$S_2 = P_1^2 + bP_1 + cP_2\,, \tag{5.3b}$$

$$S_3 = M_3^2 + dM_3 + eP_1\,, \tag{5.3c}$$

$$S_4 = \{M_3, P_1\} + fM_3 + gP_1 + hP_2\,, \tag{5.3d}$$

$$S_5 = M_3^2 + wP_1^2 + xM_3 + yP_1 + zP_2\,, \tag{5.3e}$$

where $a, b, \ldots, z$ are real constants [39]. If we let all coefficients of first-order terms (in $\mathscr{E}_1$) be zero, orbits (5.3a) and (5.3b) coalesce and we recover the results of Miller [3] pertaining to the operators which serve to find the separating co-ordinates of (5.1), namely P_1^2, M_3, $\{M_3, P_1\}$ and $M_3^2 + w^2 P_1^2$, corresponding to Cartesian, polar, parabolic and elliptic systems. It should be noted that none of the operators (5.3), when second- and first-order terms are present, leads to new separable co-ordinate systems. This seems reasonable, since we know independently ([40] p. 498-504) that the purely second-order ones yield the only four orthogonal separating systems. A supporting computation [39] shows that this is the case, including the known nonorthogonal separating systems associated with the Cartesian and polar orbits. This result should not be surprising as is known that the correspondence between second-order operators and separating co-ordinates is not one to one. (Another example can be found in [15] on the separating co-ordinates of the Klein-Gordon equation.)

The vector space $\mathscr{S}_2$ does not close into a finite-dimensional Lie algebra, yet a subspace of it does. It has been shown that out of the enveloping algebra of $\mathscr{I}\mathscr{O}_N$ [33,35] one can produce a set of operators which are at most quadratic in the generators and an ideal under Lie brackets with the generators of the semi-simple subalgebra. This method, sometimes called the Gell-Mann formula, consists in commuting the second-order Casimir operator of $\mathscr{I}\mathscr{C}_N$ with the N normal generators and adding multiples of the same. In the case of (5.2) for $N = 2$, we can define the operators

$$(5.4a)\qquad M_1^{(1,\tau)} = -\frac{i}{2k}[M_3^2, P_1] + \tau P_1 = \frac{1}{2k}\{M_3, P_2\} + \tau P_1 = \frac{1}{k}M_3P_2 + \left(\tau + \frac{i}{2k}\right)P_1,$$

$$(5.4b)\qquad M_2^{(1,\tau)} = -\frac{i}{2k}[M_3^2, P_2] + \tau P_2 = -\frac{1}{2k}\{M_3, P_1\} + \tau P_2 = -\frac{1}{k}M_3P_1 + \left(\tau + \frac{i}{2k}\right)P_2,$$

which are Hermitean, when τ is real, in $\mathscr{L}^2(R^2)$. Direct computation yields

$$(5.5a)\qquad [M_3, M_1^{(1,\tau)}] = iM_2^{(1,\tau)}, \quad [M_3, M_2^{(1,\tau)}] = -iM_1^{(1,\tau)},$$

$$(5.5b)\qquad [M_1^{(1,\tau)}, M_2^{(1,\tau)}] = -ik^{-2}M_3(P_1^2 + P_2^2).$$

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[39] F. Soto, *La ecuación de Helmholtz y el grupo tridimensional de Lorentz*, Tesis profesional, Facultad de Ciencias UNAM (1977).
[40] P. M. Morse and H. Feshbach: *Methods of Theoretical Physics* (New York, N. Y., 1953).

Acting on the solution space of the Helmholtz equation, as $P_1^2 + P_2^2$ is equivalent to $k^2 1$, (5.5) closes into a $\mathscr{S}\mathscr{C}_{2,1}$ algebra. The generators of this algebra are $M_3 \equiv \mathscr{S}_1$ and the two operators (5.4). These are among the orbit family (5.3d) in $\mathscr{S}_2$. The representation of this $\mathscr{S}\mathscr{O}_{2,1}$ algebra on the solution space of (5.1) can be obtained by noting that

$$(5.6a)\qquad M_3^2 - M_1^{(1,\tau)2} - M_2^{(1,\tau)2} = M_3(1 - k^{-2}[P_1^2 + P_2^2]) + \\ + (i[i/2k + \tau] - [i/2k + \tau]^2)(P_1^2 + P_2^2) \simeq -\tfrac{1}{4} - \tau^2 k^2 = \lambda(\lambda + 1)\,,$$

where

$$(5.6b)\qquad \lambda = -\tfrac{1}{2} + i\tau k\,.$$

For τ real, this belongs to the principal or continuous series denoted by BARGMANN (29) as $C^0_{\tau k}$.

This is not the only deformation of the Helmholtz similarity algebra. In (34) it is shown that, out of an $\mathscr{I}_2\mathscr{S}\mathscr{C}_N$ algebra composed of $\mathscr{S}\mathscr{C}_N$ in semi-direct product with an Abelian ideal of second-order symmetric unit trace components, one can produce a $\mathscr{S}\mathscr{L}_{N,R}$ algebra and a corresponding $SL_{N,R}$-group. In the case of the N-dimensional Helmholtz equation, this Abelian ideal is provided in a natural way by the operators $k^{-2}P_\mu P_\nu$ in $\mathscr{S}_2$, which on the space of solutions have unit trace. This deformation follows closely eqs. (5.4) and can be implemented for any N. For $N = 2$ we can do better: noting that $\mathscr{S}\mathscr{L}_{2,R} \simeq \mathscr{S}\mathscr{C}_{2,1}$, we construct a generalization of this process through asking for the Abelian ideal to be built out of arbitrary functions G_1, G_2 of P_1 and P_2, such that $[M_3, G_1] = cG_2$ and $[M_3, G_2] = -cG_1$. We define, with appropiate normalization $c = in$,

$$(5.7a)\qquad M_1^{(n,\tau)} = -\frac{i}{2n^2k^n}[M_3^2, G_1] + \tau G_1 = \frac{1}{nk^n}M_3G_2 + \left(\tau + \frac{i}{2k^n}\right)G_1\,,$$

$$(5.7b)\qquad M_2^{(n,\tau)} = -\frac{i}{2n^2k^n}[M_3^2, G_2] + \tau G_2 = -\frac{1}{nk^n}M_3G_1 + \left(\tau + \frac{i}{2k^n}\right)G_2\,,$$

$$(5.7c)\qquad M_3^{(n,\tau)} = \frac{1}{n}M_3\,.$$

The condition that (5.7) close into a Lie algebra on the Helmholtz solution space implies that $G_1^2 + G_2^2$ be a function of $P_1^2 + P_2^2$ only. This is satisfied by the operator polynomial functions

$$(5.8a)\qquad G_1 = (P_1^2 + P_2^2)^{n/2}T_n(P_1/[P_1^2 + P_2^2]^{\frac{1}{2}})\,,$$

$$(5.8b)\qquad G_2 = (P_1^2 + P_2^2)^{(n-1)/2}P_2U_{n-1}(P_1/[P_1^2 + P_2^2]^{\frac{1}{2}})\,,$$

where $T_n(x)$ and $U_m(x)$ are the Chebyshev polynomials of the first and second kind. One can show, as in (5.6), that (5.7)-(5.8) for real τ belong to the principal series $C^0_{\tau k^n}$ irreducible representation. For $n=1$, (5.7) reduce to (5.4); for $n=2$ it reproduces the general case in ref. [34], while the deformation family presented here appears to be new. We shall denote the Lie algebra (5.7)-(5.8) by $\mathscr{SO}_{2,1,n}$.

Before proceeding to exponentiate (5.7)-(5.8) to the group, we would like to point out that part of the conclusions of sect. 2 can be applied to the Helmholtz-equation case: the Lie algebra with vector basis $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_1^2, \mathbf{P}_1\mathbf{P}_2, \mathbf{P}_2^2, \mathbf{M}_3\}$ has the structure $(\mathscr{A}_2\ominus\mathscr{A}_3)\ominus\mathscr{SO}_2$ (where $\mathscr{A}_n$ is the n-dimensional Abelian algebra) and maps solutions of this equation into new solutions. The exponentiation of this algebra will yield a group of integral transforms with Gaussian kernels (2.11) mapping solutions into solutions. This group is generated by the commutant of $\mathscr{W}_2\mathscr{Sp}_{4,R}$ with $\mathbf{H}_k$ in (5.1). The $\mathscr{SO}_{2,1,n}$ operators in (5.6), on the other hand, are outside $\mathscr{W}_2\mathscr{Sp}_{4,R}$ and are of order $n+2$ in the generators of $\mathscr{W}_2$. Explicitly, for $n=1$,

$$\text{(5.9a)}\qquad \mathbf{M}_1^{(1,\tau)} = k^{-1}(\mathbf{Q}_1\mathbf{P}_2^2 - \mathbf{Q}_2\mathbf{P}_1\mathbf{P}_2) + (\tau + i/2k)\mathbf{P}_1 = k^{-1}(-q_1\partial^2_{q_2} + q_2\partial_{q_1q_2}) - i(\tau + i/2k)\partial_{q_1},$$

$$\text{(5.9b)}\qquad \mathbf{M}_2^{(1,\tau)} = k^{-1}(\mathbf{Q}_2\mathbf{P}_1^2 - \mathbf{Q}_1\mathbf{P}_1\mathbf{P}_2) + (\tau + i2k)\mathbf{P}_2 = k^{-1}(-q^2\partial^2_{q_1} + q_1\partial q_{q_1q_2}) - i(\tau + i/2k)\partial_{q_2}.$$

The exponentiation of (5.7)-(5.8) can be achieved, nevertheless, due to the circumstance that a subspace of solutions of the Helmholtz equation can be mapped through Fourier transformation onto the space of functions $\tilde{f}(p_1, p_2) = \delta(\mathbf{p}^2 - k^2)\phi(\theta)$ with support on the unit circle, with co-ordinate θ:

$$\text{(5.10a)}\qquad f(\mathbf{q}) = 2\iint d^2\mathbf{p}\exp[i\mathbf{p}\cdot\mathbf{q}]\tilde{f}(\mathbf{p}) = \int_{-\pi}^{\pi} d\theta\exp[ik[q_1\cos\theta + q_2\sin\theta]]\phi(\theta) = (\mathbf{I}\phi)(\mathbf{q}),$$

$$\text{(5.10b)}\qquad \phi(\theta)\delta(\mathbf{p}^2 - k^2) = (8\pi^2)^{-1}\iint d^2\mathbf{q}\exp[-ip[q_1\cos\theta + q_2\sin\theta]]f(\mathbf{q}),$$

where we have defined constants, so as to agree with [3]. The space of Schwartz distributions on the unit circle is then mapped through $\mathbf{I}$ onto the space of solutions of (5.1). Operators which transform the latter space will have a corresponding realization in the former. The $\mathscr{SO}_2$ generators in (5.2) become

$$\text{(5.11)}\qquad \tilde{\mathbf{P}}_1 = k\cos\theta, \quad \tilde{\mathbf{P}}_2 = k\sin\theta, \quad \tilde{\mathbf{M}}_3 = -i\partial_\theta,$$

while the $\mathscr{S}\mathscr{O}_{2,1,n}$ generators (5.7)-(5.8) become

$$\tilde{M}_1^{(n,\tau)} = -i[n^{-1}\sin n\theta\,\partial_\theta + (\tfrac{1}{2} + i\tau k^n)\cos n\theta]\,, \tag{5.12a}$$

$$\tilde{M}_2^{(n,\tau)} = -i[-n^{-1}\cos n\theta\,\partial_\theta + (\tfrac{1}{2} + i\tau k^n)\sin n\theta]\,, \tag{5.12b}$$

$$\tilde{M}_3^{(n,\tau)} = -in^{-1}\partial_\theta\,. \tag{5.12c}$$

These are self-adjoint in $\mathscr{L}^2(S_1)$. The $SO_{2,1,n}$ action of (5.12) on the circle is now easy to find through (3.5). In fact,

$$\tilde{R}^{(n,\tau)}(\alpha,\beta,\gamma)\phi(\theta) = [\exp[i\alpha\tilde{M}_3^{(n,\tau)}]\exp[i\beta\tilde{M}_1^{(n,\tau)}]\exp[i\gamma\tilde{M}_2^{(n,\tau)}]]\phi(\theta) = \\ = \mu^{(n,\tau)}(\theta + \alpha/n, \beta)\phi(\Theta_\beta^{(n)}[\theta + \alpha/n] + \gamma/n)\,, \tag{5.13a}$$

$$\Theta_\beta^{(n)}[\zeta] = 2n^{-1}\operatorname{arctg}(\exp[\beta]\operatorname{tg}[n\zeta/2])\,, \tag{5.13b}$$

$$\mu^{(n,\tau)}(\zeta,\beta) = [\sin(n\zeta)/\sin(n\Theta_\beta^{(n)}[\zeta])]^{-\frac{1}{2}-i\tau k^n}\,. \tag{5.13c}$$

The action (5.13) is untary on $\mathscr{L}^2(S_1)$ as the Jacobian of the transformation is exactly offset by the multiplier factor in (5.13c):

$$\mathrm{d}(\Theta_\beta^{(n)}[\theta + \alpha/n] + \gamma/n)/\mathrm{d}\theta = |\mu^{(n,\tau)}(\theta + \alpha/n, \beta)|^2\,. \tag{5.14}$$

In the Euler angle decomposition (5.13a), $\tilde{M}_3^{(n,\tau)}$ generates rotations of the circle and $M_3^{(n,\tau)}$ corresponding ones in the q-plane. It is thus the action of $\tilde{M}_1^{(n,\tau)}$ and $M_1^{(n,\tau)}$ which is of specific interest, since they deform the circle through (5.13b) and (5.14).

In finding integral kernels representing the action (5.13) in the Helmholtz solution space, we can use the exponential eigenfunctions of P_1 (and P_2) appearing in (5.11a) and used in ref. (3). This set, however useful it has been, is not completely appropriate since, as eq. (5.10b) attests, integration over the two-dimensional q-plane is more than required by harmonic analysis in terms of the generalized eigenfunctions of a single operator. Now, the spectrum of P_1 in the space of solutions of the Helmholtz equation is given by $k\cos\theta$, $\theta \in (-\pi, \pi]$, *i.e.* the interval $(-k, k)$ twice, $-k$ and k. In order to distinguish between the two degenerate eigenvalue functions, we can restrict θ to the half-circle $[0, \pi]$ and introduce a dichotomic index $\sigma = \pm 1$ for $0 \neq \theta \neq \pi$, which should be the eigenvalue of an operator commuting with P_1. This can be chosen to be (symbolically) $P_2/|P_2|$, *i.e.* an operator whose eigenvalue is $+1$ (-1) when $\phi(\theta)$ in (5.8) has support on the $[0, \pi]$ $([-\pi, 0])$ half-circle. The two values $\theta = 0$ and $\theta = \pi$ are exceptional as the spectrum of P_1 is simple and no dichotomic index is necessary. Since these constitute a set of measure zero, we may disregard them for the purpose at hand. Equation (5.8) can be then cast in the form

$$f(\boldsymbol{q}) = \sum_{\sigma=\pm1} \int_0^\pi d\theta\, \hat{f}_\sigma(\theta)\, \Phi_{\theta,\sigma}(\boldsymbol{q}), \tag{5.15}$$

where

$$\Phi_{\theta,\sigma}(\boldsymbol{q}) = \exp[ik(q_1\cos\theta + \sigma q_2 \sin\theta)], \qquad \theta \in [0, \pi], \tag{5.16a}$$

$$\hat{f}_{\pm1}(\pm\theta) = \phi(\theta), \qquad \theta \in \begin{cases} [0, \pi], \\ [-\pi, 0]. \end{cases} \tag{5.16b}$$

Similarly,

$$\partial f(\boldsymbol{q})/\partial q_2 = ik \sum_{\sigma=\pm1} \sigma \int_0^\pi \sin\theta\, d\theta\, \hat{f}_\sigma(\theta)\, \Phi_{\theta,\sigma}(\boldsymbol{q}). \tag{5.17}$$

The harmonic coefficients $\hat{f}_\sigma(\theta)$ can now be found in terms of (5.15) and (5.17) through a single integral over q_1, by using

$$\int_{-\infty}^{\infty} dq_1 \exp[ikq_1\cos\theta] \exp[-ikq_1\cos\theta'] = 2\pi\delta(\theta-\theta')/k\sin\theta, \tag{5.18}$$

as

$$\hat{f}_\sigma(\theta) = (4\pi)^{-1} \int_{-\infty}^{\infty} dq_1 [k\sin\theta f(\boldsymbol{q}) - i\sigma\, \partial f(\boldsymbol{q})/\partial q_2]\, \Phi_{\theta,\sigma}(\boldsymbol{q})^*. \tag{5.19}$$

We should note that (5.19) does not depend on q_2. This expression for $\hat{f}_\sigma(\theta)$ thus substitutes (5.10*b*) as the transform inverse to (5.10*a*)-(5.17). The action of $\exp[i\beta \mathbf{M}_1^{(n,\tau)}]$ on $f(\boldsymbol{q})$ can be now found through (5.13), (5.15) and (5.19) as an integral transform—in q_1—of the « initial conditions » at any point q_2' as

$$\exp[i\beta\mathbf{M}_1^{(n,\tau)}] f(\boldsymbol{q}) = \sum_{\sigma=\pm1} \int_0^\pi d\theta\, \mu^{(n,\tau)}(\theta,\beta) \hat{f}_\sigma(\Theta_\beta^{(n)}(\theta))\, \Phi_{\theta,\sigma}(\boldsymbol{q}) = \tag{5.20}$$

$$= \int_{-\infty}^{\infty} dq_1' f(\boldsymbol{q}')\, \partial K^{(n,\tau)}(\boldsymbol{q},\boldsymbol{q}';\beta)/\partial q_2' + \int_{-\infty}^{\infty} dq_1'\, \partial f(\boldsymbol{q}')/\partial q_2'\, K^{(n,\tau)}(\boldsymbol{q},\boldsymbol{q}';\beta),$$

where the integral kernel representing the group action is

$$K^{(n,\tau)}(\boldsymbol{q},\boldsymbol{q}';\beta) = -i(4\pi)^{-1} \sum_{\sigma=\pm1} \sigma \int_0^\pi d\theta\, \mu^{(n,\tau)}(\theta,\beta)\, \Phi_{\Theta_\beta^{(n)}(\theta),\sigma}(\boldsymbol{q}')^*\, \Phi_{\theta,\sigma}(\boldsymbol{q}) = \tag{5.21}$$

$$= (2\pi)^{-1} \int_0^\pi d\theta [\sin(n\theta)/\sin(n\Theta_\beta^{(n)}(\theta))]^{-\frac{1}{2}-i\tau\kappa^n} \cdot$$

$$\cdot \exp[ik(q_1\cos\theta - q_1'\cos\Theta_\beta^{(n)}(\theta))] \sin[k(q_2\sin\theta - q_2'\sin\Theta_\beta^{(n)}(\theta))].$$

Although we cannot provide a closed expression in terms of elementary functions for the kernel (5.21) in the general case, we shall proceed to show that the integral action (5.20) is quite transparent.

First consider $\beta = 0$. In this case,

$$K^{(n,\tau)}(\mathbf{q}, \mathbf{q}'; 0) = K_{\mathrm{H}}(q_1 - q_1', q_2 - q_2')$$

is the Helmholtz-equation Green's function which provides the solution $f(q_1, q_2)$ in terms of the boundary conditions and normal derivative at any line $q_2' = \text{constant}$. In fact, for $q_2 = q'$, $K_{\mathrm{H}}(q_1 - q_1', 0) = 0$, while the derivative can be integrated as

$$\partial K_{\mathrm{H}}(q_1 - q_1', q)/\partial q|_{q=0} = \pi^{-1} \sin(k[q_1 - q_1'])/(q_1 - q_1') . \tag{5.22}$$

This function acts in (5.22) as the reproducing kernel for the space of solutions of the Helmholtz equation: if we Fourier transform with respect to q_1 the convolution of $f(q_1, q_2)$ and (5.22), the result is the Fourier transform $\bar{f}(p_1, q_2)$, which has support on $p_1 \in [-k, k]$ times the Fourier transform of (5.22), which is a rectangle function with the same support.

For values of $\beta \neq 0$, the integral transform (5.20) is no longer a convolution for $q_2 = q_2'$, yet it expresses the transformed $f(q_1, q_2)$ in terms of the boundary conditions $f(q_1', q_2)$ and its normal derivative $\partial f(q_1', q_2)/\partial q_2$ integrated along q_1'. This is due to the fact that $\mathsf{M}_1^{(n,\tau)}$ is a differential operator of second order in q_1, while only of first order in q_2—all higher derivatives in q_2 correspond, due to (5.1), to higher derivatives in q_1 times a first-order derivative in q_2. The value of the transformed $f(q_1, q_2)$ depends thus only on the values of the original function and normal derivative along a line $q_2 = \text{constant}$. The action of any other one-parameter subgroup conjugate to (5.20) will clearly involve only integrations along lines in the $\mathbf{q}$-plane at an angle determined by the conjugating rotation group element. As expected, the kernel $K^{(n,\tau)}(\mathbf{q}, \mathbf{q}'; \beta)$ is a solution to the Helmholtz equation (5.1) both in the $\mathbf{q}$ and in the $\mathbf{q}'$ variables. The composition and associativity properties can be verified by making use of the reproducing property of (5.22) and the measure transformation (5.14). The latter also shows that

$$K^{(n,\tau)}(\mathbf{q}, \mathbf{q}'; -\beta) = K^{(n,\tau)}(\mathbf{q}', \mathbf{q}; \beta)^* . \tag{5.23}$$

Lastly, one can verify that the infinitesimal integral transform action (5.20) for $\beta \to 0$ is correctly given by the $(n+2)$-th order operator (5.7*a*)-(5.8). Indeed, using

$$\partial \Theta_\beta^{(n)}(\theta)/\partial \beta = n^{-1} \sin[n\Theta_\beta^{(n)}(\theta)] , \tag{5.34a}$$

$$\partial \mu^{(n,\tau)}(\theta, \beta)/\partial \beta = (\tfrac{1}{2} + i\tau k^n)\mu^{(n,\tau)}(\theta, \beta) \cos[n\Theta_\beta^{(n)}(\theta)] , \tag{5.24b}$$

one can apply $\partial/\partial\beta$ on (5.20). Differentiating the kernel (5.21) and evaluating at $\beta = 0$, one finds

$$(5.25) \qquad \partial K^{(n,\tau)}(\boldsymbol{q}, \boldsymbol{q}'; \beta)/\partial\beta|_{\beta=0} = \\ = i\left\{-\frac{i}{nk}(q_1'\partial/\partial q_2 - q_2'\partial/\partial q_1)\,\boldsymbol{G}_2 + \left(\tau - \frac{i}{2k^n}\right)\boldsymbol{G}_1\right\}K^{(n,\tau)}(\boldsymbol{q}, \boldsymbol{q}'; \beta)|_{\beta=0}\,.$$

On the left-hand side, extra trigonometric factors in $\Theta_\beta^{(n)}(\theta)$ enter, while on the right-hand side the Chebyshev polynomials in $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ only produce corresponding factors in $\sin n\theta$ and $\cos n\theta$. Since the kernel at $\beta = 0$ is a function of $q_1 - q_1'$ and $q_2 - q_2'$, derivatives with respect to q_i become derivatives with respect to $-q_i'$, which can be integrated by parts by noting that only first derivatives in q_2' appear and that (5.20) is independent of these variables. If we exchange $\boldsymbol{M}_3$ and $\boldsymbol{G}_2$ in the final operator acting on the boundary data, the $-i/2k^n$ factor in (5.25) is replaced by a correct $+i/2k^n$ as in (5.7a) and we are left with an expression analogous to (5.20) with $i\boldsymbol{M}_1^{(n,\tau)}f(\boldsymbol{q}')$ in place of $f(\boldsymbol{q}')$. Since the remaining kernel is the Helmholtz kernel, we can finally set $q_2 = q_2'$ and use the reproduction property of (5.22) to conclude that the generator of the transformation (5.20)-(5.21) is indeed $\boldsymbol{M}_1^{(n,\tau)}$. It must be noted that this procedure has been rather more laborous than the corresponding one for Lie geometric actions [5] which are basically Taylor expansions, and that of one-variable integral transforms [21,22], where the Dirac δ appears at $\beta = 0$.

Even though the $SO_{2,1,n}$ action (2.21)-(2.22) is an integral one, it should not obscure the fact that certain solutions to the Helmholtz equation are self-reproducing under it. Thus, for example, a plane wave $\Phi_\theta(\boldsymbol{q})$ directed along the ray θ given in (5.16a) (letting θ run over the whole circle for simplicity so as to disregard σ) will become, under the action of $R^{(n,\tau)}(\alpha, \beta, \gamma)$ a plane wave directed along $\Theta_{-\beta}^{(n)}(\theta - \gamma/n) - \alpha/n$ and be multiplied by a factor of $\mu^{(n,\tau)}(\theta - \gamma/n, \beta)^*$. Similarly, eigenfunctions of a subgroup generator of $SO_{2,1,n}$ will transform among themselves as the rows of the $C^0_{\tau k^n}$ irreducible representation matrices decomposed with respect to that subgroup. The decomposition with respect to the compact SO_2-subgroup yields the polar partial waves ([3], eq. (3.19)). Self-reproducing solutions under $\boldsymbol{M}_1^{(n,\tau)}$-transformations are given by the eigenfunctions of $\boldsymbol{M}_1^{(n,\tau)}$, which can be found in principle through obtaining the eigenfunctions of $\widetilde{\boldsymbol{M}}_1^{(n,\tau)}$ and then performing (5.16). For $\tau = 0$ these are given by the separated products of parabolic cylinder functions in parabolic co-ordinates ([3], eq. (3.33)). If τ is arbitrary [39], no such analytic expression can be given, however.

The similarity algebra deformation sketched here applies to hyperbolic and elliptic equations in any number of dimensions, which possess an inhomogeneous classical similarity algebra where the equation itself appears as the restriction of the Abelian ideal to a nondegenerate conic surface.

6. – Conclusion.

The examples we have treated in this paper have in common the quest for finite-dimensional « higher » symmetry algebras for certain common differential equations using various strategies. We have not set out to find a «universal» symmetry group, as have ANDERSON and collaborators ([41-43]), which show that all completely integrable N-dimensional dynamical systems posses the global symmetry group of the Newtonian free particle $SL_{N+2,R}$. (See also ref. ([44]).) We are also aware of the fact that the very productive applications of Lie algebras to separation of variables by MILLER, BOYER and KALNINS (see ([3]) and references within) do not make essential use of finite group properties and in fact do not require nor need that the second-order differential operators form an algebra. The first approach would seem to trivialize the study of particular examples and, in the extreme, the second would argue to obviate the introduction of Lie structures. It is our point of view that the treatment of integral transforms from a group-theoretic point of view needs at present concrete examples given in classical language. There are indications that the proper mathematical framework for the group action produced by hyperdifferential operator Lie algebras may well be the theory of Lie-Bäcklund contact transformations in the infinite-dimensional tangent space of Ibragimov and Anderson ([45]). In this brief survey of algebras and groups associated with some representative linear differential equations, we hope to have pointed out that the theory of integral transforms may be the first to benefit from the extension of Lie-theoretical methods to operators of higher degree.

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RIASSUNTO (*)

Si studia la costruzione e l'azione di certe algebre di Lie di operatori differenziali del secondo ordine o di ordine più alto su spazi di soluzioni di ben note equazioni differenziali lineari paraboliche, iperboliche ed ellittiche. Quest'ultime comprendono le equazioni di Schrödinger hamiltoniane quantiche quadratiche a N dimensioni, equazioni d'onda e di calore ad una dimensione e l'equazione di Helmholtz bidimensionale. In un primo approccio la solita algebra dell'operatore differenziale del primo ordine di similarità dell'equazione è immersa in quella più grande, che compare come un'algebra dinamica quantomeccanica. In un secondo approccio si costruisce la nuova algebra come evoluzione temporale di un'algebra a trasformazione finita sulle condizioni iniziali. In un terzo approccio l'algebra di similarità inomogenea è deformata in una classica non compatta. In ogni caso, si può integrare l'algebra ad un gruppo di Lie di trasformazioni integrali che agiscono effettivamente sullo spazio delle soluzioni dell'equazione differenziale.

(*) *Traduzione a cura della Redazione.*

Группы интегральных преобразований, образованные алгебрами Ли дифференциальных операторов второго и более высоких порядков.

Резюме (*). — Мы исследуем конструирование и действие некоторых алгебр Ли дифференциальных операторов второго и более высоких порядков на пространствах решений хорошо известных параболических, гиперболических и эллиптических линейных дифференциальных уравнений. Последние включают N-мерные квадратичные квантовые уравнения Шредингера, одномерные уравнения теплопроводности и волновые уравнения и двумерное уравнение Гельмгольца. В первом подходе, алгебра дифференциальных операторов первого порядка внедряется в большую алгебру, которая выступает как квантовомеханическая динамическая алгебра. Во втором подходе, новая алгебра строится, как временная эволюция алгебры конечных преобразований, исходя из начальных условий. В третьем подходе, алгебра деформируется в некомпактную классическую алгебру. В каждом случае мы можем проинтегрировать алгебру в группу Ли интегральных преобразований, действуюших эффективно на пространстве решений дифференциального уравнения.

(*) *Переведено редакцией.*