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Recursive Method for the Computation of the SO_n, $SO_{n,1}$, and ISO_n Representation Matrix Elements

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We find a procedure whereby the matrix elements of the finite $SO_{n,1}$ transformations (principal series) can be expressed as a single integral, over a compact domain, of two matrix elements of the SO_n subgroup and a multiplier. In this way we automatically obtain their classification by the canonical chain $SO_{n,1} \supset SO_n \supset \cdots \supset SO_2$. Analytic continuation yields the SO_{n+1} matrix elements in a recursive form. We obtain the asymptotic behavior of the boost matrix elements. The Inönü–Wigner contraction yields the ISO_n representation matrix elements classified by the chain $ISO_n \supset SO_n \supset \cdots \supset SO_2$.

1. INTRODUCTION

The unitary irreducible representation (UIR) matrix elements of the unimodular orthogonal (SO_n), pseudo-orthogonal ($SO_{n,1}$) and inhomogeneous orthogonal (ISO_n) groups have been a fertile field of research due to their repeated appearance in mathematical physics: For SO_2, they are the partial waves of a periodic function; for SO_3, they are the $D^J_{mm'}(\alpha, \beta, \gamma)$ functions. These and the Wigner $d^J_{mm'}(\beta)$ functions[1] have been so extensively used in angular momentum theory that no further remark is needed.

Bargmann's $d^l_{mm'}(\zeta)$ functions for[2] $SO_{2,1}$ have been used in Toller's cross-channel partial wave expansion.[3,4] The SO_4 d matrices[5] were used by Freedman and Wang in order to find the quantum numbers of the daughter Regge poles which belong to a given Toller pole. This, plus the important high-energy behavior of the corresponding scattering amplitude, were found by Sciarrino and Toller[4] using the $SO_{3,1}$ boost matrix elements $d^{M\lambda}_{mjj'}(\zeta)$.[6,7]

Going further, the $SO_{4,1}$ UIR matrix elements have also been calculated.[8–10] In particular, Ström[9] performed the contraction[11] $SO_{4,1} \to ISO_{3,1}$, whereby the D-matrix elements classified by the canonical chain become the matrix elements of Poincaré transformations[10,12] in the chain of subgroups which includes the homogeneous Lorentz group. The matrix elements of[13] SO_5 representations have found applications in nuclear physics,[14] and the theory of master analytic representations[15] has given a method of reaching higher groups.

The importance of the matrix elements of the general SO_n, $SO_{n,1}$, and ISO_n UIR's lies presently in mathematical physics: As group representations, they constitute an orthogonal and complete set of functions[16] on the group manifold, and any well-behaved, square-integrable function on the group can be expanded in terms of them.[16–18]

Thus far, however, they have remained as "certain" functions, some of whose relevant properties were known, but for which one could not write explicit expressions. The reason for this is not difficult to see: The straightforward procedure of obtaining them as eigenfunctions of the set of Casimir operators of the group and its subgroups involves setting up a set of simultaneous differential equations which, together with difference and recursion relations,[1,9,19] gives rise to rather involved expressions which are still under investigation[20] for SO_n and $SO_{n-1,1}$, $n > 5$.

Bargmann's[2] and Toller's[4] work, however, did not involve the solution of differential equations, but rather an integration over the compact subgroup. This was reduced further to a single integral, which has been successfully performed. In this article we set up a procedure which generalizes the above two cases. We shall work, however, only with the component of the group connected to the identity. We thus disregard the parity indices in the UIR labels.

In Sec. 2 we remind the reader how a complete and orthogonal set of functions on a homogeneous space X can be used to set up a multiplier representation of a group G whose action on X is known. The space X is here the SO_n group manifold. The properties and labels of a complete and orthogonal set of functions, the UIR matrix elements for SO_n classified by the canonical[21] chain, are reviewed in Sec. 3. The group G which acts on this space may be, however, larger then SO_n.

In Sec. 4, using a generalization of what is known in the literature as the Gell-Mann operator,[22] we can apply $G = SO_{n,1}$ in such a way that, while the transformations in the SO_n subgroup give rise to "rigid" mappings of the X manifold, the boosts in $SO_{n,1}$ generated by the Gell-Mann operator "deform" X.

In Sec. 5, the complete and orthogonal set of functions over SO_n introduced in Sec. 3 is used to set up a multiplier representation. The matrix elements of the Gell-Mann operator, proportional to the generalized Wigner $d^J_{LML'}(\zeta)$ functions for $SO_{n,1}$, are thus expressed as an integral over the SO_n subgroup (which is reduced to a single integral over one angle) of two UIR matrix elements of SO_n (simplified to the Wigner d functions for SO_n) and a multiplier.

The asymptotic behavior of the $SO_{n,1}$ d functions as $\zeta \to \infty$ and the contraction[11] $SO_{n,1} \to ISO_n$ can be seen already from the integral form. In fact, from the contraction of $SO_{n,1}$ we obtain the UIR matrix elements of ISO_n classified by the chain $ISO_n \supset SO_n \supset \cdots \supset SO_2$.

The geometrical meaning of the deformation effected on SO_n by the generators built through the Gell-Mann operator is shown, in Appendix A, to be but the natural action of the group $SO_{n,1}$ (in its Iwasawa decomposition $G = KAN$) on itself, modulo AN. A useful integral is calculated in Appendix B.

We want to emphasize that in our procedure

(a) the UIR matrix elements are classified by the canonical chain,

(b) several key properties are apparent from the integral form,

(c) the integration is performed over a compact domain and can be expressed in terms of a sum of products of trigonometric and hypergeometric functions.

We can point also to the possibilities of extending this method, taking a complete and orthogonal set of functions over other groups or homogeneous spaces—noncompact ones, for instance—and considering multiplier representations of a larger group of deformations of it, thus obtaining expressions for the representation matrix elements of noncompact groups classified by chains which can thus include noncompact subgroups.[4,23]

2. MULTIPLIER REPRESENTATIONS

In order to fix our notation, we shall make some well-known definitions.

Let X be a homogeneous space under the group of transformations G, and put $x_1, x_2, \cdots \in X$. A set of functions $\{\phi_n(x)\}$, $n \in N$, discrete, is *orthogonal* on X if

$$\int_X d\mu(x)\overline{\phi_n(x)}\phi_{n'}(x) = \delta_N(n, n'), \tag{2.1}$$

where $d\mu(x)$ is an appropriate measure on X and $\delta_N(n, n') = 0$ for $n \neq n'$ and will be detailed below.

Furthermore, the set $\{\phi_n(x)\}$ is *complete* on X if

$$\sum_{n\in N} \omega(n)\overline{\phi_n(x_1)}\phi_n(x_2) = \delta_X(x_1, x_2), \tag{2.2}$$

where $\omega(n)$ is the Plancherel weight on N; $\delta_X(x_1, x_2) = 0$ for $x_1 \neq x_2$ and is normalized in such a way that the integral (2.1) (which is a sum, if X is discrete) fulfills

$$\int_X d\mu(x_1) f(x_1)\delta_X(x_1, x_2) = f(x_2), \tag{2.3}$$

for any continuous test function f over X. The normalization of (2.1) and (2.2) can be arranged to be such that[17]

$$\sum_{n\in N} \omega(n)\tilde{f}_n \delta_N(n, n') = \tilde{f}_{n'}, \tag{2.4}$$

and hence $\delta_N(n, n') = [\omega(n)]^{-1}\delta_{n,n'}$.

Any well-behaved function f over X can be expanded in the complete and orthogonal set $\{\phi_n(x)\}$ as

$$f(x) = \sum_{n\in N} \omega(n)\tilde{f}_n\phi_n(x), \tag{2.5}$$

where $\tilde{f}_n = (\phi_n, f)_X$ is the scalar product between two functions on X, defined as

$$\begin{aligned}(f, f')_X &= \int_X d\mu(x)\overline{f(x)}f'(x) \\ &= \sum_{n\in N} \omega(n)\bar{\tilde{f}}_n\tilde{f}'_n = (\tilde{f}, \tilde{f}')_N.\end{aligned} \tag{2.6}$$

The action of G on X, $x \xrightarrow{g} x'(x, g)$, is assumed to be defined such that

$$x'(x''(x, g_1), g_2) = x'(x, g_1g_2) \quad \text{and} \quad x'(x, e) = x$$

for the unit e of the group. When $X = G$, this is satisfied if either $x'(x, g) = xg$ or $x'(x, g) = g^{-1}x$, but may be of a more general nature when $X \neq G$.

The action of G on $f(x)$ is defined through

$$f(x) \xrightarrow{g} U^{(\lambda)}(g)f(x) = M^{(\lambda)}(x, g)f(x'(x, g)), \tag{2.7}$$

where the multiplier[2] $M^{(\lambda)}(x, g)$ satisfies

$$M^{(\lambda)}(x, g_1)M^{(\lambda)}(x'(x, g_1), g_2) = M^{(\lambda)}(x, g_1g_2)$$

and $M^{(\lambda)}(x, e) = 1$ and does not vanish over $X \times G$.

A multiplier can be written in the form[2,16]

$$M^{(\lambda)}(x, g) = [\rho(x)/\rho(x'(x, g))]^\lambda, \tag{2.8}$$

where $\rho(x)$ is some function over X.

The requirement of *unitarity* of the representation

$$(U^{(\lambda)}(g)f, U^{(\lambda)}(g)f) = (f, f')$$

implies, through (2.6) and (2.7),

$$\frac{d\mu(x'(x, g))}{d\mu(x)} = |M^{(\lambda)}(x, g)|^2, \tag{2.9}$$

if we restrict the form of the multiplier and the possible values of λ in (2.8). In particular, if $X = G$ and $d\mu(x)$ is the Haar measure, the ratio (2.9) is unity and the multiplier may only be a phase.

We can construct a matrix representation of G as

$$D^\lambda_{nn'}(g) = [\omega(n)\omega(n')]^{\frac{3}{2}}(\phi_n, U^{(\lambda)}(g)\phi_{n'}), \quad (2.10)$$

where the rows and columns are labeled by the (discrete) index $n \in N$. We can check through (2.2) that (2.10) follows the group multiplication law and that

$$D^\lambda_{nn'}(e) = \delta_{n,n'}$$

while, if (2.9) is satisfied, the representation (2.10) is unitary, i.e.,

$$D^\lambda_{nn'}(g^{-1}) = \overline{D^\lambda_{n'n}(g)}.$$

At this stage, however, we cannot make any statement as to the irreducibility of (2.10) nor as to whether we can find all unitary representations in this way.

Next, we want to express the transformation (2.7) as generated by a Lie algebra of operators.[2] Assume $g(\zeta)$ belongs to a 1-dimensional subgroup of G parametrized by a variable ζ, whose generator is $N^{(\lambda)}$, i.e.,

$$U^{(\lambda)}(g(\zeta))f(x) = \exp(\zeta N^{(\lambda)})f(x). \quad (2.11)$$

The differential form of $N^{(\lambda)}$ is thus

$$N^{(\lambda)}f(x) = \frac{d}{d\zeta}[M^{(\lambda)}(x, g(\zeta))f(x'[x, g(\zeta)])]_{\zeta=0}. \quad (2.12)$$

When the multiplier $M^{(\lambda)}(x, g)$ is taken in its form (2.8), it is straightforward to see that $N^{(\lambda)}$ can be written as

$$N^{(\lambda)}f(x) = (N^{(0)} - \lambda Q)f(x), \quad (2.13)$$

where $N^{(0)}$ is the generator of the vector representation

$$\exp(\zeta N^{(0)})f(x) = f(x'[x, g(\zeta)]) \quad (2.14)$$

and

$$Q = [N^{(0)}\rho(x)]/\rho(x). \quad (2.15)$$

3. THE ORTHOGONAL GROUP

The n-dimensional (unimodular) orthogonal group SO_n has been extensively studied.[21] We need, however, a brief survey of its properties in order to define the problem.

We introduce the "Euler-angle" parameters[17,18,24] in SO_n (enclosing collective variables in braces) by

$$R_n(\{\theta\}^{(n)}) = R_{n-1}(\{\theta\}^{(n-1)})H_n(\{\theta^{(n)}\}),$$
$$H_n(\{\theta^{(n)}\}) = r_{n-1,n}(\theta^{(n)}_{n-1,n}) \times \cdots \times r_{23}(\theta^{(n)}_{23})r_{12}(\theta^{(n)}_{12}), \quad (3.1)$$

where $R_k \in SO_k$ and $r_{pq}(\theta)$ is a rotation by θ in the (p, q) plane; the ranges are $0 \le \theta_{12} < 2\pi$ and $0 \le \theta_{k-1,k} \le \pi$, $k = 3, 4, \cdots, n$. In this way, we express the SO_n manifold as the product of the SO_{n-1} manifold with [the $(n-1)$-dimensional surface of] the n-dimensional sphere S_n. Notice that $R_3(\alpha, \beta, \gamma) = r_{12}(\alpha)r_{23}(\beta)r_{12}(\gamma)$ differs from the more general usage which writes $r_{13}(\beta)$ as the middle factor. This will cause no inconvenience, however.

The Haar measure can now be split according to (3.1) as $dR_n = dR_{n-1}\, dH_n$, where

$$dH_n = \sin^{n-2}\theta_{n-1,n}\, d\theta_{n-1,n}\, dH_{n-1}, \quad dH_2 = d\theta_{12}. \quad (3.2)$$

From (3.2) and the ranges specified above, it can be seen that the volume of SO_n is $\mathrm{vol}\, SO_n = \mathrm{vol}\, SO_{n-1}S_n$, where $S_n = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$, and $\mathrm{vol}\, SO_2 = 2\pi$.

The basis vectors for the unitary irreducible representations of SO_n, classified by the canonical chain of subgroups $SO_n \supset SO_{n-1} \supset \cdots \supset SO_2$, are labeled with the Gel'fand–Tsetlin[21] kets

$$\left| \begin{matrix} J_{n,1} & J_{n,2} & \cdots & J_{n,[n/2]} \\ J_{n-1,1} & J_{n-1,2} & \cdots & J_{n-1,[(n-1)/2]} \\ \cdots & \cdots & \cdots & \cdots \\ J_{4,1} & J_{4,2} & & \\ J_{3,1} & & & \\ J_{2,1} & & & \end{matrix} \right\rangle, \quad (3.3)$$

where $[k/2]$ is the largest integer smaller or equal to $k/2$. This ket transforms as the $J_k \equiv \{J_{k,1}, J_{k,2}, \cdots, J_{k,[k/2]}\}$ UIR of SO_k, $k = 2, \cdots, n$, while the representation row is labeled $\overline{J_{k-1}} \equiv \{J_{k-1}, J_{k-2}, \cdots, J_2\}$. For the single-valued UIR's of SO_n, all J_{ab} are integers constrained by the "zigzag" inequalities

$$\begin{matrix} J_{k,1} & & & & \\ \vee\!\!| & & & & \\ J_{k-1,1} \ge J_{k,2} & & & & \\ & \vee\!\!| & & & \\ & J_{k-1,2} \ge J_{k,3} & & & \\ & & \vee\!\!| & & \\ & & J_{k-1,3} \ge \cdot & & \\ & & & \cdot \quad \cdot & \\ & & & & \cdot \end{matrix}, \quad k = 3, \cdots, n, \quad (3.4a)$$

which end, to the right of (3.3), as

$$\begin{matrix} \cdot & & \\ \cdot & & \\ \cdot & & \\ \ge J_{k,[k/2]-1} & & \\ \cdot & & \\ \cdot & \vee\!\!| & \\ \cdot & & \\ & J_{k-1,[(k-1)/2]-1} \ge J_{k,[k/2]} & \\ & & \vee\!\!| \\ & & |J_{k-1,[(k-1)/2]}|, \quad k \text{ odd}, \end{matrix} \quad (3.4b)$$

or

$$\begin{array}{c} \vdots \\ J_{k,[k/2]-1} \\ \vdots \\ \geq \\ \vdots \end{array}$$

$$J_{k-1,[(k-1)/2]} \geq |J_{k,[k/2]}|, \quad k \text{ even}. \tag{3.4c}$$

In order to economize subindices, we will agree on the following notation: Let J (resp. L and M) stand for J_n (resp. J_{n-1} and J_{n-2}), the UIR label of $G = SO_n$ (resp. $H = SO_{n-1}$ and $K = SO_{n-2}$). The row labels are $\bar{L} = \bar{J}_{n-1}$ (resp. $\bar{M} = \bar{J}_{n-2}$ and $\bar{N} = \bar{J}_{n-3}$), and hence $\bar{L} = \{L, \bar{M}\}$ and $\bar{M} = \{M, \bar{N}\}$; $\dim_n J$ (resp. $\dim_{n-1} L$ and $\dim_{n-2} M$) denote the dimension of the UIR. The scalar representation of SO_k, $k = 2, \cdots, n$, is $J_. = 0 = \{0, \cdots, 0\}$.

The representation D-matrices for SO_n are defined as

$$D^J_{\bar{L},\bar{L}'}(R_n(\{\theta\}^{(n)})) = \langle J\bar{L}|\, R_n(\{\theta\}^{(n)})\, |J\bar{L}'\rangle, \tag{3.5}$$

where we have written the ket and bra (3.3) horizontally. The generalized Wigner d matrices (to be calculated in Sec. 5) are defined through

$$d^J_{LML'}(\theta) = \langle JLM\bar{N}|\, r_{n-1,n}(\theta)\, |JL'M\bar{N}\rangle \tag{3.6a}$$

and are seen to be diagonal in M, the representation label of K, and independent of its row-label $\bar{N}$, since $r_{n-1,n}(\theta)$ commutes with all transformations in K. Similarly,

$$D^J_{L\bar{M},L'\bar{M}'}(R_{n-1}) = \delta_{L,L'} D^L_{\bar{M},\bar{M}'}(R_{n-1}) \tag{3.7}$$

is independent of J, the UIR label of G, and diagonal in that of H.

In particular, for SO_2 the $d^J_{LML'}(\theta)$ are $d^J(\theta) = e^{iJ\theta}$ i.e., the indices L, M, and L' are absent; for SO_3, the d matrices are $d^J_{LL'}(\theta)$, the usual Wigner d matrices[1] for rotations around the x axis. For SO_n, $n \geq 4$, we have the general expression (3.6).

Equations (3.1), (3.5), (3.6a), and (3.7) allow us to write (omitting arguments in an obvious way)

$$D^J_{\bar{L},\bar{L}'}(R_n) = \sum_{\bar{M}''} D^L_{\bar{M},\bar{M}''}(R_{n-1}) E^J_{L\bar{M}'',L'\bar{M}'}(H_n), \tag{3.8}$$

where

$$E^J_{L\bar{M}'',L'\bar{M}'}(H_n) = d^J_{LM''L'}(\theta^{(n)}_{n-1,n}) E^{L'}_{M''\bar{N}'',M'\bar{N}'}(H_{n-1}) \tag{3.9}$$

and $E^L(H_2(\theta)) = d^L(\theta)$. Thus we see that the D-matrix elements (3.8) can be expressed in terms of the Wigner d-matrix elements (3.6). Only the latter need therefore be calculated explicitly.

Most of the interesting properties of the D and d matrices can be found before their explicit calculation. Chief among them are the orthogonality and completeness relations (2.1) and (2.2), which read[16,17]

$$\int_G dR_n \overline{D^J_{\bar{L}_1\bar{L}_2}(R_n)} D^{J'}_{\bar{L}_1'\bar{L}_2'}(R_n) = \delta_{\bar{L}_1,\bar{L}_1'}\delta_{\bar{L}_2,\bar{L}_2'}\delta_{J,J'} \frac{\mathrm{vol}\, G}{\dim_{n'} J}, \tag{3.10}$$

$$\sum_J \frac{\dim_n J}{\mathrm{vol}\, G} \sum_{\bar{L},\bar{L}'} \overline{D^J_{\bar{L},\bar{L}'}(R_n)} D^J_{\bar{L},\bar{L}'}(R_n') = \delta_G(R_n, R_{n'}), \tag{3.11}$$

where $\delta_{\bar{L},\bar{L}'}$, etc., stand for a product of Kronecker δ's in the individual indices. The Plancherel weight is seen to be $w(J) = \dim_n J/\mathrm{vol}\, G$, while the role of the index n in (2.1) is taken by the triad of collective indices $(J, \bar{L}, \bar{L}')$.

From (3.8) and the splitting of the Haar measure, we can find the "orthogonality" relations for the E matrices as

$$\int_{S_n} dH_n \sum_{\bar{M}_1} \overline{E^J_{L_1\bar{M}_1,L_2\bar{M}_2}(H_n)} E_{L_1\bar{M}_1,L_2'\bar{M}_2'}(H_n) = \delta_{L_2,L_2'}\delta_{\bar{M}_2,\bar{M}_2'}\delta_{J,J'} \frac{\mathrm{vol}\, G}{\mathrm{vol}\, H} \frac{\dim_{n-1}(L_1)}{\dim_n(J)}, \tag{3.12}$$

while (3.2), (3.9), and (3.12) yield, for the d matrices,

$$\int_0^\pi \sin^{n-2}\theta\, d\theta \sum_M \frac{\dim_{n-2} M}{\mathrm{vol}\, K} \overline{d^J_{L_1ML_2}(\theta)}\, d^{J'}_{L_1ML_2}(\theta) = \delta_{J,J'} \frac{\dim_{n-1} L_1 \dim_{n-2} L_2}{\dim_n J} \frac{\mathrm{vol}\, G}{(\mathrm{vol}\, H)^2}. \tag{3.13}$$

Thus, while for SO_2 we have in $\{d^J(\theta)\}$ a complete and orthogonal set of functions, for SO_3 the Wigner $\{d^J_{LL'}(\theta)\}$ constitute an orthogonal set in the index J. The set is complete for $L = L' = 0$.[17,18] For SO_n, $n \geq 4$, the general result is (3.13), and this includes the sum over M-labels. Indeed, it is not difficult to show that $\{E_{0\bar{0}}, J_{L\bar{M}}(H_n)\}$ is an orthogonal and complete set of functions on $SO_{n-1}\backslash SO_n$ and the same result holds for $\{d^J_{000}(\theta)\}$ on $SO_{n-1}\backslash SO_n/SO_{n-1}$.[17]

We shall use (3.10), (3.11), and (3.13) in order to build the D matrices (2.10) after we have defined, in the next section, the group of transformations we wish to represent.

The parametrization of $R^P_n \in SO_{n-1,1}$ follows the definitions (3.1), (3.8), and (3.9) with

$$R^P_n = R_{n-1}H^P_n = R_{n-1}b_{n-1,n}(\zeta)H_{n-1}, \tag{3.14}$$

where $b_{n-1,n}(\zeta)$ is a boost in the $(n-1)$th direction through a hyperbolic angle ζ, $0 \leq \zeta < \infty$. The metric tensor has nonzero components $g_{11} = \cdots = g_{n-1,n-1} = -g_{nn} = 1$, and the $SO_{n-1,1}$ manifold is expressed as the product of the SO_{n-1} manifold and [the $(n-1)$-dimensional surface of] the n-dimensional hyperboloid. The Haar measure is $dR_n^P = dR_{n-1}\, dH_n^P$, where

$$dH_n^P = \sinh^{n-2}\zeta\, d\zeta\, dH_{n-1}. \tag{3.15}$$

The Gel'fand–Tsetlin kets for $SO_{n-1,1}$, classified by the canonical chain[25] $SO_{n-1,1} \supset SO_{n-1} \supset \cdots \supset SO_2$, can also be written in the form (3.3), where, for one-valued representations, all indices (except $J_{n,1}$) are integer and follow the "zigzag" inequalities (3.4). The domain of the index $J_{n,1} \equiv \lambda$ is the complex plane. We are at present interested in the principal series[25,26] of UIR's which corresponds to $\lambda = -\frac{1}{2}(n-1) + i\tau$, τ real, and the finite-dimensional (nonunitary) representations which lie at $\lambda = 0, 1, 2, \cdots$ and for which the rest of the inequalities (3.4) hold.

The symbol ${}^P d^J_{LML'}(\zeta)$ will be used for the boost matrix elements of the pseudo-orthogonal group defined, in analogy to (3.6a), as

$${}^P d^J_{LML'}(\zeta) = \langle JLM\bar{N}|\, b_{n-1,n}(\zeta)\, |JL'M\bar{N}\rangle. \tag{3.6b}$$

The orthogonality and completeness relations (3.10) and (3.11) must be carefully justified, as both $\dim_n J$ and vol G are infinite, and an integral with the Plancherel measure $dw(J)$ takes the place of the sum in (3.11). However, we shall not come to need them.

The ISO_{n-1} group is the semidirect product of T_{n-1}, the translation group in $n-1$ dimensions, and SO_{n-1}. Its elements are commonly written as (x, R_{n-1}), $x \in T_{n-1}$ and $R_{n-1} \in SO_{n-1}$, with the usual semidirect product law. The group manifold of ISO_{n-1} is thus the product of the $(n-1)$-dimensional Euclidean space and the SO_{n-1} manifold.

We shall parametrize the former in spherical coordinates, expressing $R_n^I \in ISO_{n-1}$ as

$$R_n^I = R_{n-1}H_n^I = R_{n-1}t_{n-1}(\xi)H_{n-1}, \tag{3.16}$$

where $t_{n-1}(\xi)$ is the translation along the $(n-1)$th direction, $0 \leq \xi < \infty$. In terms of the more usual notation,

$$\begin{aligned} R_n^I &= (0, R_{n-1})(t_{n-1}(\xi), I)(0, H_{n-1}) \\ &= (R_{n-1}t_{n-1}(\xi), R_{n-1}H_{n-1}). \end{aligned}$$

The Haar measure is $dR_n^I = dR_{n-1}\, dH_n^I$, where

$$dH_n^I = \xi^{n-2}\, d\xi\, dH_{n-1}. \tag{3.17}$$

Again, we can use the Gel'fand–Tsetlin kets[24] (3.3) where, for the one-valued representations, all indices (but $J_{n,1}$) are integer and follow (3.4). The index $J_{n-1} \equiv r$ is real.

The symbol ${}^I d^J_{LML'}(\xi)$ will be used for the radial translation matrix elements written, in analogy with (3.6a) and (3.6b), as

$${}^I d^J_{LML'}(\xi) = \langle JLM\bar{N}|\, t_{n-1}(\xi)\, |JL'M\bar{N}\rangle. \tag{3.6c}$$

The remarks following Eq. (3.6b) apply to ISO_{n-1}.

4. DEFORMATION OF THE GROUP MANIFOLD

Let $M_{\mu\nu}$ be the generator of a rotation $r_{\mu\nu}(\theta)$ in the (μ, ν) plane, $\mu, \nu = 1, 2, \cdots, n$, of the n-dimensional Euclidean space ($g_{\mu\nu} = \delta_{\mu\nu}$), i.e.,

$$\exp(\theta M_{\mu\nu}) = r_{\mu\nu}(\theta).$$

In terms of the Cartesian coordinates x_μ, they may be represented as

$$M_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}, \tag{4.1}$$

and can be checked to obey the commutation relations of the generators of an so_n algebra:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} \\ + g_{\sigma\nu}M_{\rho\mu} + g_{\rho\mu}M_{\sigma\nu}, \end{aligned} \tag{4.2a}$$

while

$$[M_{\mu\nu}, x_\rho] = g_{\nu\rho}x_\mu - g_{\mu\rho}x_\nu. \tag{4.2b}$$

Finally, we build the second-order Casimir operator of so_n as

$$\Phi^{(n)} = \frac{1}{2}\sum_{\mu,\nu} M_{\mu\nu}M_{\mu\nu}. \tag{4.3}$$

Now we construct[22]

$$M_{\mu,n+1} \equiv \frac{1}{2}[x_\mu, \Phi^{(n)}] = \frac{1}{2}\sum_\nu (M_{\mu\nu}x_\nu + x_\nu M_{\mu\nu}) \tag{4.4}$$

and check that the operators (4.4) together with (4.1) satisfy (4.2a) when we enlarge the range of the indices μ, ν, etc., to $1, 2, \cdots, n, n+1$, with $g_{\mu,n+1} = -x^2\delta_{\mu,n+1}$, where $x^2 = \sum x_\mu x_\mu$. Thus we have built, for $x^2 = 1$, an $so_{n,1}$ algebra (4.2a) out of the iso_n enveloping algebra.

Furthermore, we define the operators

$$M^{(\sigma)}_{\mu,n+1} \equiv M_{\mu,n+1} + \sigma x_\mu, \tag{4.5}$$

and check that (4.5) too, together with (4.1), generates an $so_{n,1}$ algebra whose second-order Casimir operator is

$$\begin{aligned} \Phi^{(n,1)}(\sigma) &= \Phi^{(n)} - \sum_\mu M^{(\sigma)}_{\mu,n+1}M^{(\sigma)}_{\mu,n+1} \\ &= \Phi^{(n,1)}(0) - \sigma^2 x^2. \end{aligned} \tag{4.6}$$

We assume that we already know the representation D-matrices of the SO_n group, and now we want to construct a representation (2.10) for the $SO_{n,1}$ group. As we need only the ${}^Pd$ matrices for $SO_{n,1}$, we consider the boost generator

$$M^{(\sigma)}_{n,n+1} = x_n \sum_\mu x_\mu \frac{\partial}{\partial x_\mu} - x^2 \frac{\partial}{\partial x_\mu} + [\tfrac{1}{2}(n-1) + \sigma]x_\mu, \tag{4.7}$$

which we have written explicitly in terms of the Cartesian coordinates through (4.1) and (4.4).

We introduce the spherical coordinate system[24] in the n-dimensional space which is best suited for the description of the SO_n group manifold, as

$$x(\{\theta^{(n)}\}) = R_n^{-1}(\{\theta\}^{(n)})x(\{0\}), \tag{4.8a}$$

where $x(\{0\}) = (0, \cdots, 0, x)$, i.e., $x_n = x\cos\theta_{n-1}$, and

$$x_{n-p} = x \sin\theta_{n-1} \cdots \sin\theta_{n-p}\cos\theta_{n-p-1}, \quad p = 1, \cdots, n-1, \tag{4.8b}$$

where we have put $\theta_q \equiv \theta^{(n)}_{q,q+1}$ for economy.

When acting on functions $f(\{\theta\})$ of the angular coordinates, the operator (4.7) can be written as

$$M^{(\sigma)}_{n,n+1} = \sin\theta \frac{\partial}{\partial\theta} + [\tfrac{1}{2}(n-1) + \sigma]\cos\theta, \tag{4.9}$$

where we have put $\theta \equiv \theta_{n-1}$ for short. (In this connection, see Appendix A.)

In order to identify (4.9) as the generator $N^{(\lambda)}$ of a multiplier representation (2.11) in its form (2.13), we set

$$-\lambda = \tfrac{1}{2}(n-1) + \sigma, \tag{4.10}$$

$$N^{(0)} = \sin\theta\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\omega}, \tag{4.11}$$

where $\omega = \ln\tan(\tfrac{1}{2}\theta)$ and $Q = \cos\theta$, whereby

$$\rho(\theta) = \sin\theta \tag{4.12}$$

provides an appropriate construction of the multiplier (2.7).

The transformation in the parameter ω brought about by the operator $\exp(\zeta N^{(0)})$ is a translation by ζ, i.e., $\omega \to \omega' = \omega + \zeta$. Hence, in the parameter[27] θ

$$\tan\tfrac{1}{2}\theta \to \tan\tfrac{1}{2}\theta' = \tan\tfrac{1}{2}\theta\, e^{\zeta}. \tag{4.13}$$

Therefore, we can state that, while the operators (4.1) generate *rigid* rotations of the space (4.8b) and therefore on the group manifold of SO_n through (4.8a), the operator (4.9) is the generator of a *deformation* of the group manifold which affects only the parameter $\theta \equiv \theta^{(n)}_{n-1,n}$ through (4.13). (See Appendix A.)

We shall work with functions on the SO_n manifold, rather than on the n-sphere (the homogeneous space $SO_{n-1}\backslash SO_n$), since not all representations of the former can be realized on the latter.[18]

5. THE d-MATRIX ELEMENTS

As was stated in Sec. 3, we can use the set of $D^J_{L_1L_2}(R_n)$ functions which is orthogonal and complete, in order to build a representation of the group of transformations on SO_n through the general procedure (2.10). The specific transformation we are interested in is the deformation (4.13) with the multiplier (2.8) built out of (4.12). Thus, we shall build the D matrices of $SO_{n,1}$ with rows and columns specified by the UIR labels of its canonical chain.

We choose the action on the group to be $g''(g', g) = g'g$. [See text around Eq. (2.9).] Then, for $R_n \in SO_n$,

$$(D^{J_1}_{L_1,L_1'}, U^{(\lambda)}(R_n)D^{J_2}_{L_2,L_2'}) = \delta_{L_1,L_2}\delta_{J_1,J_2}\frac{\text{vol}\,G}{\dim_n J} D^{J_1}_{L_1',L_2'}(R_n), \tag{5.1}$$

because of (2.7), (2.9), and (3.10).

Hence, we can write, for any (fixed, allowed) L,

$$D^J_{L_1,L_2}(R_n) = \frac{\dim_n J}{\text{vol}\,G}(D^J_{L,L_1}, U^{(\lambda)}(R_n)D^J_{L,L_2}), \tag{5.2}$$

incorporating the requirements of (2.10) and the independence of λ and L, as in (3.7).[28]

Using the operator (4.9) and Eqs. (4.10)–(4.13), we have

$$\begin{aligned}(D^{J_1}_{L_1,L_1'}, &\exp(\zeta N^{(\lambda)})D^{J_2}_{L_2,L_2'}) \\ &= \int_R dR_n \overline{D^{J_1}_{L_1L_1'}(R_{n-1}\nu_{n-1,n}(\theta)H_{n-1})}\left(\frac{\sin\theta}{\sin\theta'}\right)^\lambda \\ &\quad\times D^{J_2}_{L_2L_2'}(R_{n-1}\nu_{n-1,n}(\theta')H_{n-1}) \\ &= \delta_{L_1,L_2}\delta_{L_1',L_2'}\frac{(\text{vol}\,H)^2}{\dim_{n-1}L_1 \cdot \dim_{n-1}L_1'}\sum_M \frac{\dim_{n-2}M}{\text{vol}\,K} \\ &\quad\times \int_0^\pi \sin^{n-2}\theta\, d\theta \overline{d^{J_1}_{L_1ML_1'}(\theta)}\left(\frac{\sin\theta}{\sin\theta'}\right)^\lambda d^{J_2}_{L_1ML_1'}(\theta'),\end{aligned} \tag{5.3}$$

where we have used (3.8) and (3.9), as well as (3.10) and (3.12), for SO_{n-1}.

In line with (5.2) and (3.6b), we set

$${}^Pd^{(\lambda,L)}_{JL'J'}(\zeta) = \frac{(\dim_n J\cdot\dim_n J')^{\frac{1}{2}}}{\text{vol}\,G} \times [D^J_{LL'}, \exp(\zeta N^{(\lambda)})D^{J'}_{LL'}], \tag{5.4}$$

and we can check that

(a) the representation property is fulfilled, i.e.,

$$\sum_{J''} {}^{P}d^{\{\lambda L\}}_{JL'J''}(\zeta_1)\, {}^{P}d^{\{\lambda L\}}_{J''L'J'}(\zeta_2) = {}^{P}d^{\{\lambda L\}}_{JL'J'}(\zeta_1 + \zeta_2),$$

(b) due to (3.13),

$${}^{P}d^{\{\lambda L\}}_{JL'J'}(0) = \delta_{J,J'}.$$

Hence (5.4) indeed provides a representation of $SO_{n,1}$.

Our method of construction gives automatically the Gel'fand–Tsetlin classification of the representation D matrices. Indeed, from (3.4) (recall $J_{n,q} \equiv J_q$, $J_{n-1,r} \equiv L_r$),

$$J_q \geq L_q \geq J_{q+1}, \quad q = 1, \cdots, [n/2] - 1,$$

and

$$0 \leq L_{[(n-1)/2]} \geq |J_{[n/2]}|, \quad n \text{ even}, \tag{5.5}$$

or

$$0 \leq J_{[n/2]} \geq |L_{[(n-1)/2]}|, \quad n \text{ odd},$$

and similar relations between L and J', L' and J, and L' and J'.

Thus, the $SO_{n,1}$ UIR labels in (5.4),

$$I = \{I_1, \cdots, I_{[(n+1)/2]}\} \equiv \{\lambda, L\} = \{\lambda, L_1, \cdots, L_{[(n-1)/2]}\}, \tag{5.6}$$

also fulfill (5.5) with respect to its row indices (J, J') through the replacements $n \to n + 1$, $J \to I$, and $L \to J$, except for the index $I_1 = \lambda$ which is, so far, allowed to be complex and unrestricted. This we shall now investigate.

In order to fulfill the unitarity condition (2.9), we notice that (4.13) yields

$$\frac{\sin^{n-2}\theta'}{\sin^{n-2}\theta}\frac{d\theta'}{d\theta} = \left(\frac{\sin\theta'}{\sin\theta}\right)^{n-1}, \tag{5.7}$$

and thus, if we demand that $-\lambda - \bar{\lambda} = n - 1$, we shall satisfy (4.10) when

$$-\lambda = \tfrac{1}{2}(n-1) + i\tau, \quad \tau \text{ real}. \tag{5.8}$$

This provides the principal series of UIR's of $SO_{n,1}$.[20] As the rest of the labels (i.e., $I_2, \cdots, I_{[(n+1)/2]}$) satisfy (3.4)–(5.5), they are, indeed, UIR labels of the $SO_{n,1}$ representation matrices.

We are also interested in the finite-dimensional nonunitary irreducible representations of $SO_{n,1}$ since, when we perform the Weyl continuation [i.e., when we consider the parameter ζ in (4.13) as $\zeta = i\theta_n$, $0 \leq \theta_n \leq \pi$], we obtain the UIR matrices of SO_{n+1}.

It is known that the so_{n+1} second-order Casimir operator (4.3) has eigenvalues

$$\phi^{(n+1)} = -l(l + n - 1) + \text{integer}, \quad l = 0, 1, 2, \cdots. \tag{5.9}$$

Under the substitution $-l = \frac{1}{2}(n-1) + \sigma$, Eq. (5.9) takes the form

$$\phi^{(n+1)} = [\tfrac{1}{2}(n-1)]^2 - \sigma^2 + \text{integer},$$
$$\sigma = -\tfrac{1}{2}(n-1), -\tfrac{1}{2}(n-1) - 1, \cdots, \tag{5.10}$$

which has the same dependence on σ as (4.6).

The values of σ in (5.10) give the UIR's of SO_{n+1}, and we can now identify l in (5.9) with λ in (5.6) and see that (5.4), with $\lambda = 0, 1, 2, \cdots$, will provide the UIR representations of SO_{n+1}.

The explicit form of the d matrices (5.4) is, from (5.3),

$${}^{P}d^{\{\lambda,L\}}_{JL'J'}(\zeta) = \frac{(\dim_n J \cdot \dim_n J')^{\frac{1}{2}}}{\dim_{n-1} L \dim_{n-1} L'} \frac{(\text{vol } H)^2}{\text{vol } G \text{ vol } K} \sum_M \dim_{n-2} M \times \int_0^\pi \sin^{n-2}\theta\, d\theta \overline{d^J_{LML'}(\theta)} \left(\frac{\sin\theta}{\sin\theta'}\right)^\lambda d^{J'}_{LML'}(\theta'). \tag{5.11}$$

The expressions for $\dim_n J$, $\dim_{n-1} L$, and $\dim_{n-2} M$ can be found from the branching relations (3.4), but are, in general, rather cumbersome to express in closed form [for SO_2, $\dim_2 J = 1$; for SO_3, $\dim_3 J = 2J + 1$; for SO_4, $\dim_4 (J, 0) = J^2$]. Therefore, we shall leave them thus indicated. The second factor in (5.11) is

$$\frac{(\text{vol } H)^2}{\text{vol } G \cdot \text{vol } K} = \frac{S_{n-1}}{S_n} = \frac{\Gamma(\frac{1}{2}n)}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}(n-1))}. \tag{5.12}$$

Equation (5.11) provides thus, when $\zeta = i\theta_n$ and $\lambda = 0, 1, 2, \cdots$, an inductive procedure whereby the d matrices of SO_{n+1} can be found in terms of those of SO_n.

The first step of the procedure is SO_2, where we have

$$d^J(\theta) = e^{iJ\theta} \quad \text{using} \int_0^{2\pi} \text{since} \quad 0 \leq \theta < 2\pi; \tag{5.13a}$$

but it can also be taken to be SO_4 (since the d matrices for $SO_{2,1}$, SO_3, and $SO_{3,1}$ are well known), since, due to the local isomorphism $SO_4 \simeq SO_3 \times SO_3$, we have the simple form[5]

$$d^{I_1 I_2}_{JLJ'}(\theta) = \sum_m C(\tfrac{1}{2}(I_1 + I_2), \tfrac{1}{2}(I_1 - I_2), J; \tfrac{1}{2}(L + m), \tfrac{1}{2}(L - m), L) \times C(\tfrac{1}{2}(I_1 + I_2), \tfrac{1}{2}(I_1 - I_2), J'; \tfrac{1}{2}(L + m), \tfrac{1}{2}(L - m), L) e^{im\theta}, \tag{5.13b}$$

where $C(\cdots)$ are the SO_3 Clebsch–Gordan coefficients.

The general form of the $d^I_{JLJ'}(\theta)$ functions for SO_n and $SO_{n,1}$ will not be attempted here beyond the recursion formula (5.11), which may be of more practical use.

The simplest cases, SO_2, $SO_{2,1}$,[2] SO_3,[1] $SO_{3,1}$,[4,6] and[5] SO_4 have been calculated as finite sums of trigonometric (and hypergeometric) functions. The expression (5.13b) for SO_4 is noteworthy for its simplicity. The appearance of 3-J symbols in the coefficient in (5.13b) and Holman's result[13] for SO_5, which involves 9-J symbols, suggests that a relatively compact expression for the general d matrices may yet be found.

The recursion formula (5.11) can be used to determine the asymptotic behavior of ${}^Pd^I_{JLJ'}(\zeta)$ as $\zeta \to \infty$. Indeed, from (4.13), we have

$$\sin\theta/\sin\theta' = \cosh\zeta - \sinh\zeta\cos\theta,$$

and hence

$$ {}^Pd^{\{\lambda,L\}}_{JL'J'}(\zeta) \xrightarrow[\zeta\to\infty]{} e^{\lambda\zeta}\Delta^{\lambda L}_{JL'J'}, \tag{5.14a}$$

where

$$\Delta^{\lambda L}_{JL'J'} = \frac{(\dim_n J \dim_n J')^{\frac12}}{\dim_{n-1} L \dim_{n-1} L'} \frac{\Gamma(\frac12 n)}{\pi^{\frac12}\Gamma(\frac12(n-1))} \times \sum_M \dim_{n-2} M d^{J'}_{LML'}(\pi) \times \int_0^\pi \sin^{n-2}\theta\, d\theta \overline{d^J_{LML'}(\theta)} \sin^{2\lambda}\tfrac12\theta. \tag{5.14b}$$

In Appendix B we perform an integral which may be helpful in the evaluation of (5.11) and (5.14).

The ${}^Id$ matrices for the ISO_n groups are found as the matrix elements (3.6c) of the transformation generated by $x_n = r\cos\theta$, $\theta \equiv \theta^{(n)}_{n-1,n}$. In its form (2.7),

$$U^{(\gamma)}(\xi)f(\{\theta\}) = \exp(i\xi\gamma\cos\theta)f(\{\theta\}) \tag{5.15}$$

is unitary [i.e., satisfies (2.9)] for r real. Notice, however, that it produces no deformation of the group and thus cannot be put in the form (2.8).

The second-order Casimir operator $R^2 = \sum x_\mu x_\mu$ has the eigenvalue r^2, whence we can write, as for (5.11),

$$ {}^Id^{\{\gamma,L\}}_{JLJ'}(\xi) = \frac{(\dim_n J\dim_n J')^{\frac12}}{\dim_{n-1} L\dim_{n-1} L'} \frac{(\mathrm{vol}\, H)^2}{\mathrm{vol}\, G\cdot \mathrm{vol}\, K} \times \sum_M \dim_{n-2} M \int_0^\pi \sin^{n-2}\theta\, d\theta\overline{d^J_{LML'}(\theta)} \times \exp(i\gamma\xi\cos\theta) d^{J'}_{LML'}(\theta). \tag{5.16}$$

Furthermore, we can check that (5.15) and (5.16) are indeed contractions[11] of the corresponding expressions (4.13) and (5.11), i.e., that

$$ {}^Pd^{\{\lambda,L\}}_{JL'J'}(\zeta) \xrightarrow[(i\lambda\zeta=\gamma\xi)]{\lambda\to i\infty} {}^Id^{\{\gamma,L\}}_{JL'J'}(\xi),$$

since

$$(\sin\theta/\sin\theta')^\lambda \xrightarrow[(i\lambda\zeta=\gamma\xi)]{\lambda\to i\infty} \exp(i\xi\gamma\cos\theta).$$

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APPENDIX A

The Iwasawa decomposition[29] of[8c] $G = SO_{n,1}$ can be written uniquely as $g = k(\{\theta\})\cdot a(\eta)\cdot n(\xi)$, that is,

$$g = \begin{pmatrix} k(\{\theta\}) & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 1\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0 & 0\\ 0 & \cosh\eta & \sinh\eta\\ 0 & \sinh\eta & \cosh\eta\end{pmatrix} \times \begin{pmatrix}\mathbb{1} & -\xi & \xi\\ \underline{\xi} & 1-\Xi & \Xi\\ \underline{\xi} & -\Xi & 1+\Xi\end{pmatrix},$$

where $k(\{\theta\}) \in K = SO_n$, the maximal compact subgroup of G parametrized as in (3.1), whereby we have $k_{nn} = \cos\theta$ (using $\theta \equiv \theta^{(n)}_{n-1,n}$), $a(\eta) \in A$, the Abelian subgroup of G which corresponds to the boost in the nth direction in (3.14), $n(\xi) \in N$, the nilpotent subgroup of G, where ξ is the column vector $(\xi_1, \xi_2, \cdots, \xi_{n-1})$, $\underline{\xi}$ its transpose, and

$$\Xi = \tfrac12(\xi_1^2 + \cdots + \xi_{n-1}^2).$$

Consider now the transformation induced by

$$g \xrightarrow{\zeta} g' = a^{-1}(\zeta)g = k(\{\theta'\})a(\eta')n(\xi'). \tag{A1}$$

Direct calculation yields

$$\cos\theta \xrightarrow{\zeta} \cos\theta' = (\cosh\zeta\cos\theta - \sinh\zeta) \times (\cosh\zeta - \sinh\zeta\cos\theta)^{-1}, \tag{A2a}$$

$$\exp\eta \xrightarrow{\zeta} \exp\eta' = \exp\eta(\cosh\zeta - \sinh\zeta\cos\theta). \tag{A2b}$$

Notice that (A2a) is the same transformation as (4.13). The infinitesimal generator [as in (2.1) and (2.2)] of the transformation (A2) on the space of functions on G/N is thus

$$N = \sin\theta \frac{\partial}{\partial\theta} - \cos\theta \frac{\partial}{\partial\eta}. \tag{A3}$$

The parameter η of the Abelian subgroup $A \subset G$ does not appear as such in (A3) and thus[30] $\partial/\partial\eta$ commutes with (A3) as well as with all the generators of K and of N [whose action can be written in terms of those of K and A through (3.1)]. Hence, we can choose that subspace of functions on $G/AN \cong K$ which corresponds to an eigenvalue λ under $\partial/\partial\eta$ and now the operator (A3) takes exactly the form (4.9)–(4.10), which was obtained from the Gell-Mann formula (4.4)–(4.5) on the space K. The deformation which the latter produces on K is thus seen to be the same as the natural action of $G = KAN$ on itself (modulo AN). [Notice, however, that this is *not* true, modulo H_n, had we taken the decomposition (3.1).]

Through a suitable choice of λ, the operator (A3) can be made anti-Hermitian,[31] and we know a complete and orthogonal set of functions on K: the D matrices for SO_n. Although it is suggestive to consider a similar set on $G/AN \cong K$ as a subset of those on G, the theory of complete sets of functions on homogeneous spaces with noncompact stabilizers is lacking. Some of the difficulties have been pointed out in Ref. 18.

APPENDIX B

Before solving the integrals in (5.11), (5.14b), and (5.16), we have to decide in which form we expect the integrand to appear and try to put the solution in the same form. The cases which are known suggest that $d^I_{JLJ'}(\theta)$ will appear as a sum of powers of $\sin\theta$ and $e^{i\theta}$ for the compact cases and $\sinh\zeta$ and e^ζ for the noncompact ones.

We will therefore perform the integral

$$I^{n,\lambda}_{p,q,p',q'}(\zeta) \equiv \int_0^\pi \sin^{n-2}\theta\, d\theta (e^{ip\theta}\sin^q\theta)\left(\frac{\sin\theta}{\sin\theta'}\right)^\lambda (e^{ip'\theta'}\sin^{q'}\theta'), \tag{B1}$$

where p, q, p', and q' are integers and where θ and θ' are related by (4.13). There ζ is real and λ, in general, complex. If we want to be able to make the analytic continuation from $SO_{n,1}$ to SO_{n+1} easily, we need a form where we can replace ζ by $i\theta_n$, $0 \leq \theta_n \leq \pi$, and then let λ be a nonnegative integer.

We express (B1), expanding the exponentials by the binomial theorem, as

$$\begin{aligned} I^{n,\lambda}_{p,q,p',q'}(\zeta) &= 2^{n+q+q'-2}\sum_{\gamma=0}^{2|p|}\sum_{\gamma'=0}^{2|p'|}\binom{2|p|}{\gamma}\binom{2|p'|}{\gamma'} \\ &\times \left(\frac{|p|}{p}i\right)^\gamma \left(\frac{|p'|}{p'}i\right)^{\gamma'} J(\lambda + n - 2 + q + \gamma, \\ &\quad -\lambda + q' + \gamma', \lambda + n - 2 + 2|p| - \gamma, \\ &\quad -\lambda + 2|p'| - \gamma'; \zeta), \end{aligned} \tag{B2}$$

where

$$J(a,b,c,d;\zeta) = \int_0^\pi d\theta \sin^a \tfrac12\theta \sin^b \tfrac12\theta' \cos^c \tfrac12\theta \cos^d \tfrac12\theta'. \tag{B3}$$

In order to solve (B3), substitute[4]

$$x \equiv \sin\theta/\sin\theta', \quad dx = \sinh\zeta \sin\theta\, d\theta,$$

and the limits of the integral $[0, \pi]$ become $[e^{-\zeta}, e^\zeta]$, and

$$\sin^m \tfrac12\theta \sin^{m'} \tfrac12\theta' = [e^{\zeta m'}(x - e^{-\zeta})^{m+m'}x^{-m'}(2\sinh\zeta)^{-m-m'}]^{\frac12},$$

$$\cos^n \tfrac12\theta \cos^{n'} \tfrac12\theta' = [e^{-\zeta n'}(e^\zeta - x)^{n+n'}x^{-n'}(2\sinh\zeta)^{-n-n'}]^{\frac12}.$$

Thus, when $a + b$ and $c + d$ are odd and positive, we can expand

$$\begin{aligned} J(a,b,c,d;\zeta) &= (2\sinh\zeta)^{-\frac12(a+b+c+d)}e^{\frac12(b-d)\zeta} \\ &\times \int_{e^{-\zeta}}^{e^\zeta} dx\, x^{-\frac12(b+d)}(x - e^{-\zeta})^{\frac12(a+b-1)}(e^\zeta - x)^{\frac12(c+d-1)}, \end{aligned}$$

using the binomial theorem, into a finite number of summands. This is the case in passing from SO_3 to[4] $SO_{3,1}$, but it does not seem to be a general property. Thus, we have to effect the further transformation

$$y = (x - e^{-\zeta})(2\sinh\zeta)^{-1},$$

in order to bring it to a form where it can be found[32] to be

$$\begin{aligned} J(a,b,c,d;\zeta) &= (2\sinh\zeta)^{-\frac12(a+b+c+d)}e^{b\zeta} \\ &\times \frac{\Gamma(\frac12(a+b+1))\Gamma(\frac12(c+d+1))}{\Gamma(\frac12(a+b+c+d)+1)} \\ &\times F(\tfrac12(b+d); \tfrac12(a+b+1); \\ &\qquad \tfrac12(a+b+c+d)+1; 1 - e^{2\zeta}). \end{aligned} \tag{B4}$$

When we use (B1)–(B4) in order to find the d-matrix elements for SO_{n+1}, we obtain them in terms of trigonometric (and hypergeometric) functions, i.e., in the same form as we assumed them to be when we choose to construct the form (B1).

[1] E. P. Wigner, *Gruppentheorie* (Vieweg, Braunschweig, 1931), pp. 180, 232.
[2] V. Bargmann, Ann. Math. **48**, 568 (1947).
[3] M. Toller, Nuovo Cimento **37**, 631 (1965); **53**, 671 (1968).
[4] A. Sciarrino and M. Toller, J. Math. Phys. **8**, 1252 (1967).
[5] L. C. Biedenharn, J. Math. Phys. **2**, 433 (1961); D. Z. Friedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).
[6] A. Z. Dolginov, Zh. Eksp. Teor. Fiz. **30**, 746 (1956) [Sov. Phys. JETP **3**, 589 (1956)]; S. Ström, Arkiv Fysik **29**, 467 (1965); **33**, 465 (1966).
[7] In the notation used in this article, they appear as $d^{\lambda M}_{jmj'}(\zeta)$.
[8] (a) J. Dixmier, Bull. Soc. Math. France **89**, 9 (1960); (b) A. Kihlberg and S. Ström, Arkiv Fysik **31**, 491 (1966); (c) R. Takahashi, Bull. Soc. Math. France **91**, 289 (1963).
[9] S. Ström, Arkiv Fysik **30**, 455 (1965).
[10] W. J. Holman III, J. Math. Phys. **10**, 1888 (1969).
[11] E. Inönü and E. P. Wigner, Proc. Natl. Acad. Sci. (U.S.) **39**, 510 (1953).
[12] J. S. Lomont and Harry E. Moses, J. Math. Phys. **5**, 294 (1964); **5**, 1438 (1964).
[13] W. J. Holman III, J. Math. Phys. **10**, 1710 (1969).
[14] N. Kammer, D. L. Pursey, and S. A. Williams, J. Math. Phys. **9**, 1224 (1968): S. A. Williams and D. L. Pursey, *ibid.*, 1230 (1968); R. T. Sharp and S. C. Pieker, *ibid.*, 663 (1968).
[15] J. G. Kuriyan, M. Mukunda, and E. C. G. Sudarshan, Commun. Math. Phys. **9**, 204 (1968).
[16] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, Mathematical Monographs Vol. 22, translated by V. N. Singh (American Mathematical Society, Providence, R.I., 1968).
[17] K. B. Wolf, Ph.D. thesis, Tel-Aviv University, 1969; Tel-Aviv University, Preprint 112-69, 1969.
[18] R. L. Anderson and K. B. Wolf, J. Math. Phys. **11**, 3176 (1970).
[19] J. Fisher, J. Niederle, and R. Rączka, J. Math. Phys. **7**, 816 (1966); R. Rączka, N. Limić, and J. Niederle, *ibid.*, 1861 (1966); N. Limić, J. Niederle, and R. Rączka, *ibid.*, 2026 (1966); **8**, 1079 (1967); R. Delbourgo, J. Koller, and R. M. Williams, *ibid.* **10**, 957 (1969).
[20] U. Ottoson, private communication.
[21] I. M. Gel'fand and M. L. Tsetlin, Dokl. Akad. Nauk SSSR **71**, 840 (1950); S. C. Pang and K. T. Hecht, J. Math. Phys. **8**, 1233 (1967); M. K. F. Wong, *ibid.*, 1899 (1967).
[22] A. Sankaranarayanan, Nuovo Cimento **38**, 1441 (1965); A. Böhm, Phys. Rev. **145**, 1212 (1966); J. Rosen and P. Roman, J. Math. Phys. **7**, 2072 (1966).
[23] N. Mukunda, J. Math. Phys. **8**, 2210 (1967); **9**, 50 (1968).
[24] See Ref. 16, where a slightly different convention is used.
[25] A. Chakrabarti, J. Math. Phys. **9**, 2087 (1968).
[26] U. Ottoson, Commun. Math. Phys. **8**, 228 (1968).
[27] This is essentially the relation which appears in Refs. 2, 4, 8c, and 9.
[28] Notice that, had we taken the second choice in Sec. 2 [see text before Eq. (2.7)], i.e., $g''(g',g) = g^{-1}g'$, we would have

$$D^J_{L_1L_2}(R_n) = \frac{\dim_n J}{\mathrm{vol}\, G}(D^J_{L_2,L}, U^{(\lambda)}(R_n^{-1})D^J_{L_1,L})$$

and a similar involution for (5.4).
[29] K. Iwasawa, Ann. Math. **50**, 507 (1949); R. Hermann, *Lie Groups for Physicists* (Benjamin, New York, 1966), Chap. 7.
[30] In fact, it can be checked that $\partial/\partial\eta$ is the infinitesimal generator of the transformation $g \to g'' = ga(\zeta)$.
[31] R. Harish-Chandra, Proc. Natl. Acad. Sci. (U.S.) **37**, 697 (1951); **40**, 1076 (1952); for applications, see A. Kihlberg. Arkiv Fysik **30**, 121 (1965); **32**, 241 (1966); A. Kihlberg, V. F. Müller, and F. Halbwachs, Commun. Math. Phys. **3**, 194 (1966).
[32] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Series, Products and Integrals* (Academic, New York, 1966), Eq. 3.181.

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Iterated Integral-Transform Trial Functions

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The concept of iterated integral-transform trial functions is introduced. Its formal correspondence with the iterative solution of integral equations is established. Extensions and generalizations are indicated, and some of the advantages of the approach are discussed. Ways are suggested to make tractable the multidimensional integrals that arise in the method.

1. INTRODUCTION

Recently I proposed[1] the use of integral-transform (IT) trial functions in quantum-mechanical calculations. The conceptual simplicity of the basic idea enhances the computational successes that we achieved with IT trial functions.[2–6] This simplicity makes possible extensions and generalizations quite natural. The systematic construction of special, correlated many-particle wavefunctions,[7] various generalizations of the conventional scaling procedure and their natural relation to correlation,[8] and the construction of new *molecular* functions from *atomic* bases[9] are the most important examples of such extensions. In this work, a further generalization will be introduced, the concept of *iterated* IT trial functions.

2. INTEGRAL TRANSFORM FUNCTIONS

Integral-transform trial functions may be constructed by the prescription

$$F_1(x) = \int_{D_0} S_0(t)F_0(tx)\, d\mu(t). \tag{2.1}$$

In Eq. (2.1) $F_1(x)$ is an approximation to $F(x)$, the exact solution to the eigenvalue equation $HF(x) = EF(x)$, $F_0(x)$ is the *known* exact solution of a model