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On the phase space description of quantum nonlinear dynamics

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Abstract

We analyze the difference between classical dynamics (geometric optics) and quantum dynamics (wave optics) by calculating the time history of the Wigner function for the simplest nonlinear Hamiltonians which are fourth-degree polynomials in p and q. It is shown that the moments of the Wigner function carry important information about the state of a system and can be used to distinguish between quasiclassical and quantum evolution.

1. Introduction

The standard formulation of quantum mechanics in either the Schrödinger or the Heisenberg pictures may create the impression that quantum and classical dynamics are completely different. However, there are representations [1] in which quantum dynamics seems to resemble classical statistical mechanics, and where the state of a quantum system may be represented by the quasiprobability distribution in the phase space of the corresponding classical system. Of course, there are at least two important differences. Firstly, quasiprobability distributions may take negative values (unlike the true probability distributions). Secondly, the classical distribution can be localized at a point in phase space, whereas the quantum distribution must always be spread in a finite phase volume, in agreement with uncertainty relations. Let us take an initial distribution which is consistent with the uncertainty relations and describes a real quantum particle. Then, *what is the difference between classical and quantum dynamics in phase space?*

The classical dynamical law is very simple. Every element of the phase space moves along the classical trajectory while preserving its volume. If at time $t = 0$ the probability to find a particle in a unit volume at the point q_0, p_0 was $W_{cl}(q_0, p_0)$, then at time t the probability distribution is

$$W_{cl}(q,p;t) = W_{cl}\left(q_0(q,p,t), p_0(q,p,t)\right), \qquad (1)$$

where $q(t), p(t)$ is the classical trajectory passing through the point q_0, p_0 at time $t = 0$. How much can this image help us to understand quantum dynamics? This question is important also for optical applications, since the optical Helmholtz equation in the paraxial approximation is reduced to the Schrödinger one. Then the distance along the optical axis plays the role of the time t in mechanics. For simplicity, we consider here the two-dimensional case and denote a coordinate perpendicular to the optical axis by q. Then the canonically conjugate momentum p describes the direction of the ray at the point q, t. The classical limit corresponds to geometric optics.

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2. Linear transformations

Linear homogeneous canonical transformations [2] are generated by Hamiltonians which are second-degree polynomials in p and q. They lead to linear equations of motion which are identical in classical and quantum mechanics. Therefore the solutions to the quantum equations (the Heisenberg operators) have a form similar to the classical trajectories $p(t), q(t)$. In optics, linear transformations describe Gaussian systems.

Among the various quasiprobability distributions, there is only one for which the linear quantum evolution law coincides with the classical one in Eq. (1) [3]. This is the Wigner function,

$$W(q,p;t) = 2 \int_{-\infty}^{+\infty} \mathrm{d}r\, \mathrm{e}^{2\mathrm{i}pr} \Psi^*(q+r;t)\Psi(q-r;t). \tag{2}$$

Here, the wave function $\Psi(q;t)$ is a solution of the Schrödinger equation in the coordinate representation.

We shall refer to the classical probability distribution (1) evolving from the initial conditions $W(p_0, q_0; t=0)$ as the "classical Wigner function". Therefore, the classical and quantum Wigner functions evolve identically in linear dynamics. The Wigner function provides the closest possible description of quantum and classical dynamics.

3. Nonlinear dynamics

We consider here two examples of nonlinear dynamics generated by the simplest nonlinear Hamiltonians which are the fourth-degree polynomials in p and q. Such operators describe in wave optics the third-order approximation to the paraxial regime; i.e., they lead to the aberrations of images in optical devices [4]. In the nonlinear case, the classical solution does not determine the quantum dynamics, since higher moments of p and q enter into the Heisenberg equations of motion. The mean values of these products (e.g., $\langle\{pq\}(t)\rangle$) become additional variables which are absent in the classical case.

We start with the Hamiltonian

$$H = p^4,$$

which corresponds in wave optics to the first correction to paraxial free propagation (spherical aberration); it also describes the first relativistic correction to the Schrödinger equation.

We show in Fig. 1 the classical and quantum evolution of the Wigner function for the Hamiltonian p^4 when the initial state is a vacuum coherent state (a Gaussian centered at the origin of the phase plane). After p^4 evolution, the resulting state is no longer a Gaussian, but is represented by a hill that rapidly spreads in q. The difference between the classical (Fig. 1c) and quantum (Figs. 1a, 1b) cases lies in the oscillations of the Wigner function which appear in the latter. They are seen in the level plots as small islands forming in the concave part of the main hill; their area is considerably smaller than the area of the vacuum state. The presence of these "quantum oscillations" in phase space clearly indicates the quantum character of the evolution (in a comparison with the quasiclassical evolution discussed in Section 2). The stronger the oscillations, the "more quantum" state we have. These oscillations are absent for generalized coherent Wigner functions described by a Gaussian (which, therefore, are quasiclassical states).

However, it is inconvenient to plot many graphs of the Wigner function for different time instants, to conclude how nonclassical (or how quasiclassical) the evolution is. We would like to have a numeric parameter which would allow us to distinguish between classical and quantum dynamics. There are, at least, two candidates to play this role. Firstly, it is the left-hand side of the Schrödinger–Robertson uncertainty relation [5],

$$\delta = \sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geqslant \tfrac{1}{4}, \tag{3}$$

where $\bar{q}$ and $\bar{p}$ are the mean values of the coordinate and momentum; $\sigma_{qq} = (\Delta q)^2$ and $\sigma_{pp} = (\Delta p)^2$ are the variances; $\sigma_{qp} = \langle qp + pq\rangle/2 - \bar{q}\bar{p}$ describes correlations between coordinate and momentum fluctuations. Here the brackets $\langle\ \ \rangle$ denote the average over a given quantum state. The equality in Eq. (3) holds for pure Gaussian states. Dodonov and Man'ko noticed that δ in Eq. (3) is invariant under linear canonical transformations both in quantum and classical mechanics [6]. For nonclassical states δ describes the growth of quantum fluctuations. Under nonlinear transformations, however, δ is not an invariant even

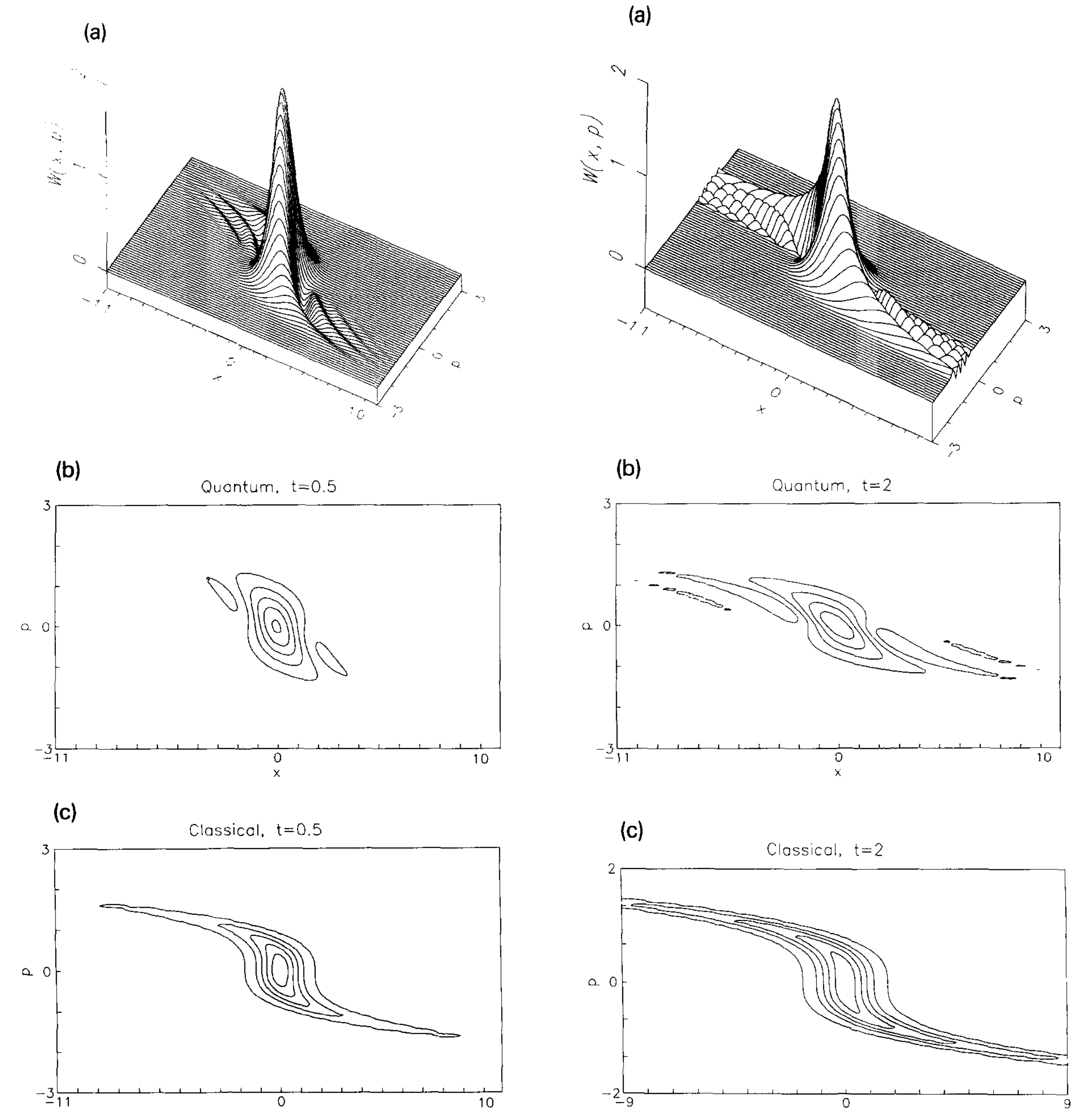

Fig. 1. Evolution of the quantum and classical Wigner functions for the Hamiltonian p^4. (a) Three-dimensional plots of the quantum Wigner functions for times $t = 0.5$ and $t = 2.0$; (b) level plots of the same quantum Wigner functions; (c) level plots of the classical Wigner functions for the same time instants.

in classical mechanics. The Dodonov–Man'ko parameter δ can thus be used to characterize the degree of nonlinearity of the system, rather than the degree of nonclassical behavior of the system, as was noted in Ref. [7]. This parameter is in fact very useful to describe the short time nonlinear behavior even if it does not feel global effects, such as Schrödinger cat states, which appear for longer times.

The difficulty in describing quantum fluctuations for Schrödinger cat states can be overcome by taking advantage of the entropy as a measure of the fluctuations. Since we have no true distribution in phase space, Wehrl [8] proposed to calculate the entropy using the nonnegative Q-function $Q(q,p)$,

$$S_Q = -\int Q \log Q \frac{dp\,dq}{2\pi}, \tag{4}$$

instead of a real distribution. The Wehrl entropy carries more precise information about the phase volume occupied by the quantum state and is especially convenient for the description of the Schrödinger cat states [9]. Unfortunately, S_Q is not invariant under the squeezing transformation. The reason is that the Q-function behaves nonclassically under squeezing transformation, generated by the quadratic generator $pq + qp$ (see, e.g., Ref. [10]).

We would have an appropriate measure of the quantum nature of the state if we could calculate the entropy using the Wigner function as a probability distribution. Unfortunately, this is impossible, since the Wigner function can have negative values. (Moreover, these negative values are known to be an important manifestation of the nonclassicality of the state.) However, we note that the moments of the Wigner function,

$$I_k(t) = \frac{k}{2^k}\int W^k(p,q;t)\frac{dp\,dq}{2\pi}, \quad k = 1,2,\ldots, \tag{5}$$

have the desired properties. The classical counterparts of these moments are invariant under any canonical transformation, as follows directly from the phase volume conservation. In quantum dynamics, these moments are preserved by any linear canonical transformation but are changed by nonlinear transformations. For all the quasiclassical states described by Gaussian wave functions, these moments (in our normalization) are equal to unity. The difference from unity may serve as a measure for the nonclassical nature of the state. On the other hand, the change in these moments in the course of nonlinear evolution reflects an extra growth of the quantum fluctuations in comparison with the corresponding classical dynamics; it also provides information about how closely the process can be described by the quasiclassical approximation.

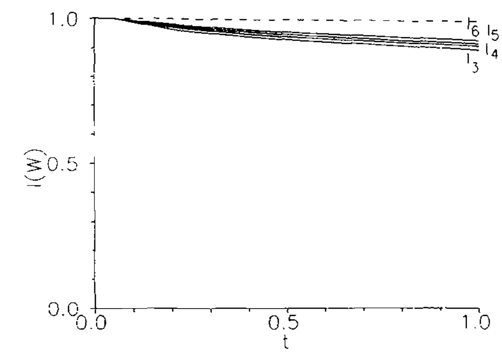

Fig. 2. Time evolution of the moments of the quantum Wigner function for the Hamiltonian p^4.

We show in Fig. 2 the time evolution of the moments of the Wigner function under the Hamiltonian p^4. It is clear that the normalization condition always implies $I_1 = 1$. Furthermore, $I_2 = 1$ for any pure state. (For mixed states described by the density matrix ρ, the second moment is equal to the purity of the state, $I_2 = \mathrm{tr}(\rho^2)$ [1].) I_2 is shown in Fig. 2 to test the quality of our numerical calculations. The decrease of the moments I_k for $k \geqslant 3$ reveals the difference between classical and quantum dynamics. We note that the behavior of the moments for the Hamiltonian p^4 is quite flat. This indicates that quasiclassical states remain a good approximation for quantum states.

4. Kerr medium

A successful model of active optical media in which self-interaction of the field takes place is the Kerr medium. Its Hamiltonian is a harmonic oscillator which describes a single quantized mode of an electromagnetic field with frequency ω, plus a self-interaction term with a coupling constant χ [11–13]. In quantum electrodynamics, the Hamiltonian is usually written in terms of the photon number operator $\hat{n} = a^\dagger a = p^2/2\omega + \omega q^2/2$ as

$$H = \omega\hat{n} + \chi\hat{n}^2, \tag{6}$$

where $\hbar = 1$. It is clear that the harmonic oscillator Hamiltonian and the total Kerr Hamiltonian have common eigenvectors. The photon number is conserved, but there is a nontrivial evolution of the phase of the field.

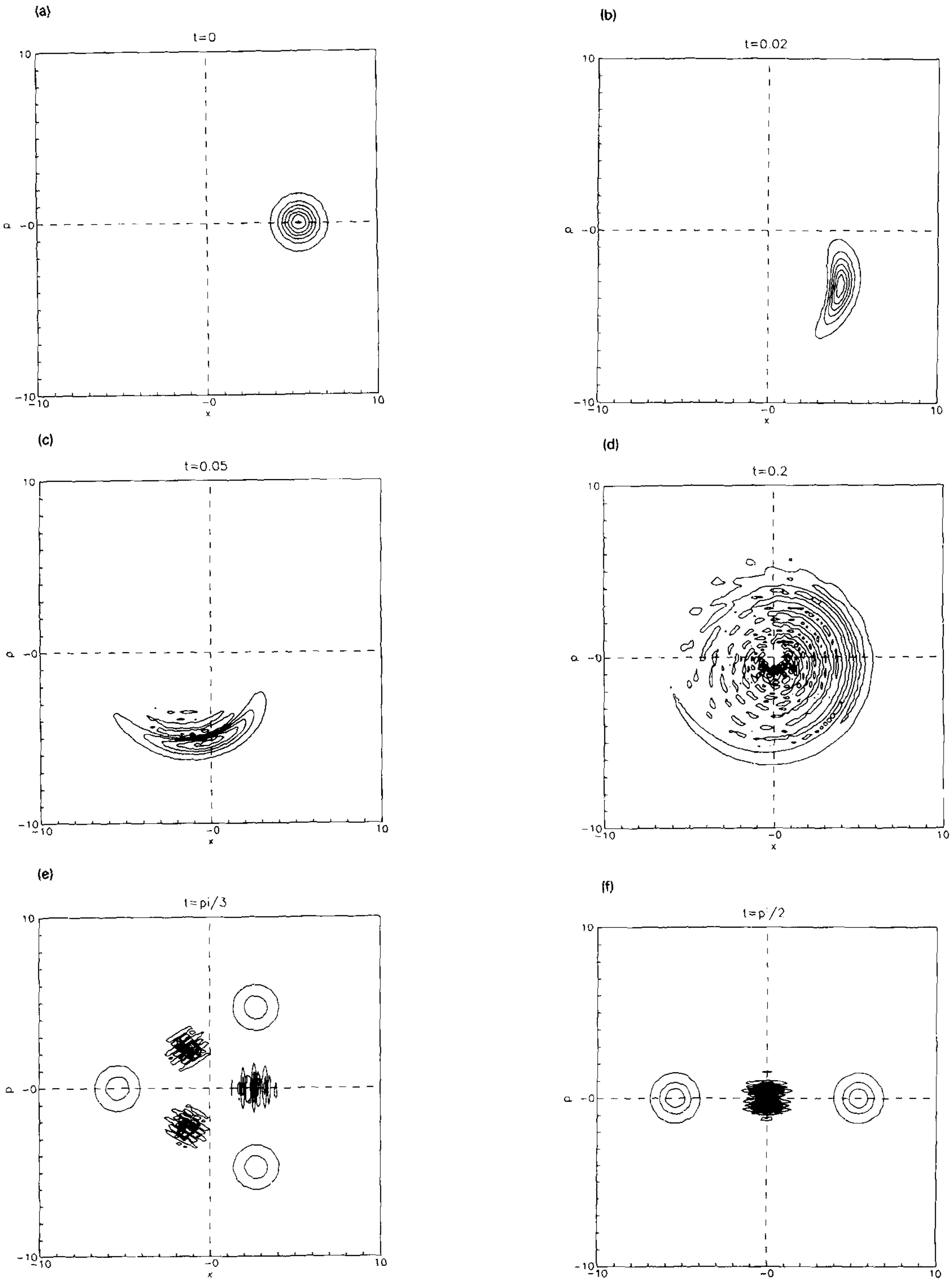

Fig. 3. Evolution of the quantum Wigner function for the Kerr medium with an initial coherent state described by a Poisson distribution with $\bar{n} = 30$. We see Schrödinger cats for times $t = \pi/3$ and $t = \pi/2$.

The time evolution of the Wigner function under the Kerr Hamiltonian (6) is shown in Fig. 3. The corresponding evolution of the moments is shown in Fig. 4. In these figures we choose $\chi = 1$. The first term in the Kerr Hamiltonian leads to a "fast" rotation of the graphs with angular frequency ω; we work in the interaction picture, which subtracts this rotation.

The initial Gaussian Wigner function is shown in Fig. 3a. For a small time $t = 0.02$, the Gaussian is stretched and rotated in the phase plane as shown in Fig. 3b; the moments – shown in Fig. 4 – are still ~ 1, so the state is still close to the semiclassical one. It is now squeezed in a definite direction in phase plane. This squeezing can clearly be seen in Fig. 3b. Note that in the graphs of the Q-function it is more difficult to visually notice squeezing, since the hills are "fatter". The Q-function overestimates the fluctuations in the squeezed states. Evolution by free propagation in a linear medium rotates the phase plane of the field by the bare harmonic oscillator Hamiltonian. One can use this additional rotation to achieve the best squeezing in the field coordinate or momentum [12].

As time advances – see Fig. 3c – it becomes clear that the hill is stretched along a circle (not along a straight line). As the phase spreads, the hill forms a crescent. The deformation of the top of the hill is still quasiclassical. However, the shape of the hill is already sufficiently bent for the "quantum oscillations" to appear. When the moments I_k are still of the order 1 in Fig. 4, these oscillations are weak and their contribution to the state volume is still small. The area of the hill increases slowly while the spread in phase grows faster, so we may expect an amplitude squeezing. This squeezing occurs slightly away from the radial direction, but it can be transformed into a radial (amplitude) squeezing if we shift the origin of the phase plane, so as to put it at the center of curvature of the crescent. Physically, this can be realized by placing the nonlinear Kerr medium inside of one arm of a Mach–Zehnder interferometer, as was proposed by Kitagawa and Yamamoto [11]. Strong squeezing in the photon number fluctuations can be achieved in this way. The Kerr amplitude squeezing can be further enhanced by taking as initial state an already squeezed state [14]. Note finally that the quantum oscillations are not visible in the graphs of the Q-function used in Ref. [11].

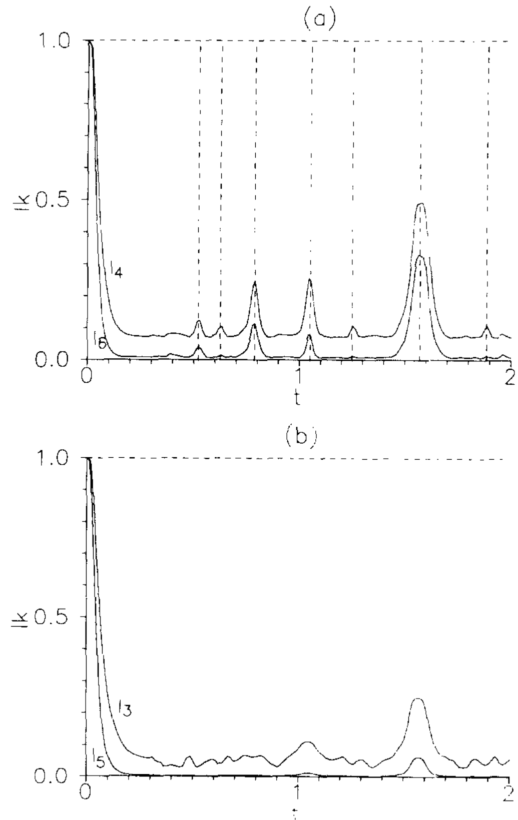

Fig. 4. Time evolution of the moments of the quantum Wigner functions for the Kerr Hamiltonian. (a) Even moments I_2 (dotted line), I_4 and I_6. (b) Odd moments I_1 (dotted line), I_3 and I_5. Dashed vertical lines correspond to time instants $\pi/6$, $\pi/5$, $\pi/4$, $\pi/3$, $2\pi/5$, $\pi/2$ and $3\pi/5$, when Schrödinger cats appear.

As time evolves further, Fig. 3d, the phase spread reaches 2π and the "quantum oscillations" become really strong. It also becomes clear that the reason for these oscillations is the self-interference in the phase space. One can say that different parts of the quasiprobability distribution create interference fringes when meeting each other. At some time instants the self-interference leads to standing waves along the circle. These waves are formed in the Kerr medium at times $\chi t = l\pi/m$, where l, m are integer numbers, $l < m \simeq \sqrt{\bar{n}}$. These are just the Schrödinger cats [15]; see Figs. 3e, 3f.

Most of the information contained in the Wigner function plots can be restored from the graphs of the moments $I_{3\text{-}6}$, Fig. 4. The time instants when we can expect squeezing or amplitude squeezing belong to the initial peak, when the moments are still close to unity. When the Wigner function shows complicated interference fringes, moments are kept in their lowest steady values. The "Schrödinger cat" times correspond to the well pronounced peaks of the moments. One can easily estimate the maximum values of the moments in these peaks for small numbers of the cat components.

5. Conclusions

The well-known difference between classical and quantum dynamics is connected with the phenomenon of self-interference in phase space for a quasi-periodic motion, which leads to the Schrödinger cat states [13,15]. It is a "global phenomenon" since the quantum state spreads over the whole phase volume allowed by the conservation laws. It reveals itself usually at times longer than the fundamental period of the oscillation of the system.

We have shown here that quantum nonlinear dynamics also differs from its classical counterpart for short times, when the state is still well localized in phase space. The higher moments of the Wigner function can be used as numerical measure of this difference, since the moments change in the quantum case but are constants in the classical picture.

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