Fractional Fourier–Kravchuk transform

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We introduce a model of multimodal waveguides with a finite number of sensor points. This is a finite oscillator whose eigenstates are Kravchuk functions, which are orthonormal on a finite set of points and satisfy a physically important difference equation. The fractional finite Fourier–Kravchuk transform is defined to self-reproduce these functions. The analysis of finite signal processing uses the representations of the ordinary rotation group SO(3). This leads naturally to a phase space for finite optics such that the continuum limit ($N \to \infty$) reproduces Fourier paraxial optics. © 1997 Optical Society of America [S0740-3232(97)01807-3]

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1. INTRODUCTION: FINITE WAVEGUIDE MODEL

We analyze the mathematical model of a planar multimodal optical waveguide carrying in parallel a finite number of amplitude signals of some color, sampled at the same number of discrete sensor points. Such waveguides may be communication channels in photonic devices (a band of higher refractive index produced by doping a transparent substratum) that could replace several thinner parallel but separate monomodal waveguides. Our model is based on Kravchuk functions—finite analogues of the harmonic oscillator wave functions—and it defines naturally a fractional Fourier transform along the waveguide, modeled as a finite harmonic oscillator. This transform acts on the position and momentum coordinates, which take a finite number of values, and transforms the Kravchuk wave functions into themselves. To distinguish this particular transform from the more common finite Fourier exponential transform, we call ours the Fourier–Kravchuk transform. As the number and density of sensor points increases, both the Fourier–Kravchuk and the Fourier exponential matrices become the common exponential Fourier integral kernel.

The finite oscillator model that we present in this paper pays attention to the following considerations:

• Waveforms used for parallel signal communication by waveguides will carry a finite number of (real or complex) data values:

$$\kappa_0, \ \kappa_1, \ ..., \ \kappa_N. \tag{1.1}$$

• Since waveguides are likely to be produced in stratified chips with two-dimensional layered design, they will be planar. We assume the y dimension of the guide to be such that it admits only the fundamental solution in this direction.

• Waveforms will be produced and sampled at a finite number of sensors, located at equidistant discrete points on the (dimensionless) ξ axis separated by $h = \sqrt{2/N}$,

$$\xi_n = (n - N/2)h, \qquad n = 0, 1, ..., N. \tag{1.2}$$

(The actual distance $x_k - x_{k-1}$ is thus given in units of $\lambdabar$.)

• We assume that we observe the values of the waveform $f(\xi)$ at these points only, i.e., that we measure the quantities

$$f_0 = f(\xi_0), \qquad f_1 = f(\xi_1), \ ..., f_N = f(\xi_N). \tag{1.3}$$

The optimal situation is, of course, when the number of sensors equals the number of data values.

• When $N \to \infty$ the points ξ_j become dense in a growing interval, and we recover the well-known quantum harmonic oscillator wave functions and the fractional Fourier integral transform.

In an isotropic optical medium of smooth refractive index, approximated by

$$n(\xi, \ \eta) = n_o - (\nu_1{}^2\xi^2 + \nu_2{}^2\eta^2) + \cdots , \tag{1.4}$$

which is independent of the optical axis coordinate ζ and close to the forward ($+z$) direction, the electric field in a transverse direction satisfies a limit form of the Maxwell equations. These are formally identical with the ζ-evolution Schrödinger equation for a scalar, dimensionless field $\phi(\xi, \eta, \zeta)$,[1] namely,

$$\hat{H}\phi := \frac{1}{2}\left(-\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} + \nu_1{}^2\xi^2 + \nu_2{}^2\eta^2\right)\phi = i\,\frac{\partial}{\partial\zeta}\,\phi. \tag{1.5}$$

We use Cartesian coordinates $(x, y, z) = \lambdabar(\xi, \eta, \zeta)$, where the latter are dimensionless with the natural scale of the reduced wavelength, $\lambdabar = \lambda/2\pi = 1/k$, which corresponds to a quantum harmonic oscillator with angular frequencies ν_1 and ν_2 in the x and y directions (see Fig. 1).

The Hamiltonian operator $\hat{H}$ in Eq. (1.5) satisfies the quantum Newton equations for the harmonic oscillator

$$[\hat{H}, [\hat{H}, \ \xi]] = \nu_1{}^2\xi, \qquad [\hat{H}, [\hat{H}, \ \eta]] = \nu_2{}^2\eta, \tag{1.6}$$

where $[\hat{A}, \ \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator Lie bracket. [Classically, Newton's equation for a unit mass in the potential $V(x) = \frac{1}{2}\omega^2x^2$ is $\ddot{x} = -\omega^2x$, where dots indicate derivatives with respect to time; upon quantization, time derivatives are replaced by i times the commutator with the Hamiltonian operator.]

We must now explicitly stress that the finite oscillator model, satisfying the above considerations and the Newton equations (1.6), is distinct from the common approximation of the physical waveguide by a quantum continuous harmonic oscillator with wave functions $\phi(\xi, \eta, \zeta)$. Whereas the common treatment disregards the nonphysical region $(\xi, \eta) \in \mathfrak{R}^2$, where the refractive index [Eq. (1.4)] is less than its vacuum value, the finite oscillator model addresses the values of the field only at a finite number of discrete sensor points inside the physical waveguide. Planar waveguides are as thin as possible but still three dimensional; all their sensor points [Eq. (1.2)] are on the x axis. We regard the Newton equations (1.6) as the fundamental dynamical definition of a waveguide; they are satisfied (of course) by the Schrödinger differential operator (1.5) but also by a difference operator [see Eq. (3.8) below]. The latter describes an intrinsically discrete and finite optical system, where the (Kravchuk) eigenfunctions are orthogonal and complete over that finite sensor set, rather than a continuous system (the common oscillator) with just a finite set of sampling points and (Hermite) eigenfunctions that are not orthogonal on the finite set of points and are overcomplete.

In Section 2 we use the common harmonic oscillator wave functions[2] to extract the data of Eq. (1.1) from the data of Eq. (1.3) and to show why the Hermite function basis is not the most adequate in this case. Section 3 introduces a proper orthonormal finite basis for the vector space of sensor points: that of the Kravchuk functions. These are the wave functions of the finite harmonic oscillator because they satisfy Newton's equation (1.6) but with a second-order difference operator for the Hamiltonian. It is a finite analog of the usual second-order differential Schrödinger operator. The main properties of these wave functions, based on the Kravchuk polynomials,[3–6] are assembled in Appendix A.

Fractionalization of an idempotent integral transform on a Hilbert space can be based on the orthonormal, complete eigenbasis of a self-adjoint operator H with equally spaced spectrum. The exponential $\exp(-i\pi H/2)$ of such an operator multiplies the eigenfunctions by phases $(-i)^n$, $n = 0, 1, \ldots, N$. Section 4 places the fractional Fourier integral transform as a closed subgroup of the group $\mathrm{Sp}(2, \mathfrak{R})$ of 2×2 matrices of unit determinant, called canonical transforms.[7,8] Just as for the oscillator wave functions—but for finite N—Section 4 fractionalizes the Fourier–Kravchuk transform to power $\alpha \in \mathfrak{R}$ (modulo 4), assuming the phase factor $\exp(-in\alpha\pi/2)$ on the basis of eigenvectors of H.
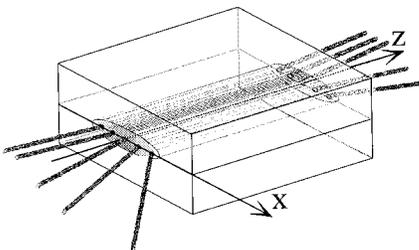

Fig. 1. Planar multimodal waveguide model with sensor points. The guide can carry only a finite number of modes; the waveforms are sampled at the same number of equidistant sensor points (five in this figure).

The fractional Fourier–Kravchuk transform kernel is again a Kravchuk function and is also a Wigner D matrix for an irreducible representation of the ordinary three-dimensional rotation group $\mathrm{SO}(3)$.[5,6,9] The symmetry thus uncovered in Section 5 is, *post factum*, not surprising. It interprets the fractional finite Fourier–Kravchuk transform as a rotation around a circle, element of $\mathrm{SO}(2) \subset \mathrm{SO}(3)$. The rotation around the other two axes, generated by position and momentum operators with a discrete, finite, equally spaced spectra is analyzed in Section 6. It embeds the fractional finite Fourier–Kravchuk transform into the higher group $\mathrm{SU}(2)$ of 2×2 unitary matrices [twofold cover of the rotation group $\mathrm{SO}(3)$], a true counterpart of (continuous) optical phase space that is the arena of finite signal theory. The concluding Section 7 compares our transform with the more generally known Fourier exponential finite transform.

2. HARMONIC OSCILLATOR WAVE FUNCTIONS AND SAMPLING POINTS

The quantum harmonic oscillator wave functions are well known to be

$$\psi_n(\xi) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(\xi)\exp(-\xi^2/2),$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x, \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

where $H_n(\xi)$ are the Hermite polynomials in the dimensionless variable ξ. This is an orthonormal set of functions in the Hilbert space $\mathscr{L}^2(\mathfrak{R})$ defined by closure with respect to the sesquilinear inner product

$$(\psi_n, \psi_m)_\mathfrak{R} = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \psi_n(\xi)^* \psi_m(\xi) = \delta_{n,m}, \tag{2.2}$$

which integrates over the full real line $\xi \in \mathfrak{R}$.

The basis of functions $\{\psi_n\}_{n=0}^{\infty}$ is dense in $\mathscr{L}^2(\mathfrak{R})$: any function $f(\xi) \in \mathscr{L}^2(\mathfrak{R})$ can be approximated (in the norm) by the expansion

$$f(\xi) = \sum_{n=0}^{\infty} c_n \psi_n(\xi). \tag{2.3}$$

The expansion coefficients $\{c_n\}_{n=0}^{\infty}$ can be determined by performing the integrals $c_n = (\psi_n, f)_\mathfrak{R}$ for $n = 0, 1, 2, \ldots$.

When the function $f(\xi)$ in Eq. (2.3) is known only through its values $f_n = f(\xi_n)$, $n = 0, 1, \ldots, N$, at $N + 1$ discrete sampling points $\xi_0, \xi_1, \ldots, \xi_N$, then any set of $N + 1$ continuous, linearly independent functions can be used to expand $f(\xi)$. If we write

$$f(\xi_j) = \sum_{n=0}^{N} c_n^{(N)} \psi_n(\xi_j), \tag{2.4}$$

or in the matrix form

$$\begin{pmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_N) \end{pmatrix} = \begin{bmatrix} \psi_0(\xi_0) & \psi_1(\xi_0) & \cdots & \psi_N(\xi_0) \\ \psi_0(\xi_1) & \psi_1(\xi_1) & \cdots & \psi_N(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(\xi_N) & \psi_1(\xi_N) & \cdots & \psi_N(\xi_N) \end{bmatrix} \begin{pmatrix} c_0^{(N)} \\ c_1^{(N)} \\ \vdots \\ c_N^{(N)} \end{pmatrix},$$

$$(2.5)$$

we see that to determine the $N + 1$ coefficients $\{c_n^{(N)}\}_{n=0}^N$ we must invert an $(N + 1) \times (N + 1)$ matrix and then apply it to the column of data values $f_n = f(\xi_n)$.

This algorithm is inefficient because we are using only the linear independence of the wave function set. The only trade-off is the freedom of choosing the sampling points ξ_j, which need not be equally spaced. A more efficient algorithm, however, should sidetrack the matrix inversion by choosing a diagonal transformation matrix corresponding to an orthonormal set of functions on the sampling points. These should be functions of the continuous coordinate ξ rather than, say, the canonical basis of Kronecker δ's, $D_j(\xi_k) = \delta_{j,k}$, because they are waveforms and the environment is that of a harmonic oscillator. Moreover, in the limit $N \to \infty$, with the interpoint separation decreasing as $h \sim N^{-1/2}$, the basis functions should become the harmonic oscillator wave functions.

3. FINITE OSCILLATOR KRAVCHUK WAVE FUNCTIONS

We remind the reader that the binomial process of random walk on the line is described in probability theory by the Kravchuk polynomials $k_n^{(p)}(s, N)$.[10] When the process is right–left symmetric, then the $p = 1/2$ case applies and the polynomials are said to be symmetric.

Symmetric Kravchuk functions $\{\phi_n(s, N)\}_{n=0}^N$ are functions of a continuous variable s in the interval $-1 - N/2 \leqslant s \leqslant 1 + N/2$ and orthonormal with respect to the inner product over the set of $N + 1$ discrete points inside that interval:

$$\sum_{j=0}^N \phi_{n'}(s_j, N)\phi_n(s_j, N) = \delta_{n',n}, \qquad (3.1a)$$

$$s_j = j - N/2, \qquad j = 0, 1, \ldots, N. \qquad (3.1b)$$

The symmetric Kravchuk functions are given in terms of symmetric Kravchuk polynomials $k_n(s, N)$ of degrees n, $0 \leqslant n \leqslant N$ in the variable s, normalized and multiplied by the root of the binomial distribution,[11,12] i.e.,

$$\phi_n(s, N) = 2^{n - N/2} k_n(s + N/2, N)$$
$$\times \left[\frac{n!(N - n)!}{\Gamma(N/2 + s + 1)\Gamma(N/2 - s + 1)} \right]^{1/2}.$$

$$(3.2)$$

Their most important properties are given in Appendix A, and they are plotted in Fig. 2. Below we emphasize further the reasons for considering the symmetric Kravchuk functions to be the best finite counterpart of the harmonic oscillator.

A wave field sensed on a section transversal to the optical axis of a physical waveguide will contain only a finite number of oscillator modes, because the refractive index

has finite range, bounded from below by the vacuum value 1. A function whose harmonic oscillator expansion is physically meaningful, and of which we know only the values on a set of $N + 1$ points ξ_j equidistant by $h = \sqrt{2/N}$ [cf. Eqs. (1.2) and (3.1b)], is expanded in terms of Kravchuk functions (3.2) of the variable $s = \xi/h$ as

$$f(\xi_j) = (N/2)^{1/4} \sum_{n=0}^N \kappa_n^{(N)} \phi_n(\sqrt{N/2}\xi_j, N),$$

$$j = 0, 1, \ldots, N. \quad (3.3)$$

The Kravchuk basis is orthonormal with respect to the discrete orthogonality relation (3.1a), and hence finding the coefficients with the $N + 1$ data values of Eq. (1.1) requires multiplying the $N + 1$ equations (3.3) by the (presumably tabulated) numbers $\phi_n(\sqrt{N/2}\xi_j, N)$ and summing over the $N + 1$ sample points ξ_j, to obtain

$$\kappa_n^{(N)} = (N/2)^{-1/4} \sum_{j=0}^N \phi_n(\sqrt{N/2}\xi_j, N) f(\xi_j). \qquad (3.4)$$

Having found the data coefficients $\kappa_n^{(N)}$, we may want to interpolate the data points by a smooth function between them by writing

$$f_A(\xi, N) = (N/2)^{1/4} \sum_{n=0}^N \kappa_n^{(N)} \phi_n(\sqrt{N/2}\xi, N). \qquad (3.5)$$

This function on the line segment $[s_0, s_N]$ can be called the finite-N approximation to the original function $f(\xi)$ whose values were known on the set of $N + 1$ equidistant sample points $\xi_j = hs_j$. From the limit relation (A10), when N increases and the separation $h = \sqrt{2/N}$ between samples decreases, we recover the usual harmonic oscillator wave function expansion.

As shown in Appendix A, Kravchuk functions satisfy a three-term recurrence relation and lead to an eigenequation (A12) with a real, equally spaced spectrum characteristic of the harmonic oscillator,

$$\mathbf{H}^{(N)}(s)\phi_n(s, N) = (n + \tfrac{1}{2})\phi_n(s, N),$$

$$n = 0, 1, \ldots, N, \quad (3.6)$$

with a finite-difference operator $\mathbf{H}^{(N)}(s)$ (see below). This suggests that we may identify the operator $\mathbf{H}^{(N)}(s)$ as the Hamiltonian of the finite system. This operator turns out also to satisfy the harmonic oscillator Newton equation (1.3), viz.,

$$[s, \mathbf{H}^{(N)}(s)] = \frac{1}{2}[\alpha(-s)\exp(\partial_s) - \alpha(s)\exp(-\partial_s)]$$

$$= iP_s^{(N)}, \qquad (3.7a)$$

$$[\mathbf{H}^{(N)}(s), P_s^{(N)}] = is. \qquad (3.7b)$$

Moreover, we know that as $N \to \infty$ the finite-dimensional eigenvectors become the usual oscillator eigenfunctions [see Eq. (A10)].

For these reasons we consider physically meaningful the one-dimensional finite oscillator described by the difference Hamiltonian,
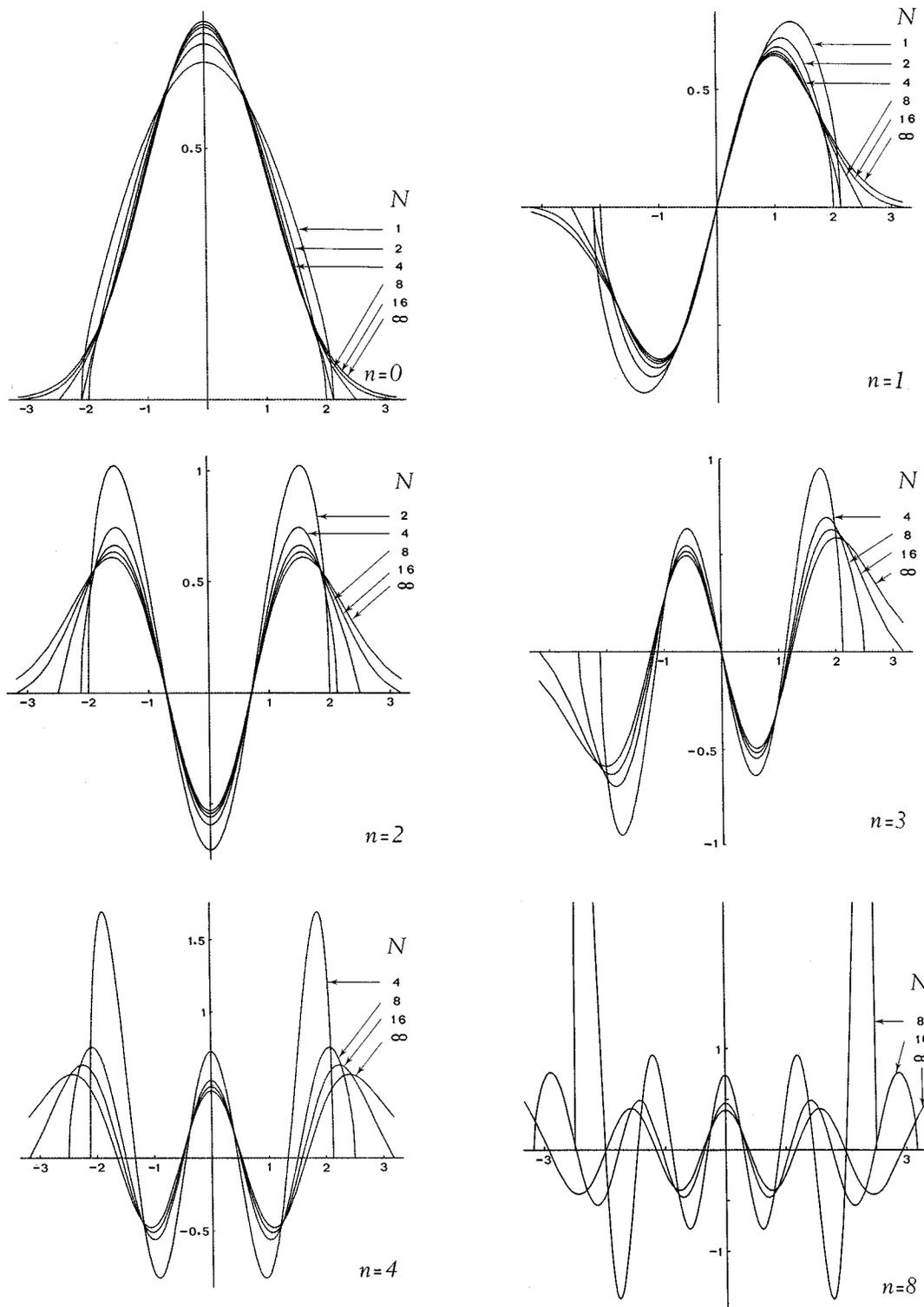
Fig. 2. Kravchuk functions $h^{-1/2}\phi_n(\xi, N)$ for $n = 0, 1, 2, 3, 4$, and 8, for the sensor spacing $h = \sqrt{2/N}$. Each figure plots the values of $N \in \{1, 2, 4, 8, 16, \ldots, \infty\}$ compatible with n. There are $N + 1$ sensor points, spaced by $h = \sqrt{2/N}$ between $-\sqrt{N/2}$ and $\sqrt{N/2}$. The end-point zeros occur one h outside this interval. The figures show that the $N \to \infty$ limit of the Kravchuk functions as the number and density of sensors increases is the (infinite) set of harmonic oscillator wave functions.

$$\mathbf{H}^{(N)}(s) = -\frac{1}{2}\left[\alpha(s)\exp(-\partial_s) + \alpha(-s)\exp(\partial_s)\right]$$

$$+ \frac{1}{2}(N + 1), \tag{3.8a}$$

where $\exp(\pm\partial_s)f(s) = f(s \pm 1)$ are the basic shift operators and

$$\alpha(s) = [(N/2 + s)(N/2 - s + 1)]^{1/2}. \tag{3.8b}$$

This Hamiltonian also has the property of factorization and participation in a spectrum-generating algebra, which turns out to be the Lie algebra of the rotation group SO(3) (for details, see Ref. 11). We propose thus the finite analog of the Schrödinger equation

$$[\mathbf{H}^{(N_1)}(s_1) + \mathbf{H}^{(N_2)}(s_2)]\phi(s_1, s_2)$$

$$= (n_1 + n_2 + 1)\phi(s_1, s_2) \quad (3.9)$$

for the $N_1 \times N_2$ two-dimensional waveguide model (1.4) in the coordinates

$$x = \lambda\xi, \qquad \xi = h_1 s_1 = \sqrt{2/N_1}\,s_1, \qquad (3.10a)$$

$$y = \lambda\eta, \qquad \eta = h_2 s_2 = \sqrt{2/N_2}\,s_2. \qquad (3.10b)$$

The $N_1 \times N_2$ eigenstates are the products of Kravchuk functions $\phi_{n_1}(s_1, N_1)\phi_{n_2}(s_2, N_2)$, $n_1 \in \{0, 1, \ldots, N_1\}$, and $n_2 \in \{0, 1, \ldots, N_2\}$, with eigenvalues $n_1 + n_2 + 1 \in \{1, 2, \ldots, N_1 + N_2 + 1\}$.

With the preceding equations we describe finite waveguides of two-dimensional x–y section. As stated among the considerations of Section 1, we are particularly interested in planar waveguides, i.e., those in which only the fundamental mode is present in the y direction and the sensors are placed on a linear array of $N_1 + 1$ equidistant points along the x axis. Because these may be the first ones used in flat photonic technology, and also because their mathematical treatment is somewhat simpler, Sections 4 and 5 will present the one-dimensional mathematical treatment, and the closing paragraph of each will detail the results for the two-dimensional planar case.

When we set $n_2 = N_2 = 0$, the eigenmodes of the discrete, planar waveguide are the functions $\phi_n(s_1, N)\phi_0(s_2, 0)$ given by

$$\phi_n(s_1, s_2; N)$$

$$= 2^{n-N/2}k_n(s_1 + N/2, N)$$

$$\times \left[\frac{n!(N-n)!\,\sin \pi s_2}{\pi s_2 \Gamma(N/2 + s_1 + 1)\Gamma(N/2 - s_1 + 1)}\right]^{1/2}. \quad (3.11)$$

These wave functions $\phi_n(s_1, s_2; N)$ in the planar discrete waveguide are eigenfunctions of the finite Hamiltonian difference operator,

$$\mathbf{H}^{(N)}(s_1, s_2) = \mathbf{H}^{(N)}(s_1) + \mathbf{H}^{(0)}(s_2), \qquad (3.12)$$

with a linear spectrum [Eq. (3.9)]. This spectrum is the same as that of the ordinary plane harmonic oscillator with wave functions $\psi_n(s_1)\psi_0(s_2)$, $n = 0, 1, \ldots, N$, but governed by the finite-difference operator (3.8a). Hence evolution along the optical axis z of the $(N + 1)$-point discrete waveguide is

$$\exp[i\zeta\,\mathbf{H}^{(N)}(s_1, s_2)]\phi_n(s_1, s_2; N)$$

$$= \exp[i\zeta(n + 1)]\phi_n(s_1, s_2; N), \quad (3.13)$$

exactly as in the corresponding continuous model. Therefore, if a continuous wave function is approximated by a sum of Kravchuk functions on $N + 1$ points, the accuracy of the approximation will be maintained independently of ζ. In this sense we regard the Kravchuk-

function expansion as the best finite-N approximation to a waveform in a quantum harmonic oscillator environment, such as that of an optical waveguide in the common paraxial treatment.

4. FRACTIONAL FOURIER TRANSFORM

The ordinary Fourier transform of Lebesgue square-integrable functions on the real line $\mathfrak{R}$ is the unitary map of the Hilbert space $\mathscr{L}^2(\mathfrak{R})$ given by

$$f(x) \mapsto \widetilde{f}(x) = (\mathscr{F}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathfrak{R}} dx'\,\exp(-ixx')f(x'). \quad (4.1)$$

Among its many well-known properties is that of being a fourth root of the unit operator, $\mathscr{F}^4 = 1$ and that it transforms the quantum harmonic oscillator wave functions (2.1) into themselves, up to a phase $(-i)^n$:

$$\mathscr{F}\psi_n(x) = \exp(-i\pi n/2)\psi_n(x). \quad (4.2)$$

To define fractional powers of $\mathscr{F}$ it is sufficient to embed (4.1) in a continuous Lie group of integral transforms. In 1971, Moshinsky and Quesne studied the two-dimensional symplectic group Sp(2, $\mathfrak{R}$) of linear canonical transformations in quantum mechanics[7]; this includes the harmonic oscillator evolution cycle. The corresponding infinitesimal Lie algebra was recognized to be that of second-order differential operators, and Hilbert spaces were constructed that made their complexification unitary. Using a notation based on Ref. 8, this one-parameter group of integral transforms is

$$\left\{\mathscr{C}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}f\right\}(x) = \exp\left[-\frac{it}{2}\left(-\frac{d^2}{dx^2} + x^2\right)\right]f(x)$$

$$= \int_{-\infty}^{\infty} dx'\,C_F(x, x'; t)f(x'), \quad (4.3a)$$

with the integral kernel family

$$C_F(x, x'; t) = \frac{\exp\left(-\frac{i}{4}\pi\,\mathrm{sgn}\,\sin t\right)}{\sqrt{2\pi|\sin t|}}$$

$$\times \exp\left[\frac{i}{2\sin t}(x^2\cos t - 2xx'\right.$$

$$\left. + x'^2\cos t)\right]$$

$$= \sum_{n=0}^{\infty}\psi_n(x)\exp[-i(n + 1/2)t]\psi_n(x'). \quad (4.3b)$$

The last line in Eq. (4.3b) displays the integral kernel as a bilinear generating function for the orthonormal family of Hermite functions [cf. Eq. (4.2)]. For the value $t = \pi/2$ we have the Fourier transform (4.1), but for a phase we have

$$\mathscr{C}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \exp(-i\pi/4)\mathscr{T}. \quad (4.4)$$

The square root of the Fourier transform was used in Ref. 8, pp. 328 and 392, to bind the repulsive oscillator and Mellin power functions. The property (4.2) led to the rediscovery of fractional Fourier and Hankel transforms by Namias in 1980.[13] The waveguide realization was proposed and tested recently by Lohmann *et al.*[14] The kernel has the key property of group composition,

$$\int_{-\infty}^{\infty} \mathrm{d}x' C_F(x, x'; t_1) C_F(x', x''; t_2)$$

$$= \sigma(t_1, t_2; t_1 + t_2) C_F(x, x''; t_1 + t_2), \quad (4.5a)$$

with the sign defined as

$$\sigma(t_1, t_2; t_1 + t_2)$$

$$= \begin{cases} 1 \\ -1, \end{cases} \quad \mathrm{sgn} \sin(t_1 + t_2) + \mathrm{sgn} \sin t_1 + \mathrm{sgn} \sin t_2$$

$$= \begin{cases} \text{positive} \\ \text{negative} \end{cases}. \quad (4.5b)$$

In the limit when $t \to 0^+$, the kernel $C_F(x, x'; t)$ coincides with $\delta(x - x')$, associativity holds, and the inverse is $C_F(x, x'; -t) = C_F(x', x; t)^*$. This one-parameter group of integral transforms in $\mathscr{L}^2(\mathfrak{R})$ reduces, however, with respect to parity. Odd and even functions constitute the irreducible components, which in Bargmann's notation are $D_{1/4}^+$ and $D_{3/4}^+$ (Ref. 15).

The phase in relation (4.4) between harmonic oscillator evolution and Fourier transforms is important: The fourth power of the Fourier integral transform $\mathscr{F}^4 = \mathscr{T}$ is the identity operator. But note that four quarter cycles of oscillation in the one-dimensional optical waveguide change the sign of the functions, i.e., $\mathscr{C}_F^4 = -\mathscr{T}$ [see Eq. (4.5b)]; only $\mathscr{C}_F^8 = \mathscr{T}$. The minus sign after t undergoes one cycle, $[0, 2\pi)$, indicates that the canonical transform operators $\mathscr{C}$ in Eqs. (4.3) follow the double cover of the symplectic group (isomorphic to the group of 2×2 real, unimodular matrices), and form a faithful representation of the metaplectic group, indicated by $\mathrm{Mp}(2, \mathfrak{R})$; there, the parameter t ranges in $[0, 4\pi)$. This metaplectic sign is a characteristic of the oscillator representation. This minus sign squares to a plus sign when we work in two dimensions and consider the usual, axially symmetric Fourier integral transform with the kernel $\exp[-i(xx' + yy')]$. Finally, on cue from one of the referees, we should stress that the metaplectic phase described above is distinct from the Berry phase of the rotation group.

5. FRACTIONAL FOURIER–KRAVCHUK TRANSFORMATIONS

We are concerned with waveforms in waveguides where a finite number of sensors will give $N + 1$ analogic data values $f(\xi_n)$, $\xi_n = (n - N/2)h$. The signal is thus a vector in $\mathfrak{R}^{N+1}$. We define the αth power of the one-dimensional finite Fourier–Kravchuk transform $\hat{\mathbf{F}}$ by a linear operator that maps the Kravchuk basis functions onto themselves:

$$\hat{\mathbf{F}}^\alpha \phi_n(s, N) = \exp(i\alpha\pi/4)\exp\left[-\frac{i\pi}{2}\alpha\mathbf{H}^{(N)}(s)\right]\phi_n(s, N)$$

$$= \exp(-i\alpha\pi n/2)\phi_n(s, N), \quad (5.1)$$

where $n = 0, 1, \ldots, N$. This operator can be represented in the Kravchuk basis by a matrix $\mathbf{F}$ that is diagonal. In the basis of discrete sensor points $\xi_n = hs_n$, however [cf. Eqs. (3.3) and (3.4)], $\hat{\mathbf{F}}^\alpha$ will be represented by the nondiagonal matrix $\mathbf{F}^\alpha = \|F_{n,n'}^\alpha\|$ in

$$\hat{\mathbf{F}}^\alpha f(\xi_n) = \sum_{n'=0}^{N} F_{n,n'}^\alpha f(\xi_{n'})$$

$$= (N/2)^{1/4}\sum_{n'=0}^{N} \kappa_{n'}^{(N)}\hat{\mathbf{F}}^\alpha \phi_{n'}(h\xi_n, N)$$

$$= \sum_{n'=0}^{N}\left[\sum_{j=0}^{N}\phi_{n'}(h\xi_j, N)f(\xi_j)\right]$$

$$\times \exp(-i\alpha\pi n'/2)\phi_{n'}(h\xi_n, N), \quad (5.2)$$

where

$$F_{n,n'}^\alpha = \sum_{j=0}^{N}\phi_j(h\xi_{n'}, N)\exp(-ij\alpha\pi/2)\phi_j(h\xi_n, N) \quad (5.3a)$$

$$= \exp[i(\pi/2)(n + n' - N\alpha/2)]$$

$$\times \sqrt{C_N^n C_N^{n'}}\cos^N(\pi\alpha/4)\tan^{n+n'}$$

$$\times (\pi\alpha/4)_2F_1[-n, -n'; -N; \sin^{-2}(\pi\alpha/4)] \quad (5.3b)$$

$$= \exp[i(\pi/2)(n' - n - N\alpha/2)]$$

$$\times \sqrt{\frac{n!(N - n)!}{n'!(N - n')!}}\sin^{n'-n}(\pi\alpha/4)$$

$$\times \cos^{N-n'-n}(\pi\alpha/4)k_n^{[\sin^2(\pi\alpha/4)]}(n', N). \quad (5.3c)$$

The bilinear generating function for Kravchuk polynomials used in Eq. (5.2) can be found in Eqs. (21)–(23) of Ref. 16. We call it the finite Fourier–Kravchuk kernel in the $(N + 1)$-dimensional position space of functions $f(\xi_n)$, $n = 0, 1, \ldots, N$.

The finite Fourier–Kravchuk matrix $\mathbf{F} = \|F_{n,n'}^1\|$ in Eq. (5.3) has the following properties:

- It is a fourth root of unity: $\mathbf{F}^4 = \mathbf{1}$.
- Its square is the inversion matrix: $\mathbf{F}^2 = \mathbf{I}$, $I_{n,n'} = \delta_{n,N-n'}$.
- The matrices satisfy

$$\mathbf{F}^{\alpha_1}\mathbf{F}^{\alpha_2}f(\xi_j) = \mathbf{F}^{\alpha_2}\sum_{n'=0}^{N} F_{j,n'}^{\alpha_1}f(\xi_{n'})$$

$$= \sum_{n'=0}^{N} F_{j,n'}^{\alpha_1}\sum_{n=0}^{N} F_{n',n}^{\alpha_2}f(\xi_n)$$

$$= \sum_{n',n=0}^{N} F_{j,n'}^{\alpha_1}F_{n',n}^{\alpha_2}f(\xi_n)$$

$$= \mathbf{F}^{\alpha_1+\alpha_2}f(\xi_j). \quad (5.4)$$

Hence $\mathbf{F}^\alpha$ qualifies as the αth power of $\mathbf{F}$, with $\mathbf{F}^0 = \mathbf{1}$, and the finite Fourier–Kravchuk transform is thus extended for fractional α.

- The matrices $\mathbf{F}^\alpha$ are unitary: $(\mathbf{F}^\alpha)^\dagger = \mathbf{F}^{-\alpha}$.
- Finally, we check that for $\alpha = 1$ we obtain the explicit form of the reproducing kernel for the symmetric Kravchuk functions[16]:

$$F_{n,n'}^{\alpha=1} = \exp[i(\pi/2)(n + n' - N/2)]$$

$$\times \sqrt{2^{-N}C_N^n C_N^{n'}}\,{}_2F_1(-n, -n'; -N; 2)$$

$$= \exp[i(\pi/2)(n' - n - N/2)]\phi_n(\xi_{n'}, N). \quad (5.5)$$

The fractional finite Fourier–Kravchuk matrices $\mathbf{F}^\alpha$ have a natural embedding as irreducible representations of rotations for vectors and spinors; the Lie group is the rotation group SO(3) [more properly, its twofold cover SU(2)], as we now proceed to show. From the theory of general Kravchuk polynomials $k_n^{(p)}(s, N)$[4–6] and of angular momentum,[9] we refer to the following relation between them (see Appendix A) and the Wigner little-d matrix elements for rotations around the y axis (here the 2 axis) generated by $\hat{J}_2$ in the standard notation:

$$d_{m,m'}^l(\beta) = \langle lm|\exp(-i\beta\hat{J}_2)|lm'\rangle$$

$$= D_{m,m'}^l(0, \beta, 0)$$

$$= (-1)^{m-m'}d_n^{-1}\sqrt{\varrho(s)}\,k_n^{(p)}(s, N), \quad (5.6a)$$

where $n = l - m$, $s = l - m'$, $N = 2l$, and $p = \sin^2\beta/2$, while d_n and $\varrho(s)$ are given in Eqs. (A1). In the specific case of our interest we take $\beta = \pi/2$, then $p = \frac{1}{2}$ and thus

$$d_{m,m'}^l(\pi/2) = \frac{(-1)^{m-m'}}{2^m}\sqrt{\frac{(l+m)!(l-m)!}{(l+m')!(l-m')!}}$$

$$\times k_{l-m}(l - m', N)$$

$$= d_{m',m}^l(-\pi/2)$$

$$= (-1)^{j-m'}\phi_{j-m}(m', N)$$

$$= (-1)^{m-m'}\phi_{j-m}(-m', N). \quad (5.6b)$$

Finally, we use the big-D notation for Euler angle rotations to write

$$D_{m,m'}^l(\alpha, \beta, \gamma) = \langle lm|\exp(-i\alpha\hat{J}_3)\exp(-i\beta\hat{J}_2)$$

$$\times \exp(-i\gamma\hat{J}_3)|lm'\rangle$$

$$= \exp(-im\alpha)d_{m,m'}^l(\beta)\exp(-im'\gamma)$$

$$(5.7a)$$

and the particular case

$$D_{m,m'}^l(\alpha, 0, 0) = \langle lm|\exp(-i\alpha\hat{J}_3)|lm'\rangle$$

$$= \exp(-im\alpha)\delta_{m,m'}. \quad (5.7b)$$

Thus the finite Fourier–Kravchuk transform (3.3)–(3.4) is related to the Wigner rotation functions and, as Eq. (3.6) shows, it is diagonal in the Kravchuk basis asso-

ciated with the Hamiltonian difference operator (5.1). The manipulations in Eq. (5.3) have thus a simple group-theoretical meaning for angular momentum $l = N/2$, namely,

$$\exp(il\alpha\pi/2)F_{n,n'}^\alpha$$

$$= \sum_{m,m'=-l}^l D_{l-n,m}^l(0, \pi/2, 0)D_{m,m'}^l$$

$$\times (-\alpha\pi/2, 0, 0)D_{m',l-n'}^l(0, -\pi/2, 0)$$

$$= \langle l, l-n|\exp[-i(\pi/2)\hat{J}_2]\exp[i(\pi/2)\alpha\hat{J}_3]$$

$$\times \exp[i(\pi/2)\hat{J}_2]|l, l-n'\rangle \quad (5.8a)$$

$$= \langle l, l-n|\exp[i(\pi/2)\alpha\hat{J}_1]|l, l-n'\rangle \quad (5.8b)$$

$$= \sum_{k=0}^N \phi_k(s_n)\exp[i(l-k)\alpha\pi/2]\phi_k(s_{n'}),$$

$$s_n = n - l. \quad (5.8c)$$

In Eq. (5.8c) we display the kernel as a bilinear generating function.

The properties that we expect of a proper fractional power of the finite Fourier transform $\mathbf{F}^\alpha$ listed above are easily checked to hold. When $N \to \infty$ we keep fixed in view the origin $F_{0,0}^\alpha$ and let the matrix grow down and right. In this limit, the Kravchuk functions $\phi_n(h\xi_{n'}, N)$ in Eq. (5.2) tend to the Hermite functions $\psi_n(\xi)$ in Eq. (2.1), and the bilinear generating functions (summed thereafter) have the limit

$$\lim_{N\to\infty}\sqrt{N/2}\,F_{n,n'}^\alpha = \exp(i\pi\alpha/4)C(\xi, \xi', \pi\alpha/2). \quad (5.9)$$

A mathematically precise formulation of this limit in terms of Hilbert spaces should be made, but we leave this rather technical matter for further research.

Returning finally to the physical flat waveguide with the small but nonzero thickness of Fig. 1, we define the corresponding planar fractional Fourier–Kravchuk transform as Eq. (5.1) in the 1 coordinate s_1 multiplied by the same transform in the 2 coordinate s_2, but for $N_2 = 0$. Thus we are led to

$$\hat{\mathbf{F}}^\alpha\phi_n(s_1, s_2; N)$$

$$= \exp(i\alpha\pi/2)\exp\left[-\frac{i\pi}{2}\alpha\mathbf{H}^{(N)}(s_1, s_2)\right]\phi_n(s_1, s_2; N)$$

$$= \exp(-i\alpha\pi n/2)\phi_n(s_1, s_2; N), \quad (5.10)$$

where $n = 0, 1, \dots, N$. This is formally the same as Eq. (5.1) and validates the previous discussion for thin waveguides.

6. THE FOURIER–KRAVCHUK ROTATIONS

The placement of the one parameter group of fractional finite Fourier–Kravchuk transformations within the three-dimensional rotation group will reveal the role of Kravchuk functions in finite signal analysis. First note that

the commutator of the position s and the difference operator $P_s^{(N)}$ defined in Eq. (3.7a) is

$$[s, iP_s^{(N)}] = -\frac{1}{2}\left[\alpha(s)\exp(-\partial_s) + \alpha(-s)\exp(\partial_s)\right]$$

$$= \mathbf{H}^{(N)}(s) - \frac{1}{2}(N+1). \qquad (6.1)$$

Let us now define the following operators:

$$J_1 = s \cdot ,$$

$$J_2 = -P_s^{(N)},$$

$$J_3 = \mathbf{H}^{(N)}(s) - \frac{1}{2}(N+1). \qquad (6.2)$$

Then the commutation relations (3.7) and (6.1) are those of self-adjoint generators of the rotation Lie algebra SO(3),

$$[J_1, J_2] = iJ_3, \qquad [J_2, J_3] = iJ_1, \qquad [J_3\ J_1] = iJ_2, \qquad (6.3a)$$

and the Casimir operator is a number,

$$J^2 = J_1{}^2 + J_2{}^2 + J_3{}^2 = l(l+1), \qquad l = N/2. \qquad (6.3b)$$

This indicates that we are in the irreducible representation of spin $l = N/2$: integer or half-integer. The spectrum $\{\mu\}$ of J_1, J_2, J_3 (or any $\hat{v} \cdot \mathbf{J}$ with $\hat{v}$ a unit norm vector) is therefore discrete and finite: $\mu = -l, -l + 1, \ldots, l$.

When J_1 is chosen as the diagonal operator, we classify the eigenfunctions in the reduction SO(3) $\supset$ SO$_1$(2). The equally spaced eigenvalues are $s_n = n - N/2$, corresponding to the discrete sensor positions. The eigenfunctions of J_1 are a Kronecker basis for the discrete sensor points, i.e., functions $\Lambda_k(\xi, N)$ such that $\Lambda_k(\xi_j, N) = \delta_{k,j}$; these have been studied in Ref. 12. On the other hand, when J_3 is chosen as the diagonal operator, we have the usual reduction SO(3) $\supset$ SO$_3$(2); the measured quantities (eigenvalues) are the indices n of the functions and are directly related to their oscillation frequency in the waveguide. The eigenfunctions are the Kravchuk functions valued on the sensor set.

The fractional finite Fourier–Kravchuk transform $\hat{\mathbf{F}}^\alpha$ is a rotation around the 3 axis as shown by Eqs. (4.3) and (6.2). By definition, it multiplies the Kravchuk eigenfunctions of J_3 only by phases. The input signal vector $\mathbf{f} \in \mathfrak{R}^{N+1}$ is measured in the *position* eigenbasis (of J_1) by the analogic values of the coefficients $f_j = f(\xi_j)$. The fractional finite Fourier–Kravchuk transform is a rotation by $\pi\alpha/2$ generated by J_3. According to Eq. (5.8), it will mix the data as

$$(\hat{\mathbf{F}}^\alpha \mathbf{f})_j = \sum_{j'=0}^{2l} \langle l,\ l - j|\exp[i(\pi/2)\alpha\hat{J}_1]|l,\ l - j'\rangle f_{j'},$$

$$l = N/2. \quad (6.4)$$

For $\alpha = 1$ we have a quarter rotation of $\pi/2$, as shown in Fig. 3. The J_1 position eigenfunctions (Kronecker δ's) become eigenfunctions of J_2. The latter we may identify as eigenfunctions of definite momentum, in analogy with
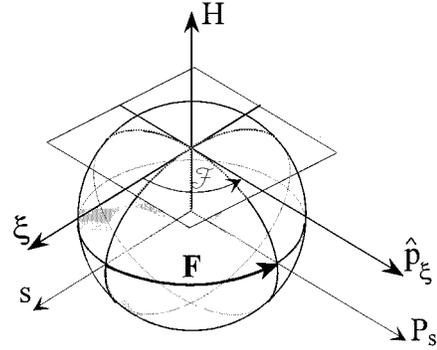

Fig. 3. The Fourier transform is a rotation by $\frac{1}{2}\pi$ generated by the oscillator Hamiltonian. It transforms the position operator ξ into the momentum operator $\hat{p}_\xi = -i\partial_\xi$ in the quantum mechanical case (plane, top); both operators have spectrum $\mathfrak{R}$, and each generates (noncommuting) translations of phase space in $\mathscr{L}^2(\mathfrak{R})$. The finite Fourier–Kravchuk transform similarly rotates the position s onto the momentum $P_s^{(N)}$ operators, whose spectrum is finite and equally spaced. The latter two close with the Hamiltonian into the Lie algebra of the rotation group.

the Heisenberg–Weyl commutation relations (3.7) of the ordinary quantum mechanical operators of position ξ, momentum $\hat{p}_\xi$, and oscillator Hamiltonian $H(\xi) = H(x)/\hbar\omega$, which satisfy the commutation relations $[\xi, H(\xi)] = i\hat{p}_\xi$ and $[H(\xi), \hat{p}_\xi] = i\xi$. In the third commutator there is a striking difference: While $[\xi, \hat{p}_\xi] = i$ in the continuous case, Eq. (6.1) holds in the finite case instead. This variant commutator indicates that we have a deformation of the original Heisenberg–Weyl algebra to the Lie algebra of rotations. From Eqs. (5.5) and (6.4) we see that these discrete, finite-momentum eigenfunctions are also given by Kravchuk polynomials. The function that is the finite Fourier–Kravchuk transform of a position eigenfunction at ξ_j will be a momentum eigenfunction of the same eigenvalue $p_j = x_j$, and, sensed at the points $\{x_k\}$, it will have values $\sim k_{\xi_k}(p_j)$.

Rotations around the 1 axis by an angle θ, generated by the position operator $J_1 = s \cdot$, may have an interesting physical realization. Their effect on the waveform samples $f(s_j)$, $j = 0, 1, \ldots, N$ is to multiply them by a phase $\exp(ij\theta)$, as if a wedge of higher refractive index were placed in the waveguide.

When the number of sensor points $N + 1$ is odd, the values transform as components of integer-spin irreducible representations: Four applications of $\hat{\mathbf{F}}$ return the data to their original values. When $N + 1$ is even, however, four applications will change the sign of all f_j's, and only the eighth power will bring the one-dimensional fractional finite Fourier–Kravchuk transform back to unity. The role of SU(2) as double cover of SO(3) thus echoes the double cover of metaplectic group Mp(2, $\mathfrak{R}$) over the linear symplectic group Sp(2, $\mathfrak{R}$) in ordinary paraxial Fourier optics.

7. FOURIER–KRAVCHUK VERSUS FOURIER EXPONENTIAL TRANSFORMS

The fractionalization of an idempotent operator is a rather straightforward mathematical problem that we illustrate with the common finite exponential Fourier

transform (see, for example, Ref. 17). This transform is defined through the unitary $M \times M$ matrix $\mathbf{E} = \|E_{j,k}\|$ of elements

$$E_{j,k} = \frac{1}{\sqrt{M}} \exp(-2\pi ijk/M). \tag{7.1}$$

This matrix is a fourth root of the $M \times M$ unit matrix: $\mathbf{E}^4 = \mathbf{1}$. Therefore functions $f(\mathbf{E})$ can be produced in the space spanned by the matrices $\mathbf{1}$, $\mathbf{E}$, $\mathbf{E}^2$, and $\mathbf{E}^3 = \mathbf{E}^{-1}$. When $M \to \infty$, this finite Fourier–exponential transform becomes, after appropriate rescaling of the row and column indices and taking limits for Hilbert spaces in the weak sense, the Fourier integral transform (4.1).

For real α, the property (5.4) of fractional Fourier transforms yields the following solution:

$$\mathbf{E}^\alpha = \frac{1}{4} \sum_{j=1}^{4} \exp[5i\pi(\alpha-j)/4] \frac{\sin\pi(\alpha-j)}{\sin\pi(\alpha-j)/4} \mathbf{E}^j. \tag{7.2}$$

Indeed, we can even find its logarithm: The Fourier matrix generator is a finite Hamiltonian matrix $\mathbf{H}$,

$$\mathbf{E} = \exp(i\pi\mathbf{H}/2),$$

$$\mathbf{H} = \frac{5}{2}\mathbf{1} + \frac{1}{2}(1+i)\mathbf{E} + \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}(1-i)\mathbf{E}^3. \tag{7.3}$$

The finite Fourier exponential transform (7.1) is (one of a manifold of[18]) matrices that diagonalize the $M \times M$ second-difference matrix $\mathbf{\Delta}$ of elements $\Delta_{j,k} = \delta_{j,k+1} - 2\delta_{j,k} + \delta_{j,k-1}$ (j and k counted modulo M). The complex form of the matrix diagonalizes also the left- and right-cyclic shift matrices and thus any circulating matrix. This is widely used in finite signal analysis because the M eigenfunctions of $\mathbf{\Delta}$ are the normal modes $\varphi_k(j) = E_{j,k}$, $k = 1, 2, \ldots, M$ of a finite Brillouin lattice, i.e., a collar of equal masses and springs. Time evolution of such a system is generated by the Hamiltonian $1/2\mathbf{\Delta}$ describing a medium that is invariant—homogeneous—under dihedral transformations (finite translations and inversions). There is no reason to expect *prima facie* that the normal modes of a finite homogeneous medium will approximate the modes of a waveguide properly. In a waveguide, lower modes are concentrated around the center of the guide; in contrast, the Brillouin modes have the same evolvement everywhere (they are thus very useful for time-series analysis of signals), so we can expect them to be poor approximations of stable modes in waveguides. Finally, note that the spectrum of $\mathbf{\Delta}$ is $\lambda_k = -4\sin^2(\pi k/M)$, $k = 1, 2, \ldots, M$ (Ref. 8), and it is *not* equally spaced. This precludes placing $\mathbf{\Delta}$ as generator in a (n undeformed) Lie algebra.

Difference equations are known to have a richer solution structure than differential equations. When the latter are $N \to \infty$ limits of the former, solution classes may coalesce. This occurs with the fractional Fourier–Kravchuk and Fourier exponential transforms, both of which yield the Fourier integral transform. If the physical model is a waveguide, the fractional Fourier–Kravchuk transform that we introduced represents the optical medium better. One decisively important prop-

erty of the Fourier exponential transform is the existence of the fast-Fourier-transform algorithm, discovered by Cooley and Tukey.[19] We are confident that a fast-Kravchuk-transform algorithm exists.

APPENDIX A: PROPERTIES OF KRAVCHUK FUNCTIONS

For $N = 1, 2, \ldots,$ the $N + 1$ symmetric Kravchuk polynomials $\{k_n(\xi, N)\}_{n=0}^{N}$ are an orthogonal set with respect to the binomial distribution over the first $N + 1$ nonnegative integers,[4,5] i.e.,

$$\sum_{j=0}^{N} \varrho(j)k_m(j, N)k_n(j, N) = d_n{}^2\delta_{m,n}, \tag{A1a}$$

where, for continuous ξ, the weight function is

$$\varrho(\xi) = \frac{1}{2^N}\binom{N}{\xi} = \frac{N!}{2^N\Gamma(\xi+1)\Gamma(N-\xi+1)}, \tag{A1b}$$

and these polynomials have normalization constants

$$d_n{}^2 = \frac{1}{2^{2n}}\binom{N}{n} = \frac{N!}{2^{2n}n!(N-n)!}. \tag{A1c}$$

These polynomials satisfy the three-term recurrence relation

$$[\xi - n - \tfrac{1}{2}(N-2n)]k_n(\xi, N)$$
$$= (n+1)k_{n+1}(\xi, N) + \tfrac{1}{4}(N-n+1)k_{n-1}(\xi, N) \tag{A2}$$

and are simply related to the Gauss hypergeometric function by

$$k_n(\xi, N) = \frac{(-1)^n}{2^n}\binom{N}{n}F\left(\begin{matrix} -n, -\xi \\ -N \end{matrix}; 2\right). \tag{A3}$$

From Eq. (A3) it is evident that $k_n(\xi, N)$ is indeed a polynomial of degree n in ξ and that the index n and the integer argument ξ can be exchanged, so that

$$\frac{(-1)^n}{2^n}\binom{N}{n}k_n(\xi, N) = \frac{(-1)^\xi}{2^\xi}\binom{N}{\xi}k_\xi(n, N). \tag{A4}$$

The Kravchuk polynomials and the binomial weight function have the appropriate limits,

$$\lim_{N\to\infty}(\tfrac{1}{2}N)^{-n/2}k_n(\tfrac{1}{2}N + \sqrt{\tfrac{1}{2}N}\xi, N) = \frac{1}{2^nn!}H_n(\xi), \tag{A5a}$$

$$\lim_{N\to\infty}\sqrt{\tfrac{1}{2}N}\varrho(\tfrac{1}{2}N + \sqrt{\tfrac{1}{2}N}\xi) = \frac{1}{\sqrt{\pi}}\exp(-\xi^2), \tag{A5b}$$

to the Hermite polynomials and the Gaussian weight function, respectively.

Kravchuk functions $\{\phi_m(\xi_j)\}_{m=0}^{N}$ are defined as an orthonormal set

$$\sum_{j=0}^{N} \phi_m(\xi_j, N)\phi_n(\xi_j, N) = \delta_{m,n}, \tag{A6}$$

with respect to the unit distribution over the symmetric set of $N + 1$ integers or half-integers ξ_j.[11] The Krav-

chuk functions are obtained from the Kravchuk polynomials by multiplying the polynomials by the square root of the weight function and of the normalization constant [cf. Eq. (3.1)] and translating the argument

$$\phi_n(\xi, N) = d_n^{-1} k_n(\tfrac{1}{2}N + \xi, N)\sqrt{\varrho(\tfrac{1}{2}N + \xi)},$$

$$0 \le n \le N, \qquad -\tfrac{1}{2}N \le \xi \le \tfrac{1}{2}N. \qquad (A7)$$

Notice carefully the domain of the Kravchuk functions. Although Eq. (A6) sums only over the $N + 1$ discrete points ξ_j between $-\tfrac{1}{2}N$ and $\tfrac{1}{2}N$, the functions $\phi_n(\xi, N)$ are well defined in the slightly larger interval,

$$-1 - \tfrac{1}{2}N \le \xi \le \tfrac{1}{2}N + 1, \qquad (A8)$$

and are zero at these end points. Finally, note that in the limit $N \to \infty$, the Kravchuk functions coincide with the harmonic oscillator functions (2.1),

$$\lim_{N \to \infty} (\tfrac{1}{2}N)^{1/4} \phi_n\left(\sqrt{\tfrac{1}{2}N}\,\xi, N\right) = \psi_n(\xi). \qquad (A9)$$

Kravchuk functions obey the following difference equation in the variable ξ, in consequence of Eqs. (A2)–(A4):

$$(\tfrac{1}{2}N - n)\phi_n(\xi, N) = \tfrac{1}{2}[\alpha(\xi)\phi_n(\xi - 1, N)$$
$$+ \alpha(\xi + 1)\phi_n(\xi + 1, N)], \qquad (A10a)$$

where

$$\alpha(\xi) = [(\tfrac{1}{2}N + \xi)(\tfrac{1}{2}N - \xi + 1)]^{1/2}. \qquad (A10b)$$

This can be written in the form of an eigenvalue equation:

$$\{-\tfrac{1}{2}[\alpha(\xi)\exp(-\partial_\xi) + \alpha(\xi + 1)\exp(\partial_\xi)]$$
$$+ \tfrac{1}{2}(N + 1)\}\phi_n(\xi, N) = (n + \tfrac{1}{2})\phi_n(\xi, N), \qquad (A11)$$

with the finite-shift operators $\exp(a\partial_\xi)f(\xi) = f(\xi + a)$. In Section 3 the term in brackets is interpreted as the finite quantum harmonic oscillator Hamiltonian $\mathbf{H}^{(N)}(\xi)$.

The computation of numerical values of the Kravchuk functions for the figures in this paper was done in MATH-EMATICA. We found it convenient to calculate separately the factors of Eq. (3.2): the square root of gamma functions from that package, multiplied by the tabulated values of the symmetric Kravchuk polynomial. The symmetric Kravchuk polynomials $\widetilde{k}_n(\xi, N) = k_n(\xi + 1/2N, N)$ were generated with the three-term recurrence relation

$$(n + 1)\widetilde{k}_{n+1}(\xi, N)$$

$$= \xi\widetilde{k}_n(\xi, N) - \tfrac{1}{4}(N - n + 1)\widetilde{k}_{n-1}(\xi, N). \qquad (A12)$$

The first eleven polynomials are the following:

$$\widetilde{k}_0(\xi, N) = 1, \qquad \widetilde{k}_1(\xi, N) = \xi,$$

$$\widetilde{k}_2(\xi, N) = \tfrac{1}{2}[\xi^2 - \tfrac{1}{4}N],$$

$$\widetilde{k}_3(\xi, N) = \tfrac{1}{3!}[\xi^3 - \tfrac{1}{4}(3N - 2)\xi],$$

$$\widetilde{k}_4(\xi, N) = \tfrac{1}{4!}[\xi^4 - \tfrac{1}{2}(3N - 4)\xi^2 + \tfrac{3}{16}N(N - 2)],$$

$$\widetilde{k}_5(\xi, N) = \tfrac{1}{5!}[\xi^5 - \tfrac{5}{2}(N - 2)\xi^3 + \tfrac{1}{16}(15N^2 - 50N$$
$$+ 24)\xi],$$

$$\widetilde{k}_6(\xi, N) = \tfrac{1}{6!}[\xi^6 - \tfrac{5}{4}(3N - 8)\xi^4 + \tfrac{1}{16}(45N^2 - 210N$$
$$+ 184)\xi^2 - \tfrac{15}{64}N(N^2 - 6N + 8)],$$

$$\widetilde{k}_7(\xi, N) = \tfrac{1}{7!}[\xi^7 - \tfrac{7}{4}(3N - 10)\xi^5 + \tfrac{7}{16}(15N^2 - 90N$$
$$+ 112)\xi^3 - \tfrac{3}{64}(35N^3 - 280N^2 + 588N$$
$$- 240)\xi],$$

$$\widetilde{k}_8(\xi, N) = \tfrac{1}{8!}[\xi^8 - 7(N - 4)\xi^6 + \tfrac{7}{8}(15N^2 - 110N$$
$$+ 176)\xi^4 - \tfrac{1}{16}(105N^3 - 1050N^2 + 2968N$$
$$- 2112)\xi^2 + \tfrac{105}{256}N(N^3 - 12N^2 + 44N$$
$$- 48)],$$

$$\widetilde{k}_9(\xi, N) = \tfrac{1}{9!}[\xi^9 - 3(3N - 14)\xi^7 + \tfrac{21}{8}(9N^2 - 78N$$
$$+ 152)\xi^5 - \tfrac{1}{16}(315N^3 - 3780N^2$$
$$+ 13356N - 13088)\xi^3 + \tfrac{9}{256}(105N^4$$
$$- 1540N^3 + 7308N^2 - 12176N$$
$$+ 4480)\xi],$$

$$\widetilde{k}_{10}(\xi, N) = \tfrac{1}{10!}[\xi^{10} - \tfrac{15}{4}(3N - 16)\xi^8 + \tfrac{21}{8}(15N^2$$
$$- 150N + 344)\xi^6 - \tfrac{5}{32}(315N^3 - 4410N^2$$
$$+ 18648N - 22976)\xi^4 + \tfrac{9}{256}(525N^4$$
$$- 9100N^3 + 52780N^2 - 115600N$$
$$+ 72064)\xi^2 - \tfrac{945}{1024}N(N^4 - 20N^3$$
$$+ 140N^2 - 400N + 384)].$$

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