Carrying out the functional differentiation of $V\bar{h}(y)$, we obtain

$$\left[\frac{\delta V\bar{h}(y)}{\delta\beta v(1,2)}\right]_{y=y^*} = \mu_2(1,2\mid y^*) + \iint d^\nu r_3 d^\nu r_4 \mu_2(1,3\mid y^*) \times \beta\tilde{v}_1(3,4)\mu_2(2,4\mid y^*) + \tfrac{1}{2}\iint d^\nu r_3 d^\nu r_4 \mu_4(1,2,3,4\mid y^*) \times \beta\tilde{v}_1(3,4) + O(\gamma^{2\nu}). \quad \text{(C3)}$$

The first two terms in this expression are obtained from the first term on the right-hand side of Eq. (8). The last term is the term of lowest order obtained by differentiation of the second term in Eq. (8). Thus, we see that in order to compute $\rho^{(2)}$ through order $\gamma^{2\nu-1}$, we must take into account terms in the pressure through order $\gamma^{3\nu-1}$. $\rho(z)$ is found, from Eqs. (2) and (8), to be

$$\rho(z) = \rho_h(y^*) + \tfrac{1}{2}\iint d^\nu r_2 d^\nu r_3 \mu_3(1,2,3\mid y^*) \times \beta\tilde{v}_1(2,3) + O(\gamma^{2\nu}). \quad \text{(C4)}$$

The integral is, of course, independent of $\mathbf{r}_1$ and is the lowest term obtained from $(\partial\bar{h}(y)/\partial \ln y)_{y=y^*}$.

Combining all these results, we obtain

$$\rho^{(2)}(1,2\mid z) = \rho_h^{(2)}(1,2\mid y^*) + \tfrac{1}{2}\iint d^\nu r_3 d^\nu r_4 \beta\tilde{v}_1(3,4) \times \{\mu_4(1,2,3,4\mid y^*) + 2\rho_h(y^*)\mu_3(1,3,4\mid y^*) + 2\mu_2(1,3\mid y^*)\mu_2(2,4\mid y^*) - \delta(1-2)\mu_3(1,3,4\mid y^*)\} + O(\gamma^{2\nu}). \quad \text{(C5)}$$

Using Eq. (B9), we see that Eq. (C5) can be written

$$\rho^{(2)}(1,2\mid z) = \rho_h^{(2)}(1,2\mid y^*) + \tfrac{1}{2}\iint d^\nu r_3 d^\nu r_4 \lambda_4(3,4\mid 1,2) \times \beta\tilde{v}_1(3,4) + O(\gamma^{2\nu}). \quad \text{(C6)}$$

This is identical through order $\gamma^{2\nu-1}$ to our Eq. (21). We have checked the agreement also to order $\gamma^{3\nu-1}$ by the same procedure. Since this calculation is straightforward and laborious, we have not presented it here.

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Complete Sets of Functions on Homogeneous Spaces with Compact Stabilizers

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We formulate and solve the problem of determining a complete set of generalized functions for a wide class of homogeneous spaces with compact stabilizers. This allows us to say precisely what unitary irreducible representations can be realized on a given homogeneous space. The techniques are applied to the n-dimensional orthogonal and unitary groups.

I. INTRODUCTION

In physics, we frequently encounter homogeneous spaces. Two well-known examples are the 3-dimensional sphere $S_3 = SO(2)\backslash SO(3)$ in angular momentum theory and Minkowski space $M = ISO(3,1)/SO(3,1)$ in Wigner's classification[1] of the unitary irreducible representations (UIR's) of the Poincaré group. With the relatively recent interest in higher-symmetry groups, homogeneous spaces have also been used, e.g., by Bég and Ruegg[2] to study $SU(3)$ and by Holland[3] to investigate some of the UIR's of $SU(n)$. Rączka *et al.* have studied the most degenerate representations of $SO(p,q)$,[4] $SU(p,q)$,[5] and $Sp(n)$.[6] Further, Lurçat,[7] Nilsson and Kihlberg[8] and others[9–11] have used some of the homogeneous spaces of the Poincaré group in order to build field theories which may provide a description in which mass and spin are treated on an equal footing.

These examples illustrate an important and well-known use of homogeneous spaces in mathematics and physics: namely, that, if there exists a right G-invariant measure on a homogeneous space $X = G_0\backslash G$ (G_0 is called the right stabilizer of the homogeneous space), then, by using the action of G on X, one can realize a unitary (in general, reducible) representation U_g of G, the Hilbert space $L_2(X)$ of square-integrable functions on X serving as a carrier space.

The central mathematical question is the decomposition (as a direct sum or integral) of $L_2(X)$ into minimal U_g-invariant subspaces, i.e., the decomposition of the regular representation U_g into irreducible representations. The elements of these minimal U_g-invariant spaces are called spherical functions.

We know, from the work of Rączka,[12] the connection between the completeness of the set of matrix elements of UIR's for the space $L_2(G)$ for locally compact, semisimple Lie groups and nuclear spectral theory: namely, that the UIR matrix elements are the generalized eigenfunctions of a complete set of operators built from the right and left universal enveloping Lie algebras. In fact, we obtain not only the spherical functions but their decomposition into 1-dimensional subspaces and, furthermore, the complete classification of all possible UIR's that appear in the regular representation of G along with their multiplicities.

Our purpose is to exploit this fact and obtain analogous results for the homogeneous spaces of the form $G_0\backslash G$, where G is an arbitrary locally compact unimodular Lie group with a complete set of UIR matrix elements. In this paper we shall discuss in detail the case when $G_0 = K$ is any (closed) compact subgroup of G.

The formalism is set up in Sec. II, where the central question "What constitutes a complete set of functions on the homogeneous space $X = K\backslash G$?" is answered. This leads immediately to the solution of the related problem in Sec. III: "What UIR's can be realized on a given homogeneous space?"

The solution is given explicitly for the orthogonal groups in Sec. IV: $SO(n)$ is treated in detail, and the extensions to $SO(n-1, 1)$ and $ISO(n)$ are indicated. In Sec. V we treat the unitary groups $U(n)$ and $SU(n)$. The parametrization of these groups and of the homogeneous space manifolds is essential for our purposes: The concrete and detailed statement of the results is simplified by the use of the Gel'fand–Tsetlin patterns.

We hope to discuss, in a future publication, homogeneous spaces where the stabilizer groups are noncompact (but locally compact).

II. COMPLETE SETS OF GENERALIZED FUNCTIONS

Let G be a locally compact Lie group and K any closed subgroup of G. We can construct the space of right (left) cosets, the homogeneous space $X = K\backslash G$ (G/K) whose points $x \in X$ are the sets $Kg(gK)$, where $g \in G$ and the topology in X is the one induced by the topology of G. The homogeneous space X is itself a group only when K is a normal subgroup of G. However, we are interested in X as a transitive manifold for G; i.e., (a) the coset Kg is mapped into the coset Kgg' under right multiplication by $g' \in G$ (gK is mapped into $g'gK$ under left multiplication by $g' \in G$), and (b) given any two cosets, there exists a $g \in G$ which maps one into the other. For the purposes of economy, we shall confine our discussion to spaces of right cosets $X = K\backslash G$.

Let $F(G)$ be the set of functions defined on the group manifold of G. Functions on the homogeneous space $X = K\backslash G$ are defined as functions which are constant on right cosets, i.e.,

$$F(X) = \{f \in F(G) \,|\, f(kg) = f(g), \quad \text{for} \quad g \in G \text{ and all } k \in K\}. \quad (1)$$

We want to point out the fact that, if $f \in L_2(X)$ and if the stabilizer K is compact, then $f \in L_2(G)$, while, if K is noncompact, $f \notin L_2(G)$. It is precisely because of this that we restrict ourselves in this paper to the case with compact stabilizers.

Since we are interested in unitary representations, we must have a right G-invariant measure on X, and we can use the following general theorem[13] to insure its existence:

Theorem: If G is a locally compact unimodular Lie group and K a subgroup of G, then there exists an invariant measure $dm(x)$ on the homogeneous space $K\backslash G$ provided that

$$|\det Ad_G(k)| = |\det Ad_K(k)| \quad \text{for all} \quad k \in K.$$

This measure $dm(x)$ is unique up to a constant factor, and

$$\int_G f(g)\, d\mu(g) = \int_{K\backslash G} dm(x) \int_K f(kg)\, d\mu(k)$$

for every $f \in C_0(G)$.

We know that $C_0(G)$, the space of all continuous functions on G with compact support, is dense in $L_2(G)$. Since all continuous functions with compact support which are constants on cosets with respect to a compact subgroup K belong to $C_0(G)$ and are dense in $L_2(X)$, the theorem is applicable to our case and covers a "sufficiently" large class of elements of $L_2(X)$.

We now restrict ourselves further to locally compact unimodular Lie groups, such that the UIR matrix elements or some subset of them constitute a complete set of generalized eigenfunctions as described by Rączka, i.e., such that they are the generalized eigenfunctions (in the sense of nuclear spectral theory) of a complete set of strongly commuting operators built from the right and left universal enveloping Lie algebras. We shall henceforth call this "the completeness requirement."

Presently, there is no general, mathematically rigorous statement characterizing all groups which satisfy the completeness requirement. The compact Lie groups are, of course, rigorously known to satisfy it by virtue of the Peter–Weyl–von Neumann theory.[14] The theory has also been developed for some of the noncompact groups, specifically $SL(2, C)$,[15] $IO(2)$,[16] and $SO(2, 1)$.[17] The work of Rączka[12] represents a generalization of the Peter–Weyl–von Neumann theorem to noncompact, semisimple Lie groups.

For groups which satisfy this completeness requirement, we can associate, with each function $f(g) \in L_2(G)$, one matrix function $F(\lambda)$ whose domain is the space of a complete set of UIR's of the group, which we denote by $\hat{G}$. The points of this space are characterized by the eigenvalues λ of a complete set of Casimir operators[18] of the group. For every λ, the rows and columns of this matrix are labeled in the same fashion as the UIR matrices $D^\lambda(g)$ themselves (as will be detailed below), which become the transformation kernels which relate the two functions[11,19]; i.e.,

$$F_{pq}(\lambda) = \int_G d\mu(g) f(g) D^\lambda_{pq}(g), \tag{2a}$$

$$f(g) = \int_{\hat{G}} d\hat{\mu}(\lambda) \sum_{p,q} F_{qp}(\lambda) D^\lambda_{pq}(g^{-1}), \tag{2b}$$

where $d\hat{\mu}(\lambda)$ is the Plancherel measure on $\hat{G}$. For compact groups G, the space $\hat{G}$ is a set of isolated points, and the integration in (2b) becomes a sum

$$\int_{\hat{G}} d\hat{\mu}(\lambda) \to \sum_{\lambda \in \hat{G}} \dim(\lambda) \Big/ V(G),$$

where $\dim(\lambda)$ is the dimension of the UIR labeled by λ and $V(G)$ is the volume of the group.

The norms of the two functions in (2a) and (2b) can be related by the Parseval identity

$$\int_G d\mu(g) \, |f(g)|^2 = \int_{\hat{G}} d\hat{\mu}(\lambda) \sum_{p,q} |F_{pq}(\lambda)|^2.$$

Let $G \supset G_r \supset G_{r-1} \supset \cdots \supset G_1$ be a chain of subgroups which includes the compact subgroup $K = G_k$ for some k, whose UIR labels can be used to classify completely the components of the UIR basis vectors [as, e.g., the canonical chains $SO(n) \supset SO(n-1) \supset \cdots \supset SO(2)$ and $U(n) \supset U(n-1) \supset \cdots \supset U(1)$ for the orthogonal and unitary groups, giving rise to the Gel'fand–Tsetlin kets; see Secs. IV and V].

It will prove convenient to regard the index labeling the rows (columns) of the representation matrices $D_{pq}(g)$, as standing for the sets $\{p_r, p_{r-1}, \cdots, p_1\}$ ($\{q_r, q_{r-1}, \cdots, q_1\}$), where p_j (q_j) is the collective label which denotes the eigenvalues of a complete set[18] of Casimir operators of the subgroup G_j along the chain. For convenience, we define $\overline{p_j} \equiv \{p_j, p_{j-1}, \cdots, p_1\}$ ($\overline{q_j} \equiv \{q_j, q_{j-1}, \cdots, q_1\}$), i.e., the row (column)-index for the UIR matrices of G_{j+1}, and

$$D^\lambda_{pq}(k_0) = D^{p_k}_{\bar{p}_{k-1}\bar{q}_{k-1}}(k_0) \prod_{j=k}^{r} \delta_{p_j, q_j} \tag{3}$$

for $k_0 \in K$, where the Kronecker δ in the collective labels p_j and q_j is to be regarded as a product of the Kronecker δ's in the individual indices.

Let us first examine the pair of functions (2) when $f(g)$ has the property (1) (i.e., when it is a function on $X = K\backslash G$), in order to determine the subset of $\{D^\lambda_{pq}(g^{-1})\}$ which constitutes a complete set in X: We find that only those D's which satisfy

$$D^\lambda_{pq}(g^{-1}) = D^\lambda_{pq}((kg)^{-1}) \quad \text{for all} \quad k \in K \quad \text{and} \quad g \in G \tag{4}$$

appear in the expansion of a $f \in L_2(X)$. We have employed the following reasoning here: $\{D^\lambda_{pq}(g^{-1})\}$ is a complete set for $L_2(G)$ and, since $f \in L_2(X)$ implies $f \in L_2(G)$, $\{D_{pq}(g^{-1})\}$ is also a complete set for $L_2(X)$; $f \in L_2(X)$ means $f(kg) = f(g)$ for all $k \in K$, and thus we find that the subset of $\{D_{pq}(g^{-1})\}$ which satisfies condition (4) is a complete set on X.

We can integrate (4) over $k \in K$ in both sides: The left-hand side, independent of k, will be multiplied by the volume $V(K)$ of the subgroup. The right-hand side can be written as $\sum_s D^\lambda_{ps}(g^{-1}) D^\lambda_{sq}(k^{-1})$, and the integration performed only over the last factor, where (3) can be used.

Furthermore, since the scalar representation

$$D^0_{00}(k) = 1$$

(which is a constant function on K) is a member of the complete set of UIR matrices for K, we have

$$\int_K d\mu(k) D^{q_k}_{\bar{s}_{k-1}\bar{q}_{k-1}}(k^{-1}) = \delta_{\bar{s}_{k-1},0}\delta_{\bar{q}_{k-1},0}\delta_{\hat{K}}(q_k, 0), \quad (5)$$

where the last factor is such that

$$\int_{\hat{K}} d\hat{\mu}(q)A(q)\delta_{\hat{K}}(q, q_0) = A(q_0)$$

for any "well-behaved" function $A(q)$ on the space $\hat{K}$. For compact groups, it can be written as

$$\delta_{\hat{K}}(q, q_0) = \delta_{q,q_0}V(K)/\dim(q_0). \quad (6)$$

As before, dim (q) is the dimension of the UIR of K labeled by q, and dim $(0) = 1$. Hence, from (3), (5), and (6) it can be seen that those D's which satisfy (4) must also satisfy

$$D^\lambda_{pq}(g^{-1}) = D^\lambda_{pq}(g^{-1})\delta_{\bar{q}_k,0} \quad (7)$$

and, hence, the complete set of functions on $X = K\backslash G$ is the set $\{D^\lambda_{p,q(k)}(g^{-1})\}$, where

$$q(k) \equiv \{q_r, q_{r-1}, \cdots, q_{k+1}, 0, \cdots, 0\}.$$

It is important to understand that $q(k)$ restricts the allowed values of λ to those UIR's which contain the scalar representation of K. This, in turn, restricts the allowed values of p. Concrete cases will be presented in Secs. IV and V.

There is another point of interest: namely, that we know that, since G is a locally compact unimodular Lie group, the right and left regular representations are simultaneously defined on the group and that they commute, since there exists a left and right G-invariant measure on the group manifold. The question arises as to what happens, e.g., with the action of the group (from the left) T^{L}_g for the right quasiregular representation on X. The answer is seen by acting with T^{L}_g on (2b) where the D's are restricted by (7). One finds that under T^{L}_g all elements of a given right coset are mapped into another right coset with respect to the subgroup $g^{-1}Kg$, so that functions constant on the right coset space $K\backslash G$ are mapped into functions constant on the right coset space $g^{-1}Kg\backslash G$.

III. DECOMPOSITION OF THE QUASIREGULAR REPRESENTATION

An important consequence of our knowledge of a complete set of generalized functions on X, as a subset of the UIR matrix elements of the group G, is that we automatically obtain a decomposition of the unitary quasiregular right representation into its UIR's, along with their multiplicities. Recall that the quasiregular right representation T^{R} of a locally compact Lie group G on $L_2(X)$ is defined as

$$T^{\mathrm{R}}_g f(x) = f(xg), \quad (8)$$

where $f \in L_2(X)$, $x \in X$, and $g \in G$. It is well known that, if there exists a right G-invariant measure on X, then T^{R}_g is a unitary representation.

In fact, any $f \in L_2(X)$ can be decomposed as

$$f(x) = \int_{\hat{G}(\text{restricted})} d\hat{\mu}(\lambda) \sum_{p,q(k)} F_{q(k),p}(\lambda) D^\lambda_{p,q(k)}(g^{-1}), \quad (9)$$

where $q(k)$ was defined above. But T^{R}_g acting on (9) transforms all $D^\lambda_{p,q(k)}(g^{-1})$ with a fixed value of $q(k)$ among themselves. Hence, for each fixed value of $q(k)$ there exists one UIR in the direct-integral decomposition of $L_2(X)$, and thus the multiplicity is exactly the number of different values of $q(k)$ constrained by a fixed (allowable) λ and $\bar{q}_k = 0$. This number may, or may not, be denumerable.

IV. APPLICATION: THE ORTHOGONAL GROUP

We shall give first a brief description of the group and representation spaces of the orthogonal groups $SO(n)$. The group manifold of $SO(n)$ can be parametrized inductively by "Euler" angles (enclosing collective variables in curly brackets) as

$$\begin{aligned} R_n(\{\vartheta\}^{(n)}) &= R_{n-1}(\{\vartheta\}^{(n-1)})S_n(\{\vartheta^{(n)}\}),\\ S_n(\{\vartheta^{(n)}\}) &= r_{n-1,n}(\vartheta^{(n)}_{n-1,n})\\ &\quad\times\cdots\times r_{23}(\vartheta^{(n)}_{23}) \times r_{12}(\vartheta^{(n)}_{12}), \end{aligned} \quad (10)$$

where $R_k \in SO(k)$ and $r_{ab}(\vartheta)$ is a rotation by ϑ in the (a, b) plane. The ranges of the variables are $0 \leq \vartheta_{12} < 2\pi$ and $0 \leq \vartheta_{k-1,k} \leq \pi$, $k = 2, 4, \cdots, n$. Thus, the $SO(n)$ manifold is the product of the $SO(n-1)$ manifold and the n-dimensional unit sphere: the homogeneous space $SO(n-1)\backslash SO(n)$, parametrized by the $n-1$ angles $\{\vartheta^{(n)}\}$.

Notice that, for $SO(3)$, $R_3(\alpha, \beta, \gamma) = r_{12}(\alpha)r_{23}(\beta) \times r_{12}(\gamma)$. This differs from the more general usage[20] in that the second rotation is made around the 1-axis rather than the 2-axis. This will cause no difficulty, however.

The Haar measure on $SO(n)$ can be split according to the parametrization (10) as $d\mu(R_n) = d\mu(R_{n-1})\,dS_n$, where

$$dS_n = \sin^{n-2}\vartheta_{n-1,n}\,d\vartheta_{n-1,1}\cdots\sin\vartheta_{23}\,d\vartheta_{23}\,d\vartheta_{12} \quad (11)$$

is the measure on the space $SO(n-1)\backslash SO(n)$. The volume of the group can be seen to be given by $V(SO(n)) = V(SO(n-1))A_n$ [and $V(SO(2)) = 2\pi$], where $A_n = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$ is the surface of the n-dimensional sphere.

The homogeneous space $SO(k)\backslash SO(n)$ is thus the set of points $S_{k+1}(\{\vartheta^{(k+1)}\})\cdots S_n(\{\vartheta^{(n)}\})$ and is a $[\frac{1}{2}n(n-1)-\frac{1}{2}k(k-1)]$-dimensional manifold with Haar measure $dS_{k+1}\cdots dS_n$.

From the work of Gel'fand and Tsetlin[21,22] we know that the bases for UIR's of $SO(n)$ classified by the canonical chain $SO(n-1)\supset\cdots\supset SO(2)$ can be labeled as

$$\left|\begin{array}{llll} J_{n,1} & J_{n,2} & \cdots & J_{n,[\frac{1}{2}n]} \\ J_{n-1,1} & J_{n-1,2} & \cdots & J_{n-1,\frac{1}{2}[(n-1)]} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ J_{4,1} & J_{4,2} & & \\ J_{3,1} & & & \\ J_{2,1} & & & \end{array}\right\rangle, \tag{12}$$

where $[\frac{1}{2}k]$ is the largest integer less than or equal to $\frac{1}{2}k$. This ket transforms as the $J_k=(J_{k1},J_{k2},\cdots J_{k[\frac{1}{2}k]})$ UIR of $SO(k)$. The J_{ab} are either all integer or all half-integer and are constrained by the inequalities

$$\begin{aligned} &J_{2k+1,j}\geq J_{2k,j}\geq J_{2k+1,j+1}, \\ &\qquad\qquad j=1,\cdots,k^{-1}, \\ &\qquad\qquad k=1,\cdots,[\tfrac{1}{2}(n-1)], \\ &J_{2k,j}\geq J_{2k-1,j}\geq J_{2k,j+1}, \\ &\qquad\qquad j=1,\cdots,k-1, \\ &\qquad\qquad k=1,\cdots,[\tfrac{1}{2}n], \\ &J_{2k+1,k}\geq|J_{2k,k}|, \quad k=1,\cdots,[\tfrac{1}{2}(n-1)], \\ &J_{n,[\frac{1}{2}n]}\geq 0, \quad n \text{ odd}, \\ &J_{n-1,[\frac{1}{2}n]-1}\geq|J_{n,[\frac{1}{2}n]}|, \quad n \text{ even}. \end{aligned} \tag{13}$$

The number of labels in the ket (12) is

$$\begin{aligned} L^0(n)&=\sum_{m=2}^{n}[\tfrac{1}{2}m]=\tfrac{1}{4}(n^2-1), \quad n \text{ odd}, \\ &=\tfrac{1}{4}n^2, \qquad n \text{ even}, \end{aligned}$$

and the number of UIR, row, and column labels of the $D^{J_n}_{\bar{J}_{n-1}\bar{J}'_{n-1}}(R_n^{-1})$ (by using the notation of Sec. 2, namely, $\bar{J}_k\equiv\{J_k,J_{k-1},\cdots,J_2\}$) is thus $[\frac{1}{2}n]+2L^0(n-1)=\frac{1}{2}n(n-1)$, i.e., the same as the number of parameters of the group.

The scalar representation of $SO(k)$ is $J_k=(0,0,\cdots,0)\equiv 0$. Notice, then, that, if we have zeros in the J_k row, $k>2$, of (12), the inequalities (13) imply that the J_{k+1} row must consist of zeros, except for $J_{k+1,1}$, which is only constrained to be integer. In the J_{k+2} row, all except $J_{k+2,1}$ and $J_{k+2,2}$ must be zero, etc. Thus we can see that only the "most symmetric" $(J,0,\cdots,0)$ UIR'S of $SO(n)$, $n>3$, can be realized on the homogeneous space $SO(n-1)\backslash SO(n)$, a result familiar from the theory of spherical functions.[4]

For $SO(2)\backslash SO(3)$, $J_{21}=0$ implies only that J_{31} must be integer and thus the functions on the 3-dimensional sphere[23] can realize all—but only—the single-valued representations of the group $SO(3)$.

For $SO(k)\backslash SO(n)$, the ket (12) will have $\bar{J}_k=0$. These zeros will "propagate" upwards under a diagonal to the right (see Fig. 1), and there will be $[\frac{1}{2}(2k-n)]$ zeros in the row of the UIR labels. Thus, if $k\leq\frac{1}{2}(n+1)$ (n odd) or $k\leq\frac{1}{2}n$ (n even), it will be possible to realize all the single-valued UIR's of $SO(n)$ in this space.

It is not difficult to see that the number of indices $J_{q,r}$, $q\geq k+1$, which are forced to be zero is $L^0(k-1)$, so that there are $[\frac{1}{2}k]+2L^0(k-1)=\frac{1}{2}k(k-1)$ zeros in the pattern (12), and the number of remaining free labels is $\frac{1}{2}n(n-1)-\frac{1}{2}k(k-1)$, equal to the number of parameters of the space $SO(k)\backslash SO(n)$.

On the other hand, if some of the UIR labels are forced to be zero, these zeros may propagate into the row (p) indices as well, downwards along a vertical line (see Fig. 1). There will thus be $2L^0(k-1)+[\frac{1}{2}k]-L^0(2k-n)$ zeros in the column (q) pattern (12), $[\frac{1}{2}(2k-n)]$ in the UIR labels, and $L^0(2k-n-1)$ in the row (p) pattern (12). The number of remaining free labels is again equal to the number of parameters of the homogeneous space.

As far as the $SO(n-1,1)$ group is concerned, the same procedure can be applied to the chain of subgroups $SO(n-1,1)\supset SO(n-1)\supset\cdots\supset SO(2)$. The only differences lie in (10), where $r_{n-1,n}(\xi)$ is now a boost in the $(n-1)$ direction so that $0\leq\zeta<\infty$.

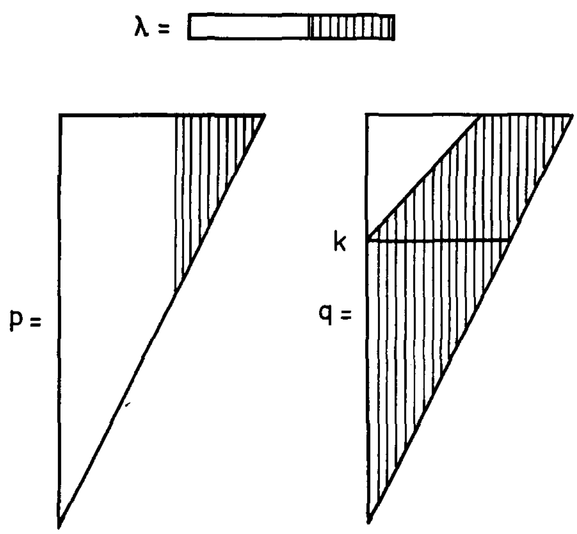

FIG. 1. Graphical representations of the zeros (shaded areas) in the UIR, row, and column labels of $D^{\lambda}_{p,q(k)}(R^{-1})$ for the orthogonal group.

The homogeneous space $SO(n-1)\backslash SO(n-1,1)$ is an n-dimensional revolution hyperboloid with a measure (11), where the trigonometric function in $\vartheta_{n-1,1}$ must be replaced by a hyperbolic one in ζ. As shown by Chakrabarti,[24] the ket (12) replaces its discrete $J_{n,1}$ index by a continuous one, which is not subject to the restrictions (13). Our knowledge of these groups[25] is still unsatisfactory regarding the UIR matrix elements themselves[26]; however, the statements made regarding the complete set of functions on the homogeneous spaces considered above do not depend on their detailed knowledge.

The inhomogeneous orthogonal group $ISO(n)$ is the semidirect product of the n-dimensional translation group $T(n)$ and $SO(n)$. Its elements are the points $g = (x, R)$, $x \in T(n)$, $R \in SO(n)$, with the product $(x_2, R_2)(x_1, R_1) = (x_2 + R_2x_1, R_2R_1)$.

The $ISO(n)$ manifold is thus the direct product of the $T(n)$ and $SO(n)$ manifolds, its Haar measure being $d^nx\, d\mu(R)$. Kets similar to (12) which classify the components of the UIR vectors using the chain $ISO(n) \supset SO(n) \supset \cdots \supset SO(2)$ have been set up by Chakrabarti.[24] The more common (and physically relevant) classification of the UIR's of $ISO(n)$ [and of $ISO(n-1, 1)$] is the one which follows Wigner's "little group" method.[1] Harmonic analysis on these groups, with their UIR's classified by the mass-M spin-J pair of labels (J is a collective index for $n \geq 5$), has been carried out,[10,11] and the case of functions on the space of cosets of the type $ISO(n)/SO(k)$ developed in Ref. 11. This does not fall, however, within the bounds of our formalism, which so far requires the use of the "canonical" chain. This subject, then, requires further investigation.

V. APPLICATION: THE UNITARY GROUP

We shall present one parametrization of the $U(n)$ group manifold which we consider convenient because of its inductive definition, which makes it similar to the "Euler" angle parametrization of the orthogonal groups seen in the previous section.[26] It can be conveniently used to parametrize the homogeneous space $SU(k)\backslash SU(n)$.

Enclosing collective labels in curly brackets, we define

$$U_n(\{\varphi, \vartheta\}^{(n)}) = U_{n-1}(\{\varphi, \vartheta\}^{(n-1)})C_n(\{\varphi^{(n)}, \vartheta^{(n)}\}),$$
$$C_n(\{\varphi^{(n)}, \vartheta^{(n)}\}) = \Phi_n(\varphi_n^{(n)})r_{n-1,n}(\vartheta_n^{(n)}) \times \cdots \times \Phi_2(\varphi_2^{(n)})r_{1,2}(\vartheta_2^{(n)}) \times \Phi_1(\varphi_1^{(n)}), \quad (14)$$

where $U_k \in U(k)$; $\Phi_k(\varphi)$ is a diagonal matrix with elements $e^{[i\varphi/(1-k)]}$ in the (q, q) positions, $q = 1, 2, \cdots, k-1$, $e^{(i\varphi)}$ in the (k, k) position, and 1 in the remaining places on the diagonal. It is unimodular for $k \geq 2$.

As in the previous section, $r_{ab}(\vartheta)$ is a matrix with $\cos\vartheta$ in the (a, a) and (b, b) positions, 1 elsewhere on the diagonal, $-\sin\vartheta$ and $\sin\vartheta$ in the (a, b) and (b, a) positions, and 0 in the remaining places.

The number of parameters of C_n is $2n - 1$, so that the parameters of $U(n)$ are n^2. The condition of unimodularity implies

$$\sum_{k=1}^{n} \varphi_1^{(k)} = 0,$$

since

$$\det [C_k(\{\varphi^{(k)}, \vartheta^{(k)}\})] = \exp (i\varphi_1^{(k)}),$$

and this restricts by one the number of parameters. The ranges of the "rotation" angles are $0 \leq \vartheta_j^{(k)} \leq \frac{1}{2}\pi$, $j = 2, \cdots, k$, and of the "phases" $0 \leq \varphi_j^{(k)} < 2\pi$, $j = 1, \cdots, k$. The Haar measure on $U(n)$ decomposes as for the orthogonal group: $d\mu(U_n) = d\mu(U_{n-1})\, dC_n$.

We can see that $C_n^{-1}(\{\varphi^{(n)}, \vartheta^{(n)}\})$ acting on the n-dimensional complex vector $(0, \cdots, 0, 1)$ generates $(z_1, \cdots, z_n)$, the surface of the n-dimensional complex unit sphere. The surface element of this can be found by using the fact that, for each coordinate,

$$d^2z = d\,\mathrm{Re}\,(z)\, d\,\mathrm{Im}\,(z) = r\, dr\, d\psi,$$

where $r = \mathrm{mod}\,(z)$ and $\psi = \arg\,(z)$. The modulus is thus only a function of the ϑ variables, the argument only of the φ variables, and

$$dC_n = (r_1 \cdots r_n)(dr_1 \cdots dr_n)(d\psi_1 \cdots d\psi_n).$$

The first term is $\sin^{n-1}\vartheta_n \cos\vartheta_n \cdots \sin\vartheta_2 \cos\vartheta_2$, the second one is (11), the measure on the n-dimensional real sphere, and the third one is just $d\varphi_1 \cdots d\varphi_n$ since the Jacobian is unity.

The measure on the n-dimensional complex sphere is therefore

$$dC_n = \sin^{2n-3}\vartheta_n^{(n)} \cos\vartheta_n^{(n)}\, d\vartheta_n^{(n)}\, d\varphi_n^{(n)} \times \cdots \times \sin\vartheta_2^{(n)} \cos\vartheta_2^{(n)}\, d\vartheta_2^{(n)}\, d\varphi_2^{(n)} \times d\varphi_1^{(n)}$$
$$= d\varphi_n^{(n)} s_n^{2n-3}\, ds_n \times \cdots \times d\varphi_2^{(n)} s_2\, ds_2 \times d\varphi_1^{(n)}, \quad (15)$$

where $s_k = \sin_k^{(n)}$; thus, this is the measure on the homogeneous space $SU(n-1)\backslash SU(n)$. Correspondingly, the space $SU(k)\backslash SU(n)$ is parametrized by

$$C_{k+1}(\{\varphi^{(k+1)}, \vartheta^{(k+1)}\}) \cdots C_n(\{\varphi^{(n)}, \vartheta^{(n)}\}),$$

has $n^2 - k^2$ parameters, and its measure is $dC_{k+1} \cdots dC_n$. The volume of the group is $V(U(n)) = V(U(n-1))B_n$ [and $V(U(1)) = 2\pi$], where $B_n = 2\pi^n/\Gamma(n) = A_{2n}$, the surface of the n-dimensional complex sphere.

Using again the work of Gel'fand and Tsetlin,[27] we can write the bases for UIR's of $U(n)$, classified by the canonical chain[15] $U(n) \supset U(n-1) \supset \cdots \supset U(1)$, as

$$\left| \begin{matrix} K_{n,1} & K_{n,2} & K_{n,3} & \cdots & K_{n,n} \\ K_{n-1,1} & K_{n-1,2} & \cdots & K_{n-1,n-1} & \\ \cdot & & & \cdot & \\ \cdot & & \cdots & \cdot & \\ \cdot & & & \cdot & \\ & K_{2,1} & K_{2,2} & & \\ & & K_{1,1} & & \end{matrix} \right\rangle . \quad (16)$$

This ket transforms as the $K_k = (K_{k1}, K_{k2}, \cdots, K_{kk})$ UIR of $U(k)$, and these labels are constrained by the inequalities

$$K_{k,j} \geq K_{k-1,j} \geq K_{k,j+1}, \quad j = 1, \cdots, k-1, \quad k = 2, \cdots, n. \quad (17)$$

The number of labels in (16) is

$$L^u(n) = \sum_{m=1}^{n} m = \tfrac{1}{2}n(n+1),$$

and the number of UIR, row, and column labels of $D^{K_n}_{\bar{K}_{n-1}\bar{K}'_{n-1}}(U)$ (again, $\bar{K}_k \equiv \{K_k, K_{k-1}, \cdots, K_1\}$) is thus $n + 2L^u(n-1) = n^2$, i.e., the same as the number of parameters of the group $U(n)$.

The representations of $SU(n)$ have the same labeling as those of $U(n)$, except that

$$(K_{n,1}, K_{n,2}, \cdots, K_{n,n}) \equiv (K_{n,1} + K, K_{n,2} + K, \cdots, K_{n,n} + K)$$

for any K, so that we can take $K = -K_{n,n}$ and thus restrict the last UIR label to zero, having thus $n^2 - 1$ UIR, row, and column labels for the (n^2-1)-parameter group $SU(n)$.

The scalar representation of $SU(k)$ is $K_k = (0, \cdots, 0)$, but this is equivalent to $(K, \cdots, K)$ in the pattern (16), forcing through the inequalities (17), all the rows below the K_k row to be $(K, \cdots, K)$ as well (see Fig. 2). This takes the place of the UIR 'O' in the case of the orthogonal groups (Sec. IV).

If the scalar representation of $SU(k)$ is to be present in (16), the inequalities (17) imply that all but two of the labels of the K_{k+1} row must be equal: $K_{k+1,2} = \cdots = K_{k+1,k} = K_{k,1}$ and, in the row above that one, $K_{k+2,3} = \cdots = K_{k+2,k} = K_{k+1,2} = K_{k,1}$, etc. Thus, we can see that only the $(J, K, \cdots, K, 0)$ UIR's of $SU(n)$ can be realized on the homogeneous space $SU(n-1)\backslash SU(n)$. In particular, this places no restriction on the $SU(3)$ UIR's which can be realized on $SU(2)\backslash SU(3)$ homogeneous space,[2,3] as can be seen also noting that every UIR of $SU(3)$ contains the scalar representation[28] of $SU(2)$.

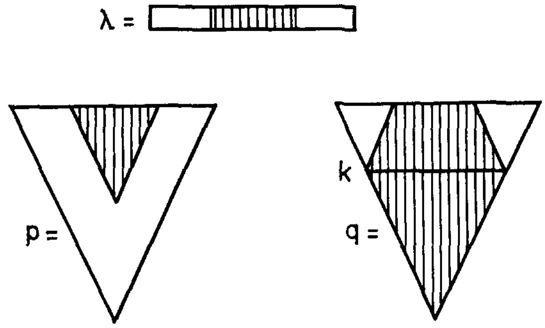

FIG. 2. Graphical representation of the regions of constant values (shaded areas) in the UIR, row, and column labels of $D^{\lambda}_{p,q(k)}(U^{-1})$ for the unitary group.

In general, all UIR's of $SU(n)$ can be realized on $SU(k)\backslash SU(n)$ when $k \leq \frac{1}{2}(n+1)$. The equality of the $2L^u(k-1) + k - 1 = k^2 - 1$ UIR labels of a scalar representation of $SU(k)$ and the triangle above it (Fig. 2) leaves $n^2 - k^2$ free parameters, i.e., the number of parameters of the space $SU(k)\backslash SU(n)$.

If $k > \frac{1}{2}(n+1)$, only the UIR's of $SU(n)$ with $K_{n,n-k+1} = K_{n,n-k+2} = \cdots = K_{n,k}$ are to be realized on $SU(k)\backslash SU(n)$. The number of free parameters can be seen (Fig. 2) to be again equal to $n^2 - k^2$.

There seems to be no fundamental difficulty in carrying out this program for the $SU(n-1, 1)$ groups[29] classified by the canonical chain $SU(n-1, 1) \supset U(n-1) \supset \cdots \supset U(1)$ nor for the inhomogeneous unitary group $ISU(n)$, whose elements are $g = (x, U)$, where $x \in T(2n)$ and $U \in SU(n)$. The kets in the Wigner "little group" chain can be constructed using the $T(2n)$ subgroup, and its representations, labeled by an n-dimensional complex vector. Again, our formalism requires that we follow a procedure parallel to Chakrabarti's,[24] in considering the "canonical" chain $ISU(n) \supset U(n) \supset U(n-1) \supset \cdots \supset U(1)$.

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Two-Dimensional Hydrogen Bonded Crystals without the Ice Rule*

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Models of 2-dimensional hydrogen bonded crystals obeying the ice rule, which previously have been solved exactly, are generalized by removing the ice rule. Many of the peculiar and unique properties of the solutions for the constrained models are now explained by showing that these models, above critical temperature, are equivalent to new unconstrained models at critical temperature. In addition to locating the critical temperature for the general but unsolved models, we locate the singularities of the ground state energy of a related ring of interacting spins.

INTRODUCTION

Much progress has been made recently in solving exactly models for hydrogen bonded crystals in 2 dimensions.[1–4] These models are constrained by the "ice rule." The essential point for the exact solution is Lieb's observation for 2-dimensional ice[1] that the transfer matrix has the same eigenvectors as a solvable 1-dimensional quantum many-body problem, the Heisenberg–Ising ring of interacting spins.[5]

However, the properties of the models, when determined, are surprising, and totally unlike the Ising problem. One can blame this, of course, on the ice rule constraint. However, this remark really does not clarify matters.It is the purpose of this paper to point out the qualitative relationship between constraint and behavior.

1. THE GENERAL EIGHT-SITE LATTICE PROBLEM

Consider a square lattice of N^2 vertices and thus $2N^2$ edges. We assume periodic boundary conditions in both directions. Place $2N^2$ arrows one to an edge, and assign an energy to each configuration of the four arrows about a vertex. In general, therefore, there will be $2^4 = 16$ possible energy assignments.

A large class of solvable models results when the possible vertex configurations obey the "ice rule"; that is, all configurations have infinite energy except the six with two arrows in, two arrows out of a vertex. This general six-site configuration is exactly soluble.[2] The allowed sites are the first six in Fig. 1, with a possible parametrization in the language of ferroelectrics shown below. In this notation, the arrows