A simple difference realization of the Heisenberg q-algebra

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A realization of the Heisenberg q-algebra whose generators are first-order difference operators on the full real line is discussed herein. The eigenfunctions of the corresponding q-oscillator Hamiltonian are given explicitly in terms of the q^{-1}-Hermite polynomials. The nonuniqueness of the measure for these q-oscillator states is also studied.

I. INTRODUCTION

The Heisenberg algebra is rooted in the phase space concept of classical mechanics and reaches into quantum mechanics, having particularly close association with the harmonic oscillator. It has three generators, a, a^+, and the constant 1; its nonzero commutator is the well-known relation

$$aa^+ - a^+a = 1. \tag{1.1}$$

This algebra has a basic realization in terms of first-order differential operators

$$a^+ = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right), \quad a = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right). \tag{1.2}$$

In the enveloping algebra one has the element (called *particle number* operator)

$$N = a^+a \tag{1.3}$$

that in the standard realization (1.2) is simply related to the harmonic oscillator Hamiltonian with the coordinate $x = \sqrt{\hbar/m\omega}\,\xi$ on the full real line, mass m, and frequency ω, namely,

$$H = \hbar\omega(N + \tfrac{1}{2}) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2. \tag{1.4}$$

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In quantum mechanics one also needs a ground state $\psi_0(x)$ annihilated by the operator a to build a complete Hilbert space of physical states. This Hilbert space is the well-known space of Lebesgue square-integrable functions over the real line, $\mathscr{L}^2(\mathfrak{R})$, whose inner product is

$$(f,g)=\int_{-\infty}^{\infty} dx\ f(x)^* g(x) \tag{1.5}$$

for arbitrary functions f and g of $x\in\mathfrak{R}$.

In this Hilbert space, the Hamiltonian (1.4) has the well-known linear energy spectrum

$$E_n=\hbar\omega(n+\tfrac{1}{2}), \quad n=0,1,2,... \tag{1.6}$$

and eigenfunctions

$$\psi_n(x)=\frac{1}{\sqrt{\sqrt{\pi}2^n n!}}\ e^{-\xi^2/2}H_n(\xi). \tag{1.7}$$

The purpose of this article is to discuss an explicit realization of the Heisenberg q-algebra

$$bb^+-qb^+b=1 \tag{1.8}$$

in terms of finite difference operators, that corresponds, as closely as possible, to the standard case (1.2). By this we mean that the generators of the Heisenberg q-algebra should be functions and first-order difference operators defined on the full real line and reduce to Eq. (1.2) when $q\to 1^-$. The motivation for the study of q analogs of the Heisenberg algebra and the harmonic oscillator is discussed in Refs. 1–10, that also provide the background for this subject.

II. REALIZATION OF THE HEISENBERG q-ALGEBRA

We consider the q-commutation relation (1.8) and the basic difference operator action

$$e^{c\partial_x}f(x)=f(x+c) \tag{2.1}$$

on any smooth function f of x on the full real line. We note that under the $\mathscr{L}^2(\mathfrak{R})$ inner product (1.5), the Hermitian conjugate of $e^{c\partial_x}$ is $e^{-c\partial_x}$. We search for an operator b of the form

$$b(x)=u(x)e^{(1/2)\partial_x}+v(x)e^{-(1/2)\partial_x} \tag{2.2a}$$

and its Hermitian conjugate b^+, that under the $\mathscr{L}^2(\mathfrak{R})$ inner product (1.5) has the form

$$b^+(x)=e^{-(1/2)\partial_x}u^*(x)+e^{(1/2)\partial_x}v^*(x)=u^*(x-\tfrac{1}{2})e^{-(1/2)\partial_x}+v^*(x+\tfrac{1}{2})e^{(1/2)\partial_x}. \tag{2.2b}$$

When we substitute the forms (2.2) into (1.8), the left-hand side of that equation will be a sum of the shift operators $e^{+\partial_x}$, 1, and $e^{-\partial_x}$. The coefficient of the first one is

$$u(x)v^*(x+1)-qv^*(x+\tfrac{1}{2})u(x+\tfrac{1}{2}) \tag{2.3}$$

and the coefficient of $e^{-\partial_x}$ is the complex conjugate of Eq. (2.3), shifted by -1, i.e., $x\mapsto x-1$. Equating (2.3) to zero yields the difference equation for v^*/u given by

$$e^{(1/2)\partial_x}\ \frac{v^*(x+(1/2))}{u(x)}=q\ \frac{v^*(x+(1/2))}{u(x)}\ . \tag{2.4}$$

The solution of this is

$$v^*(x+\tfrac{1}{2})=kq^{2x}u(x), \tag{2.5}$$

where k is a constant.

Using this result, the no-shift coefficient is of the form

$$(1-q^{4x+1}|k|^2)|u(x)|^2-(q-q^{4x-2}|k|^2)|u(x-\tfrac{1}{2})|^2. \tag{2.6}$$

Defining $g(x)=|u(x)|^2$ and $|k|^2=q^\alpha$, Eq. (1.8) becomes

$$(1-q^{4x+\alpha+1})g(x)-q(1-q^{4x+\alpha-3})g(x-\tfrac{1}{2})=1. \tag{2.7}$$

Since the right-hand side of this equality is a constant, in the left-hand side we expect $g(x)$ to contain as denominators the factors that multiply it above. Note that one may factorize $1-q^{4x+\alpha+1}=[1-q^{2x+(\alpha+1)/2}][1+q^{2x+(\alpha+1)/2}]$ but, since $g(x)$ is positive, it can contain only the second factor. A similar reasoning applied to the second term in Eq. (2.7) shows that $g(x)$ should also contain the factor $[1+q^{2x+(\alpha-1)/2}]$. Thus we look for solutions of the form

$$g(x)=f(x)/[1+q^{2x+(\alpha+1)/2}][1+q^{2x+(\alpha-1)/2}]. \tag{2.8}$$

Substitution of Eq. (2.8) into Eq. (2.7) yields the difference equation

$$[1-q^{2x+(\alpha+1)/2}]f(x)-q[1-q^{2x+(\alpha-3)/2}]f(x-\tfrac{1}{2})=1+q^{2x+(\alpha-1)/2} \tag{2.9}$$

that has the constant solution $f(x)=1/(1-q)$. From this solution we find

$$u(x)^{-1}=\sqrt{(1-q)[1+q^{2x+(\alpha+1)/2}][1+q^{2x+(\alpha-1)/2}]} \tag{2.10}$$

and the function $v(x)$ related to $u(x)$ through Eq. (2.5). Up to now, the parameter $k(|k|=q^{\alpha/2})$ is arbitrary. By consistency with the limit case when $q\to 1^-$, we demand that the q-creation and q-annihilation operators have *definite parity*: $b^+(-x;q)=-b^+(x;q)$ and $b(-x;q)=-b(x;q)$. This means that $v(-x)=-u(x)$, leading to $\alpha=1$ and $k=-\sqrt{q}$. Consequently we find

$$u(x)^{-1}=\sqrt{(1-q)(1+q^{2x})(1+q^{2x+1})} \tag{2.11a}$$

$$=2q^{x+1/4}\sqrt{(1-q)\cosh\kappa x\,\cosh\kappa(x+(1/2))}, \tag{2.11b}$$

where in the last row we have used $q=e^{-\kappa}$ $(0\leqslant\kappa<\infty)$.

Substitution of $u(x)$ and $v(x)$ into Eq. (2.2) yields the realization of the q-annihilation and q-creation operators given by

$$b(x;q)=\frac{1}{2q^{1/4}\sqrt{1-q}}\sqrt{\operatorname{sech}\kappa x}\,(q^{-x}e^{(1/2)\partial_x}-q^x e^{-(1/2)\partial_x})\sqrt{\operatorname{sech}\kappa x}, \tag{2.12a}$$

$$b^+(x;q)=\frac{1}{2q^{1/4}\sqrt{1-q}}\sqrt{\operatorname{sech}\kappa x}\,(e^{-(1/2)\partial_x}q^{-x}-e^{(1/2)\partial_x}q^x)\sqrt{\operatorname{sech}\kappa x}. \tag{2.12b}$$

The limit $q\to 1^-(\kappa\to 0^+)$ reproduces Eq. (1.2) provided that one substitutes

$$x=\xi/\sqrt{2(1-q)}. \tag{2.13}$$

For concrete calculations one can use a more convenient form for these difference operators obtained by the similarity transformation

$$\tilde{b}(x;q)=\frac{1}{\sqrt{\cosh \kappa x}}\, b(x;q)\sqrt{\cosh \kappa x}=\frac{\operatorname{sech} \kappa x}{2q^{1/4}\sqrt{1-q}}\,(q^{-x}e^{(1/2)\partial_x}-q^{x}e^{-(1/2)\partial_x}), \tag{2.14a}$$

$$\tilde{b}^{+}(x;q)=\frac{1}{\sqrt{\cosh \kappa x}}\, b^{+}(x;q)\sqrt{\cosh \kappa x}=\frac{q^{1/4}\operatorname{sech} \kappa x}{2\sqrt{1-q}}\,(q^{-x}e^{-(1/2)\partial_x}-q^{x}e^{(1/2)\partial_x}). \tag{2.14b}$$

It is evident that in the limit when $q \to 1^-$ (i.e., $\kappa \to 0^+$), $\tilde{b}$ and $\tilde{b}^+$ coincide with b and b^+. We should keep in mind, however, that the "physical" operators and wave functions for the q-harmonic oscillator are the untilded ones.

III. THE q-OSCILLATOR HAMILTONIAN

Given a concrete realization of the Heisenberg q-algebra, one has a corresponding realization for the Hamiltonian of the q oscillator

$$H(x;q)=b^{+}(x;q)b(x;q) \tag{3.1}$$

that is self-adjoint under the $\mathscr{L}^2(\Re)$ inner product (1.5). For further use we define

$$\begin{aligned}\tilde{H}(x;q)&=\sqrt{\operatorname{sech} \kappa x}H(x;q)\sqrt{\cosh \kappa x}\\&=\tilde{b}^{+}(x;q)\tilde{b}(x;q)\\&=\frac{1}{1-q}\left[1-\frac{1+q}{4q}\operatorname{sech} \kappa\left(x+\frac{1}{2}\right)\operatorname{sech} \kappa\left(x-\frac{1}{2}\right)\right]\\&\quad-\frac{\operatorname{sech} \kappa x}{4\sqrt{q}(1-q)}\left[\operatorname{sech} \kappa\left(x+\frac{1}{2}\right)e^{\partial_x}+\operatorname{sech} \kappa\left(x-\frac{1}{2}\right)e^{-\partial_x}\right].\end{aligned} \tag{3.2}$$

The Hamiltonian $\tilde{H}(x,q)$ is self-adjoint in a Hilbert space $\mathscr{L}^2_{\cosh}(\Re)$ with measure $\cosh \kappa x\, dx$ instead of dx as in $\mathscr{L}^2(\Re)$.

The eigenvalue problem for the Hamiltonian $\tilde{H}(x,q)$ is

$$\tilde{H}(x;q)\tilde{\psi}_n(x;q)=e_n(q)\tilde{\psi}_n(x;q), \tag{3.3}$$

where the spectrum $\{e_n(q)\}_{n=0}^{\infty}$ is independent of the realization and well known (cf. Refs. 1 and 2) to be

$$e_n(q)=\frac{1-q^n}{1-q}\,. \tag{3.4}$$

The eigenfunctions $\tilde{\psi}_n(x;q)$, on the other hand, do depend on the concrete realization and are chosen under the criteria of simplicity below. The q-annihilation and q-creation operators (2.13) act as ladder operators between the eigenfunction set, namely,

$$\tilde{b}(x;q)\tilde{\psi}_n(x;q)=\sqrt{e_n(q)}\,\tilde{\psi}_{n-1}(x;q), \tag{3.5a}$$

$$\tilde{b}^{+}(x;q)\tilde{\psi}_n(x;q)=\sqrt{e_{n+1}(q)}\,\tilde{\psi}_{n+1}(x;q). \tag{3.5b}$$

From here it follows that the nth state can be expressed in terms of the ground state as

$$\tilde{\psi}_n(x;q)=c_n(q)[\tilde{b}^{+}(x;q)]^n\tilde{\psi}_0(x;q), \quad c_n(q)=\Gamma_q^{-1/2}(n+1), \tag{3.6}$$

where we have used the standard notations (see, for example, Ref. 11) $\Gamma_q(n+1)=(q;q)_n/(1-q)^n$ and $(a;q)_n=\Pi_{j=0}^{n-1}(1-aq^j)$.

In the Hilbert space $\mathscr{L}^2_{\cosh}(\mathfrak{R})$, where $\tilde{H}$ is self-adjoint, the eigenvalue equation (3.3) leads through standard arguments to the orthogonality of the eigenstates $\tilde{\psi}_n(x;q)$ and $\tilde{\psi}_m(x;q)$ for $n \neq m$. We should remark that although the difference operator realizations (2.2) and (3.2) require the values of the function only at discrete points spaced by 1, our $\mathscr{L}^2$-inner products integrate over $x \in \mathfrak{R}$. It is indeed possible to propose a discrete (sum) inner product over the points $x_n=n$, $-N \leq n \leq N$, N positive integer. However, this leads to a different realization of the Heisenberg q-algebra in terms of vectors and matrices.

The particle number operator $N(x;q)$, satisfying the standard raising and lowering commutation relations $[N(x;q),b^+(x;q)]=b^+(x;q)$ and $[N(x;q),b(x;q)]=-b(x;q)$, and whose spectrum is therefore 0,1,2,..., is related with the "physical" Hamiltonian (3.1) through

$$N(x;q)=\frac{1}{\log q}\log[1-(1-q)H(x;q)]. \tag{3.7}$$

To close this section, we note that the operators $b(x;q)$ and $b^+(x;q)$ are bounded and adjoint to each other with respect to the measure of orthogonality dx for the wave functions $\psi_n(x;q)$. Their boundedness is clear from the explicit realizations (2.12a) and (2.12b), or from (3.4) and (3.5), and the fact that for $0<q<1$ the eigenvalues $e_n(q)$ have the finite limit $(1-q)^{-1}$ when $n \to \infty$. In a Hilbert space of square-integrable real-valued functions with the scalar product

$$(f,\phi)=\int_{-\infty}^{\infty} dx\ f(x)\phi(x) \tag{3.8}$$

from the evident identity

$$\int_{-\infty}^{\infty} dx\ y(x-a)f(x-a)\phi(x)=\int_{-\infty}^{\infty} dx\ y(x)f(x)\phi(x+a) \tag{3.9a}$$

it follows that the operators $y(x)\exp(a\partial_x)$ and $\exp(-a\partial_x)\ y(x)\equiv y(x-a)\exp(-a\partial_x)$ are mutually adjoint for any real a, i.e.,

$$(e^{-a\partial_x}yf,\phi)=(f,ye^{a\partial_x}\phi). \tag{3.9b}$$

Thus a simple inspection of the formulas (2.12a) and (2.12b) confirms that the operators $b(x;q)$ and $b^+(x;q)$ are adjoint to each other with respect to the measure dx. In particular, this means that the "position" $Q=b(x;q)+b^+(x;q)$ and "momentum" $P=i[b(x;q)-b^+(x;q)]$ operators are bounded and self-adjoint. The self-adjointness properties of the operators Q and P for various types of q-oscillator algebras are treated in detail in Ref. 12.

IV. THE q-OSCILLATOR STATES

The ground state of the q-oscillator is obtained from the difference equation

$$\tilde{b}(x;q)\tilde{\psi}_0(x;q)=0, \tag{4.1a}$$

that is,

$$\tilde{\psi}_0(x+1;q)=q^{2x+1}\tilde{\psi}_0(x;q). \tag{4.1b}$$

It has the simple solution

$$\tilde{\psi}_0(x;q)=cq^{x^2}=ce^{-\kappa x^2}, \tag{4.2}$$

where c is a normalization "constant" to which we shall return below. Taking into account Eq. (2.13) it is easy to verify that $q^{x^2} \to e^{-\xi^2/2}$ when $q \to 1^-$.

From Eq. (4.2) we return to the "physical" ground state undoing the similarity transformation (2.14)

$$\psi_0(x;q)=\sqrt{\cosh \kappa x}\,\tilde{\psi}_0(x;q)=c\sqrt{\cosh \kappa x}\,e^{-\kappa x^2}. \tag{4.3}$$

The constant c is fixed through requiring the $\mathcal{L}^2(\mathfrak{R})$ normalization of this wave function in the variable x, related to ξ through Eq. (2.13). The integral

$$\int_{-\infty}^{\infty} dx\ \psi_0(x;q)^2=c^2\int_{-\infty}^{\infty} dx \cosh \kappa x e^{-2\kappa x^2}=c^2\sqrt{\frac{\pi}{2\kappa}}\,e^{\kappa/8}=\frac{1}{\sqrt{2(1-q)}} \tag{4.4}$$

is easily calculated (see, for example, Ref. 13, p. 367) and therefore

$$c=[\kappa/(1-q)\pi]^{1/4}e^{-\kappa/16}=[\kappa/(1-q)\pi]^{1/4}q^{1/16}. \tag{4.5}$$

The other q-oscillator states are now found from the ground state (4.3) by means of the repeated application of the q-creation operator as in Eq. (3.6), that leads to the difference analog of the Rodrigues formula for the q^{-1}-Hermite polynomials.[8] They are of the form

$$\psi_n(x;q)=d_n^{-1}\sqrt{\cosh \kappa x}\,h_n(\sinh \kappa x|q)e^{-\kappa x^2} \tag{4.6a}$$

and are $\mathcal{L}^2(\mathfrak{R})$ orthonormal when the normalization constants are

$$d_n^{-1}=\left(\frac{\kappa}{\pi(1-q)}\right)^{1/4}\frac{q^{(n+1/2)^2/4}}{\sqrt{(q,q)_n}}. \tag{4.6b}$$

In particular note that $d_0^{-1}=c$, as given by Eq. (4.5). The q^{-1}-Hermite polynomials are defined[8] through $h_0(x|q)=1$ and by the three-term recurrence relation $h_{n+1}(x|q)=2xh_n(x|q)+(1-q^{-n})h_{n-1}(x|q)$ for $n\geqslant 0$. From the limit relation

$$\lim_{q\to 1^-}\left(\frac{2}{1-q}\right)^{n/2}h_n\left(\sqrt{\tfrac{1}{2}(1-q)}\,\xi|q\right)=H_n(\xi) \tag{4.7}$$

it follows that the functions (4.6a) coincide with the standard harmonic oscillator wave functions (1.7) when $q \to 1^-$. We also note that the wave functions that were considered in Ref. 3 for a model of the harmonic oscillator in the relativistic configurational space, are the Fourier transforms of the functions (4.6a).

V. NONUNIQUENESS OF THE MEASURE

The q^{-1}-Hermite polynomials $h_n(y|q)$ were studied by Askey[8] and shown to be $\mathcal{L}^2_\omega(\mathfrak{R})$ orthogonal with respect to the measure

$$w_A(y)dy=dy\Big/\sqrt{1+y^2}\prod_{j=0}^{\infty}[1+2(1+2y^2)q^{j+1}+q^{2j+2}]. \tag{5.1}$$

After the change of variable $y=\sinh \kappa x$ in accordance with Eq. (4.6a), we note that the measure (5.1) is *not* equal to $\psi_0(x;q)^2\,dx$, the squared ground state (4.3) which, in view of Eq. (4.6), should be the orthogonality measure for the polynomials. This is not surprising since it is known that the moment problem associated with the q^{-1}-Hermite polynomials is indeterminate,[8,10] and therefore they are orthogonal with respect to an infinite class of measures. This class may include both continuous and discrete measures. We are considering here only the former case since it is more appropriate for the physical interpretation of the model at hand. The detailed discussion of the latter case can be found, for example, in Refs. 11 and 14.

The reason for having an infinite class of measures is that there is a fundamental indeterminacy in the construction of the ground state because the solution of the difference equation (4.1) is not uniquely defined: one can multiply a solution by any function $c(x;q)$ that satisfies the periodicity requirement $c(x+m;q)=c(x;q)$ for integer m. The measure

$$w_c(x;q)dx=\psi_0(x;q)^2\ dx=c^2\cosh\kappa x e^{-2\kappa x^2}\ dx \tag{5.2}$$

corresponds to the simplest solution of Eq. (4.1), when the function c is chosen to be a true constant.

An appropriate choice of the normalization "constant" $c(x;q)$ also enables one to reproduce Askey's measure (5.1). To find it explicitly we can use Jacobi's expression for the theta function $\vartheta_2(z,q)\equiv\vartheta_2(z|\tau)$, $q=\exp\pi i\tau$, as an infinite product,[15,16] to put Eq. (5.1) in the form

$$w_A(y)dy=\frac{4\kappa q^{1/8}(q;q)_\infty\cosh\kappa x}{\vartheta_2(i\kappa x,e^{-\kappa/2})}\,dx, \tag{5.3}$$

where we recall that $y=\sinh\kappa x$. With Jacobi's imaginary transformation (see, for example, Ref. 16, p. 370)

$$\vartheta_2(z|\tau)=\frac{e^{-iz^2/\pi\tau}}{\sqrt{-i\tau}}\,\vartheta_4(z\tau^{-1}|-\tau^{-1}) \tag{5.4}$$

one can write Eq. (5.3) as

$$w_A(y)dy=\frac{2\kappa\sqrt{2(1-q)}(q;q)_\infty}{\vartheta_4(2\pi x,e^{-2\pi^2/\kappa})}\,w_c(x;q)dx. \tag{5.5}$$

Thus, the measures $w_A(y)dy$ and $w_c(x;q)dx$, with $y=\sinh\kappa x$, differ only by a factor that is a periodic function under the shift $x\mapsto x+\frac{1}{2}m$ for integer m, due to the periodicity of the Jacobi theta function $\vartheta_4(z,q)=\vartheta_4(z+\pi,q)$. In other words, the choice of

$$c(x;q)=1/\sqrt{\vartheta_4(2\pi x,e^{-2\pi^2/\kappa})},\quad q=e^{-\kappa} \tag{5.6}$$

leads to Askey's measure (5.1).

To close this section, we emphasize that the nonuniqueness of the measure for the polynomial part of the wave functions $\psi_n(x;q)$ [*not* to be confused with the orthogonality measure dx for the full wave functions—see formulas (3.6) and (4.6a)] derives from the fact that the ground state wave function $\psi_0(x;q)$ is not uniquely defined by the difference equation (4.1a). In other words, contrary to the case of the classical harmonic oscillator in quantum mechanics, one has an infinite number of realizations for the q-oscillator model under discussion. But the ground state wave functions from these realizations may differ only by some periodic factors $c(x;q)$, which are not affected by the q-lowering and q-raising difference operators $b(x;q)$ and $b^+(x;q)$, defined by (2.12a) and (2.12b), respectively. Therefore the formulas (3.5a) and (3.5b) and, consequently, the boundedness and mutual adjointness property of the operators $b(x;q)$ and $b^+(x;q)$ with respect to the measure dx (cf. the end of section III) remain valid for all such realizations.

VI. CONCLUDING REMARKS

The harmonic oscillator wave functions provide a very convenient realization for the action of the Heisenberg Lie algebra generators. This basis is known to have a wide range of applications beyond the algebra itself, so it is worthwhile to study the q analogs of this system and its wave functions.

Explicit realizations of the Heisenberg q-algebra have been considered in terms of Rogers–Szegő,[1] q-Hermite,[6] and in terms of Stieltjes–Wigert polynomials.[7] In the first and second cases, the orthogonality interval is the unit circle and the line segment $[-1, 1]$, whereas in the third case, the q-raising and q-lowering operators are difference operators of second order. It seems to us that the realization discussed here deserves some attention because the interval is the full real line and these operators are of first order. In this sense we believe that it is the closest analog of the harmonic oscillator. In addition, this realization provides the possibility of studying other constructions such as the q analogs of systems with mixed spectra[17] that have both bound and scattering states.

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