Supersymmetry in Helmholtz optics *

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Abstract

A planar waveguide with a specific refractive index profile admits supersymmetry. Physically, optical supersymmetry describes two light beams of different colors that form a standing periodic interference pattern along the waveguide axis. The supersymmetry is exact in the paraxial approximation and broken beyond.

1. Introduction

There exists a correspondence between mechanics and optics (classical or quantum, geometric or wave), in the paraxial regime [1] and in the global one [2]. Consequently, many notions can be transported from one to the other, such as coherent and squeezed states (see, e.g., Refs. [3,4]). Here we describe an optical system that exhibits supersymmetry in the sense of supersymmetrical quantum mechanics (SUSY QM).

Supersymmetry in quantum mechanics connects two Hamiltonians with the same spectrum except for the ground state (unbroken supersymmetry [5], see Ref. [6], and references therein.) SUSY has been successfully applied in atomic [7] and nuclear physics [8]. The supersymmetric form of the Dirac equation in an external field [9–11] presents the Dirac equation as the square root of the Klein–Gordon equation; the kind of supersymmetry we have here is analogous and finds the square root of the Helmholtz

equation. The supersymmetric structure of Helmholtz optics describes light propagation in a waveguide and admits a very clear physical interpretation.

We consider optical waveguides along the z-axis, i.e., media that are inhomogeneous only in the x-direction. From the wave equation in $2+1$ dimensions for solutions of time frequency ν we have the Helmholtz equation

$$[\partial_z^2 + \partial_x^2 + \nu^2 n^2(x)/c^2]f(x, z) = 0 . \qquad (1)$$

In Section 2 we present the supersymmetric structure of (1), in Section 3 a physical reinterpretation, and some concluding remarks in Section 4. Note that until Section 3 we do not use the paraxial approximation.

2. Supersymmetric structure

We write a system of two first-order equations for a two-component wave function of the x- and z-coordinates,

$$\partial_z \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} ik & v_+ \\ v_- & -ik \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{2}$$

where k is a constant and

$$v_\pm \equiv \pm \partial_x + W(x),$$

with an arbitrary function $W(x)$. The second z-derivative involves the square of this 2×2 matrix,

$$\partial_z^2 \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} -k^2 + v_+ v_- & 0 \\ 0 & -k^2 + v_- v_+ \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Therefore we have two different Helmholtz equations for the components,

$$(\partial_z^2 + \partial_x^2 + k^2 - W_x - W^2)\Psi_1 = 0,$$
$$(\partial_z^2 + \partial_x^2 + k^2 + W_x - W^2)\Psi_2 = 0, \tag{3}$$

where $W_x \equiv dW/dx$. Now, for a given wavenumber κ in the z direction that is common to both wavefunctions, we write

$$\Psi_{1,2}(x, z) = \Phi_{1,2}(x)\, e^{-i\kappa z}. \tag{4}$$

Eqs. (3) then become the eigenvalue equations for the components,

$$(-\partial_x^2 - W_x + W^2)\Phi_1 = (k^2 - \kappa^2)\Phi_1,$$
$$(-\partial_x^2 - W_x + W^2)\Phi_2 = (k^2 - \kappa^2)\Phi_2. \tag{5}$$

Now we introduce the supersymmetric structure by considering the component Φ_1 as representing the "boson" and Φ_2 the "fermion" sectors of the supersymmetric Hamiltonian eigenfunctions. The supercharge operators are

$$Q_- = \begin{pmatrix} 0 & 0 \\ v_+ & 0 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} 0 & v_- \\ 0 & 0 \end{pmatrix}.$$

The supersymmetric Hamiltonian, defined as the anticommutator of the two supercharges, is

$$H_s = \{Q_-, Q_+\} = \begin{pmatrix} v_- v_+ & 0 \\ 0 & v_+ v_- \end{pmatrix}, \quad [H_s, Q_\pm] = 0.$$

The eigenvalue equations for H_s are then

$$v_- v_+ \Phi_1 = (k^2 - \kappa^2)\Phi_1,$$
$$v_+ v_- \Phi_2 = (k^2 - \kappa^2)\Phi_2.$$

These forms coincide with Eqs. (5) and have the same spectrum except for the ground state, which has

to be a normalizable solution of one of the two equations $v_+ \phi_1 = 0$ or $v_- \phi_2 = 0$. Therefore, SUSY connects solutions of Helmholtz equations of two different waveguides with index profiles

$$\nu^2 n^2(x)/c^2 = k^2 \mp W_x - W^2. \tag{6}$$

3. Physical reinterpretation

We now reinterpret these formulas in a second way to describe two *different* beams in the *same* waveguide. We choose

$$W(x) = wx. \tag{7}$$

The eigenvalue equations (5) are then two Schrödinger equations for harmonic oscillators with displaced energy levels,

$$(-\partial_x^2 + w^2 x^2)\Phi_{1,2} = (k^2 \mp w - \kappa^2)\Phi_{1,2}. \tag{8}$$

Meanwhile, seen as Helmholtz equations, Eqs. (8) correspond to

$$\nu^2 n^2(x)/c^2 = k^2 \mp w - w^2 x^2, \tag{9}$$

with the same $n(x)$. In the absence of material dispersion, n is independent of ν. This is an *elliptic* index-profile waveguide; it approximates the usual parabolic index-profile waveguide in the paraxial regime [12]. The eigenvalues of the operator on the left of (8) are well known to be

$$E_m = w(2m + 1), \quad m = 0, 1, 2, \dots . \tag{10}$$

We now find conditions for a supersymmetric pair of Helmholtz solutions in the same waveguide. Into this waveguide $n^2(x) = n_0^2 - n_1^2 x^2$ we inject two light beams with slightly different time frequencies

$$\nu_{1,2}^2 = \nu_0^2(1 \pm \epsilon),$$

and will call $\kappa_{1,2}$ the correspondingly slightly different space frequencies along the z-axis. To fulfill Eqs. (9) we substitute $k = n_0 \nu_0/c$, $w = n_1 \nu_0/c$ and find the conditions

$$\epsilon = \frac{n_1}{n_0 k} = \frac{w}{k^2}, \quad \frac{\nu_1^2 - \nu_2^2}{\nu_0^2} = 2\epsilon. \tag{11}$$

This determines the frequency shift in terms of the transverse index profile of the waveguide.

From the standard quantum harmonic oscillator solution we know that the Gaussian beam width in the x-direction is $\Delta x = (2w)^{-1/2}$. Therefore, the parameter $\epsilon = [2k^2(\Delta x)^2]^{-1} \sim$ (wavelength/beam width)^2 should be small. The two light beams with the frequencies $\nu_{1,2}$ obey the Helmholtz equations with

$$\nu_{1,2}^2 n^2(x)/c^2 = k^2 \mp w - w^2 x^2 (1 \mp \epsilon) .$$

When $\epsilon \to 0$, these two Helmholtz equations become a supersymmetric pair that is slightly broken by the term $\epsilon w^2 x^2$.

It is easy to see that SUSY has the same accuracy as the paraxial approximation: examination of Eqs. (8) and (10) shows that the z-propagation wavenumbers of the two beams are

$$\kappa_{1,2}^2(m) = k^2 - w[(1 \pm \epsilon)(2m+1) \pm 1] ,$$

$$m = 0, 1, ...$$

or

$$\frac{\kappa_{1,2}(m)}{k} \simeq 1 - \epsilon(m + \tfrac{1}{2} \pm \tfrac{1}{2}) - \tfrac{1}{2}\epsilon^2(m + \tfrac{1}{2} \pm \tfrac{1}{2})^2$$

$$\mp \epsilon^2(m + \tfrac{1}{2}) + ...$$

The term of order $\epsilon = w/k^2$ exhibits exact SUSY; except for the ground state $\kappa_2(0) = k$, the z-wavenumbers of the two beams coincide,

$$\kappa_1(m-1) = \kappa_2(m) + O(\epsilon^2), \quad m = 1, 2, ... , \qquad (12)$$

The two terms of order ϵ^2 respectively give a nonlinear correction to the paraxial approximation and break supersymmetry. Therefore, SUSY is exact in the paraxial approximation and is broken beyond this regime.

Thus, supersymmetry (12) connects light beams of different frequencies $\nu_{1,2}^2$ in the same waveguide (Eq. (11)), but having the same wavelength $2\pi/\kappa$ in the propagation direction z (Eq. (4)). These two beams form a stable interference pattern along the waveguide axis.

4. Conclusions

We have considered the propagation of light in a planar waveguide ruled by the two-dimensional Helmholtz equation. Two Helmholtz equations with different refractive indices (3) form a supersymmetric pair in the sense of SUSY QM.

Supersymmetry *also* describes the propagation of light beams of *different* colors in the *same* waveguide when the transversal index profile is elliptic. This profile determines the difference of light frequencies. SUSY is exact in the paraxial approximation, and broken beyond. In the paraxial regime, supersymmetric light beams of different colors have the same wavelength along the axis of the waveguide, giving rise to a stable interference pattern.

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