Kinam, Vol 6, 141-156 (1984)

An Euclidean algebra of Hamiltonian observables in Lie optics

Kurt Bernardo Wolf *

Departamento de Matemáticas
Universidad Autónoma Metropolitana
Iztapalapa DF, México.

(Recibido julio 19, 1985)

Optical phase space has a momentum coordinate which runs over a circle; a mixed Heisenberg-Weyl group embedding requires the position coordinate to range over a lattice of "sampling" points. We build on that basic algebra an Euclidean Lie algebra whose generators are the precise position, momentum, and Hamiltonian of the optical system. The free propagator function is found and we introduce "Gaussian" beams as diffusion of the propagator to imaginary width, resembling the Gaussian exponential function of paraxial Lie optics. We regain the results of paraxial Lie optics in the metaplectic group formulation when taking "forward-concentrated" beams.

I. INTRODUCTION

The R^3 Heisenberg-Weyl model of Lie (Fourier) optics [1] builds a phase-space where each coordinate and its canonically conjugate momentum, ranges over the real line R. This allows the straightforward Schrödinger quantization of the system [2], which works well in the paraxial [3, 4] (or Gaussian) approximation, and may yield, as preliminary results seem to indicate, [5] to describe aberrations.

Nevertheless, in doing so, one neglects the fact that the optical hamiltonian momentum [6] –basically *ray direction*– ranges over a sphere S_2 in two-dimensional optics, and a *circle* S_1 in a one-dimensional world of cylindric lenses. In Section II we recapitulate this fact and the current treatment of the Schrödinger quantization of paraxial [7] and metaxial optics. Section III introduces [8] our model: the Heisenberg-Weyl group $W(ZS^2)$, with a direction coordinate on S_1 and a configuration space $\lambda\!\!\!^{-} Z$, where $\lambda\!\!\!^{-} = \lambda/2\pi$ is the reduced wavelenght and Z the set of integers. The generators of the Schrödinger Heisenberg-Weyls group $W(R^3)$, Q, P and $\mathrm{I} = \lambda\!\!\!^{-}\hat{1}$, keep Q and I as before, but in $W(ZS_1^2)$, there is only a generator of finite translations $\mathrm{E} = e^{i\theta}$. Section IV presents this finite/infinitesimal algebra, and Section V constructs the self-adjoint momentum and Hamiltonian operators on $L^2(S_1)$. We introduce the Euclidean algebra generated by the position, momentum, and Hamiltonian observables in Section V.

* Permanent address: Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México. Apdo. Postal 20-726, 01000 México DF, México.

This algebra may allow, through exponentiation to the Lie group $ISO(2)$, the role of the dynamical group of the system. Through deformations of the group one may treat paraxial and metaxial optics. Section VI obtains thus the system's propagator and Section VII uses it to define the Gaussian function for the system. Free propagation is built naturally into the model since it is a subgroup of the dynamical Euclidean group for homogeneous systems. A well-known deformation of the Euclidean group is examined in section VIII, which in the paraxial approximation yields the Gaussian $Sp(2, R)$ group of linear optics. In Section IX we offer some problems of interest.

II. THE HAMILTONIAN DESCRIPTION OF OPTICS

In the Hamiltonian description of optics, [6, 9] phase-space coordinates are set up in the following manner: the *position* of the intersection of a geometric light ray with a reference plane is a two-vector **q**, while the canonically conjugate *momentum* **p**, is a two-vector in the same plane, built as the projection on the plane of the light-ray, and of magnitude

$$p = n \sin\theta \tag{1}$$

where θ is the angle between the light ray and the normal to the plane, and n is the *index of refraction* of the medium at the point of intersection. The normal to the plane at the origin of position coordinates defines the *optical axis* (drawn to the right) along which we measure the ray evolution through a variable z, which we may call optical lenght, or time. Let us from the start work here in one-dimensional optics for reasons of simplicity, which will be justified in the concluding section.

The Hamiltonian of the system [6, 9] is

$$h = -\sqrt{n^2 - p^2} = -n\cos\theta\ . \tag{2}$$

This form, we should emphasize, is *not* the usual quantum mechanical function $(1/2)\, p^2 + v(q)$, quadratic in p. Furhtermore, for homogeneous media (n constant), there is a series expansion for h in powers of p^2:

$$h = -n + \frac{1}{2n}\, p^2 + \frac{1}{8n^3}\, p^4 + \frac{1}{16n^5}\, p^6 + \ldots \tag{3}$$

which is valid for $|p| < n$. When we take only two terms in the above expansion, we are in the *paraxial* approximation, $|p| \ll n$. There, the model Hamiltonian is [10].

$$h_2 := -n + \frac{1}{2n} p^2 \ . \tag{4}$$

Terms beyond the second are used for the treatment of aberrations. [6, 11]

The paraxial Hamiltonian h_2 in [4] has the form of the mechanical Hamiltonian of free flight. Mass is here the refraction index, which may depend on position; if such is the case, the paraxial approximation still assigns to $n(q) = \nu - \rho\ q - \sigma\ q^2$, the role of a potential, but keeps the denominator in (4) equal to ν. We thus have a system which obeys a $(1/2)\ p_2 + V(Q)$ Hamiltonian, and this is easily brought to quantum mechanical form as a Schrödinger equation. This is paraxial wave optics, which yields beautifully to Lie-theoretical techniques such as the association between optical elements and elements of the metaplectic group of canonical integral transforms [3, 4, 10, 12, 13].

One has to aware, though, that this treatment of quantum/wave optical systems requires that phase space (p, q) be a *plane* R^2, so that it serve as homogeneous space for the linear symplectic $Sp(2, R)$ group action. The same assumptions are at the root of the symplectic classification of aberrations [14], where the Weyl quantization scheme has been proposed to keep the classical and operator [7] quantities to transform as the same finite-dimensional (nonunitary) irreducible representation under $Sp(2, R)$. The effect of aberrations on functions with an $Sp(2, R)$ –defined property, such as Gaussian beams and discrete coherent states [5], may be calculated in closed form. There is the necessary assumption, however, that all wave functions $\psi(q)$ be *forward-concentrated, i.e.*, that $|\widetilde{\psi}(p)|$ be strongly peaked (concentrated) around zero (in the forward direction), negligible before n, and zero beyond.

Here we want to investigate the consequences of the forms of p and h, (1) and (2), in terms of a Heisenberg-Weyl group *distinct* from the one costumarily used in quantum mechanics, and better adapted to the topology of optical phase space, *i.e.*, the physical requirement that θ be the coordinate on a circle S_1 ($\theta \equiv \theta \bmod 2\pi$), as a ray *direction.*

III. THE MIXED HEISENBERG-WEYL GROUP $W(ZS^1)$

Hermann Weyl's original formulation of quantum mechanics [17] was through what is now called the Weyl commutation relation

$$D(x)\,E(y) = E(y)\,D(x)\,e^{-i\lambda xy}\ , \tag{5}$$

where we have placed the symbol $ƛ$ (for reduced wavelenght $ƛ = \lambda/2\pi$), in place of the costumary Heisenberg constant $\hbar$. In the Schrödinger configuration representation on $L^2(R)$, the phase-space translation operators $D(x)$ and $E(y)$ are

$$D(x) = \exp(ix\mathbf{Q}),\ [D(x)f](q) = e^{ixq}\,f(q)\ , \tag{6a}$$

$$E(y) = \exp(iy\mathbf{P}),\ [E(y)f](q) = f(q + ƛ\,y)\ . \tag{6b}$$

One may then define the Heisenberg-Weyl *group* W, as composed [2] of elements whose basic presentation [16] is

$$\begin{aligned}\omega(x, y, Z) &= D(x)\,E(y)\exp(i\,ƛ\,[Z + \frac{1}{2}xy]) \\ &= E(y)\,D(x)\exp(i\,ƛ\,[Z - \frac{1}{2}xy])\end{aligned} \tag{7}$$

and which in quantum mechanics would be written $\exp i(x\mathbf{Q} + y\mathbf{P} + z\mathbf{I})$, with $\mathbf{I} = ƛ\hat{1}$. The action on functions $f(q)$ and their Fourier transforms $\tilde{f} = Ff$, is

$$\omega(x, y, Z) : f(q) = \exp i(qx + ƛ[Z + \frac{1}{2}\,xy])\,f(q + ƛ\,y)\ , \tag{8a}$$

$$F\omega(x, y, Z) : \tilde{f}(\theta) = \exp i(\theta y + ƛ\,[Z - \frac{1}{2}\,xy])\,f(\theta - ƛx)\ . \tag{8b}$$

This follows from the representation of $\mathbf{Q}$ and $\mathbf{P}$ as q. and $-iƛd/dq$ in the first equation, and $iƛd/d\theta$ and θ in the second. We want to keep distant from the Schrödinger $L^2(R)$ representation, though, noting that we have chosen "θ" (and not "p") for the Fourier conjugate variable. Only the Schrödinger operator $\mathbf{P}$ may *not* be used in what follows, *i.e.*, disregard the first, but not the second of equations (6b). In the group presentation, the composition law is

$$\begin{aligned}&\omega(x_1, y_1, Z_1)\,\omega(x_2, y_2, Z_2) = \\ &\qquad \omega(x_1 + x_2, y_1 + y_2, Z_1 + Z_2 + \frac{1}{2}\,[y_1\,x_2 - x_1\,y_2]\ .\end{aligned} \tag{9}$$

It may be verified from (5) and (7), or from (8). The group unit is $e_w = \omega\,(0, 0, 0)$, and $\omega(x, y, z)^{-1} = \omega(-x, -y, -z)$. When (x, y, z) range over R^3, we shall call this group $W(R^3)$.

The variable θ in optical phase space is the ray direction, so it ranges over S_1. Turning the system once around the optical center ($q = 0, z = 0$) should yield an optical system indistinguishable from the first one. Hence, translation by $ƛx = 2\pi$ must be congruent with e_w. These discrete x-translations generated by $x = 2\pi/ƛ$ form a group Z_D, of elements $D(2\pi k/ƛ)$, k integer. The elements in Z_D are not central in $W(R^3)$, however. They *should* be central if we insist on an *invariance* group of turns.

To resolve and apply this requirement, we draw attention in (8a) to the factor $\exp(iqx)$. This cannot be unity for $x = 2\pi/ƛ$, *unless* $q = ƛk$, with k integer. That is, position is quantized. So, let us now take the lattice of points $q = ƛk$ as a *sample* on which we can define the wavefunctions $f(q)$. This is the price we pay for demanding that $\tilde{f}(\theta)$ be a function on the circle. (This is exactly what we do when we consider Dirac comb functions to arrive at the Fourier series [13] description of periodic phenomena). Hence the two functions (f over the integers and $\tilde{f}$ over the circle) are related, as the Fourier series coefficients of a function,

$$f_k = f(ƛk) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\theta\, \tilde{f}(\theta)\, e^{-ik\theta} \,, \tag{10a}$$

$$\tilde{f}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} f_k\, e^{ik\theta} \tag{10b}$$

Regarding the group $W(R^3)$, we are thus picking the set of elements $\omega(x, y, z)$, with x modulo $2\pi/ƛ$ and y integer, to translate properly in (8a).

Let us now verify if we are consistent in these requirements. We should notice first that Weyl's commutation relation (5) tells us that a rotation generated by $D(2\pi/ƛ)$ and a translation generated by $E(1)$ commute ($ƛxy = 2\pi \times$ integer), so their effect on a wavefunction cannot be more than a constant phase. On the group level, (9), z remains a real variable, but when we reduce to the representation $ƛ$ of W, as is done in (8), z leads a phase and is meaningful only counted modulo $2\pi/ƛ$. Yet on this representation space, $D(2\pi/ƛ)$ and $E(1)$, although they are separately invariants (and commute, of course), are not without effect. Indeed

$$[\omega(2\pi/ƛ, 1, 0) : f]\,(q) = -f(q)\,. \tag{11}$$

We thus observe a *sign* multiplier, which means that the representation (8) is two-valued over the group

$$W(ZS^2) = \{\omega(x, y, z) \in W(R^3) \mid x \bmod 2\pi/ƛ, y \in Z, \text{z} \bmod 2\pi/ƛ\} . \quad (12)$$

This may be a welcome feature, in view of the need of a sign in the metaplectic representation [1] of $Sp(2, R)$.

We are thus constrained to ascribe to the position coordinate a discrete character, a lattice of points ƛ apart. This should not come as a surprise though, we are dealing with a canonically conjugate momentum $p = n \sin\theta$ whose support is bounded by n. The Whittaker-Shannon sampling theorem [17] asserts that a band-limited signal be represented by a sampling lattice of points distant by one minimum wavelengh, λ/n, in that medium.

IV. THE INFINITESIMAL AND FINITE GENERATORS

Let us now turn to the generators of $W(ZS^2)$. This is a *mixed* [2, 15] Lie group, with two continuous parameters, x and z, and one discrete parameter, y. The Lie generators of the two continuous-parameter subgroups in the configuration-space representation (8a) are

$$\text{Q} = q \cdot (q = kƛ), \quad \text{I} = \lambda\hat{1} . \quad (13a)$$

The generator of y-translation is the *finite* group element

$$\text{E} := E(1) = \omega(0, 1, 0) , \quad (13b)$$

which effects the function-space maps

$$\text{E} : f(q) = f(q + ƛ), \text{ or E} : f_k = f_{k+1} . \quad (13c)$$

This is a unitary, irreducible representation of $W(ZS^2)$ on the space ℓ^2 of square-summable sequences, with inner product

$$(f, g)_{\ell^2} := \sum_{k=-\infty}^{\infty} f_k^* g_k . \quad (14)$$

In the momentum representation (8b), the above generators have the form

$$\mathbf{Q} = i\pi \frac{d}{d\theta}, \quad (\theta \in S_1), \quad \mathbf{I} = \lambda \hat{1}, \tag{15a}$$

$$\mathbf{E} = e^{i\theta} \cdot \; . \tag{15b}$$

In the Hilbert space $L^2(S_1)$ of functions with finite inner product

$$(\tilde{f}, \tilde{g})_{S_1} := \int_{-\pi}^{\pi} d\theta \; \tilde{f}(\theta)^* \, \tilde{g}(\theta) \; , \tag{16}$$

Q and **I** are self-adjoint and **E** is unitary. There is *no* self-adjoint generator Θ to be the position operator in momentum space. Only exp $i\Theta$ is defined, and is diagonal, *i.e.* it acts by multiplication through exp $i\Theta$, translating by one unit the discrete configuration space. The angle variable, we repeat, has no "position" operator associated to it, only operators built out of convergent series of powers of **E** may be applied to $\tilde{f}(\theta)$ or $f_k = f(\lambda k)$, the two pictures of the wavefunction.

V. MOMENTUM AND HAMILTONIAN

Observables should correspond to self-adjoint operators, since expectation values must be real [18]. The two self-adjoint operators contained linearly in **E** are

$$\mathbf{P} := \frac{n}{2i} \, (\mathbf{E} - \mathbf{E}^\dagger) = n \sin\theta \cdot \; , \tag{17}$$

$$\mathbf{H} := \frac{n}{2} (\mathbf{E} + \mathbf{E}^\dagger) = -\, n \cos\theta \cdot \; . \tag{18}$$

Note that we are here *defining* **P** and **H**, but electing the literals to stand as usual for momentum and Hamiltonian, because of (1) and (2) on the direction variable; the constant n is chosen so as to give **P** and **H** the precise forms in those equations. We may expect that we shall in the future have to consider functions $n(q)$ to model inhomogeneous-medium optics such as those of fibers. Since we cannot as yet model lenses [9-11], we shall here apply these concepts to the description of Gaussian beams under free propagation within this $W(ZS^2)$ model.

VI. THE EUCLIDEAN ALGEBRA

We have thus at out disposition the three operators **Q**, **P**, and **H**, which are self-adjoint when represented on ℓ^2 and $L^2(S_1)$ in the following forms:

$$[\mathbf{Q}f](q) = q\, f(q) \ , \qquad [\mathbf{Q}\tilde{f}](\theta) = i\lambda \frac{d}{d\theta}\tilde{f}(\theta) \ , \tag{19a}$$

$$[\mathbf{P}f](q) = \frac{n}{2i}[f(g+\lambda) - f(q-\lambda)] \ , \ [\mathbf{P}\tilde{f}](\theta) = n\sin\theta\, \tilde{f}(\theta) \ , \tag{19b}$$

$$[\mathbf{H}f](q) = -\frac{n}{2}[f(q+\lambda) + f(q-\lambda)] \ , [\mathbf{H}\tilde{f}](\theta) = -n\cos\theta\, \tilde{f}(\theta) \ . \tag{19c}$$

We may also perform their commutators, and see that

$$[\mathbf{Q}, \mathbf{P}] = -i\lambda \mathbf{H} \ , \tag{20a}$$

$$[\mathbf{Q}, \mathbf{H}] = i\lambda\, \mathbf{P} \ , \tag{20b}$$

$$[\mathbf{P}, \mathbf{H}] = 0 \ . \tag{20c}$$

In contrast to the basic Schrödinger operators **Q** and **P**, which obey $[\mathbf{Q}, \mathbf{PJ} = i\lambda\hat{1}$, and for which in the paraxial approximation $\mathbf{H}_2 = -n\hat{1} + 1/2\mathbf{P}^2$, so one has the commutators $[\mathbf{Q}, \mathbf{H}] = i\lambda\mathbf{P}$ and $[\mathbf{P}, \mathbf{H}] = 0$, identical to (20b, c), the Euclidean algebra *iso*(2) exhibits the new commutator (20a).

For paraxial optics we regain the Schrödinger representation of the Heisenberg-Weyl algebra. As we remarked earlier, we take matrix elements only between *forward-concentrated* functions, *i.e.* functions such that $\tilde{f}_c(\theta) \ll 1$ outside $\theta \ll 1$. For (19b) and (19c) se see that $\mathbf{P}\tilde{f}_c \cong n\theta\tilde{f}_c$, $\mathbf{H}\tilde{f}_c \cong -n\tilde{f}_c$. In that case, the commutator (20a) becomes the usual one for quantum mechanics.

VII. THE EUCLIDEAN GROUP AND FREE PROPAGATION

The Euclidean algebra *iso*(2) of last Section, may be exponentiated to a Euclidean *group*, with subgroups generated by (19). These are, on $\ell^2(q = \lambda k)$ and $L^2(S_1)$,

$$[exp\,(ix\mathbf{Q})\,f]_k = exp\,(ixq)\,f_k\,,\quad [exp\,(ix\mathbf{Q})\,\tilde{f}\,]\,(\theta) = \tilde{f}\,(\theta - \lambda x), \tag{21a}$$

$$[exp\,(iy\mathbf{P}\,)\,f]_k = \sum_{k'} f_{k'}\,J_{k-k'}\,(yn),$$

$$[exp\,(iy\mathbf{P})\,\tilde{f}\,](\theta) = exp\,(iyn\,\sin\theta)\,\tilde{f}\,(\theta)\;, \tag{21b}$$

$$[exp(iz\mathbf{H})\,f]_k = \sum_{k'} f_{k'}\,exp\,[-\,i\pi\,(k-k')/2]\,J_{k-k'}\,(Zn),$$

$$[exp\,(iZ\mathbf{H})\,\tilde{f}\,]\,(\theta) = exp\,(izn\,\cos\theta)\,\tilde{f}\,(\theta)\;. \tag{21c}$$

In fact, by changing the sign of the exponent in (21c), we have the evolution operator of optical free propagation

$$\mathbf{F}_Z\;:= exp\;(-\,i\,z\,\mathbf{H})\;. \tag{22a}$$

The representation of $\mathbf{F}_z$ on ray direction θ is a diagonal integral kernel:

$$\tilde{F}_z\,(\theta,\theta') \;=\; \delta(\theta-\theta')\,exp\;[izn\,\cos\theta],\,\theta\in S_1 \tag{22b}$$

In coordinate space, we have the *propagator matrix* [21]

$$F_z\,(k,k') = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta'\;exp\;i\,[zn\,\cos\theta - (k-k')\,\theta]$$

$$= exp\;[i\pi\,(k'-k)/2]\,J_{k'-k}\,(zn),\quad q = \lambda k,\; k\in Z\;. \tag{22c}$$

This propagator is simple enough to allow visual examination of some features. It represents a process which is *unitary* in ℓ^2. The sum of the point illuminations, $\Sigma_k\,|\,\psi_k\,|^2$, is conserved. The process is homogeneous (invariant under q-translation, hence depending only on $k'-k$), and isotropic (only dependent on $|\,k'-k\,|$). The propagator here is related to the mechanics of a vibrating infinite lattice of masses [19], which sports a $i\kappa\,\sin\theta$ in the exponent of the integrand corresponding to (21b), in place of our $izn\,\cos\theta$ in (21c). The optical system is here "rotated" by 90° with respect to the mechanical model. This bring in the phase $\exp(i\pi(k'-k)/2)$ in front of the Bessel function; the Bessel functions which propagate the vibrating lattice have no phase, and can be seen in Ref. 20.

In figure 1, we sketch the zeros of the Bessel functions of integer index.

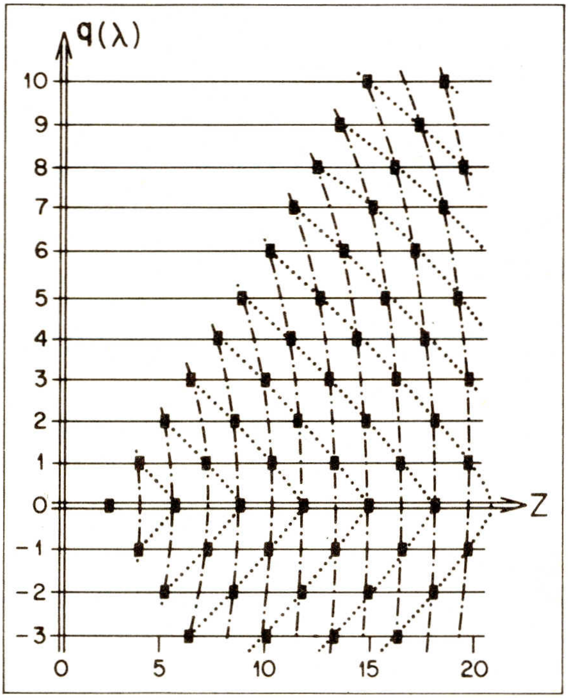

Figure 1 The zeros of the Bessel function (black rectangles), joined to outline the wavefronts of a Gaussian beam in free propagation.

These zeros mark the wavefronts of a sharply waisted beam, and may be compared with the wavefronts obtained by Gaussian exponential functions, sharply peaked, in the paraxial approximation [12]. It must be remarked, though, that wave-fronts are a continuous concept. If position coordinates are discrete, we may join their zeros in various ways to visually integrated wavefronts. In the same figure we may draw wavefronts which open asymptotically like 45° cones, or asymptotically plane fronts. In the latter, we join zeros of Bessel functions in index jumps of two.

It is also evident from the figure in Ref. 20, that the propagator here has its amplitude concentrated on a cone. This may be seen recalling that the Bessel function $J_k(zn)$ remains near zero along z until it finds its first zero; around the first zero, oscillatory (and damped) behaviour begins; for k large, this happens when $zn = k + 1.85575k^{1/3} + o(k^{-1/2})$. The damping is $\sim z^{-1/2}$. We conclude thus that the propagator $F_z(k, k')$ is significantly different from zero on the cone $zn \gtrsim q/\lambda\!\!\!^-$, or $q \lesssim n \lambda\!\!\!^- z$. The limit $\lambda\!\!\!^- \to 0$ allows for infinitely focused beams, while for every finite $\lambda\!\!\!^-$ there is a minimal unavoidable spread in q given by $n \lambda\!\!\!^- z$.

Propagator functions describe the effect on the medium due to a delta initial distribution, at an optical lenght z. Since q-space is discrete, it is a Kronecker δ. Propagators, therefore, exhibit maximal spread in momentum, since they have been completely constrained in initial position. Consider now a wavefunction completely constrained in momentum, $\widetilde{f}_{\theta_0}(\theta) = \delta(\theta - \theta_0)$, *i.e.*, a light beam with a precisely defined direction. Then, the optical lenght evolution is obtained through product with $\exp(-izn \cos \theta_0)$, still a beam with the same precise direction. The only change has been the phase. The position-space intensity is obtained from the Fourier analysis of $\widetilde{f}_{\theta_0}(\theta)$ as in (10a), yielding $f_k = (2\pi)^{-1/2} \exp(-i[zn \cos \theta_0 + k\theta_0])$, *i.e.*, it is a plane wave directed with angle θ_0 to the optical axis.

VIII. GAUSSIAN FUNCTIONS

What is the model's best choice for a Gaussian beam? Let us first propose the following definition of the centered *Gaussian function* of width w for an $W(ZS^2)$ system. This we do in terms of the propagator (22), as

$$G_w(q) := F_{iw}(k = q/\lambda\!\!\!^{-}, 0) = I_k(wn) \quad . \tag{23}$$

It is the propagator at an imaginary optical lenght $z = iw$, so that the original δ is *diffused* to G_w. The last member in (23) is simply (22c), where the modified Bessel function I_k appears on the imaginary axis of the original Bessel J_k function's argument [22].

In figure 2 we draw the profile of such a Gaussian for discrete configuration space (for a width $wn = 5$) from a standard table [23]. The profile at the integers indeed looks "Gaussian". Moreover, the virtue of these Gaussian functions is that they *stay* Gaussian under free propagation, even if with complex width, for it is evident that we have translations in a *complex* optical lenght variable:

$$\begin{aligned} \mathbf{F}_Z\, G_w(q) &= [\mathbf{F}_Z\, \mathbf{F}_{iw}\, \delta_0](q) \\ &= [\mathbf{F}_{Z+iw}\, \delta_0](q) = F_{Z+iw}(k = q/\lambda\!\!\!^{-}, 0) \\ &= G_{w-iZ}(q) \quad . \end{aligned} \tag{24}$$

We may abstract the symbols and recall that the exponential Gaussian function $w^{-1/2} \exp(-q^2/2w)$ has exactly the same property with respect to Gaussian (paraxial) free propagation [12] generated by h_2 in (4).

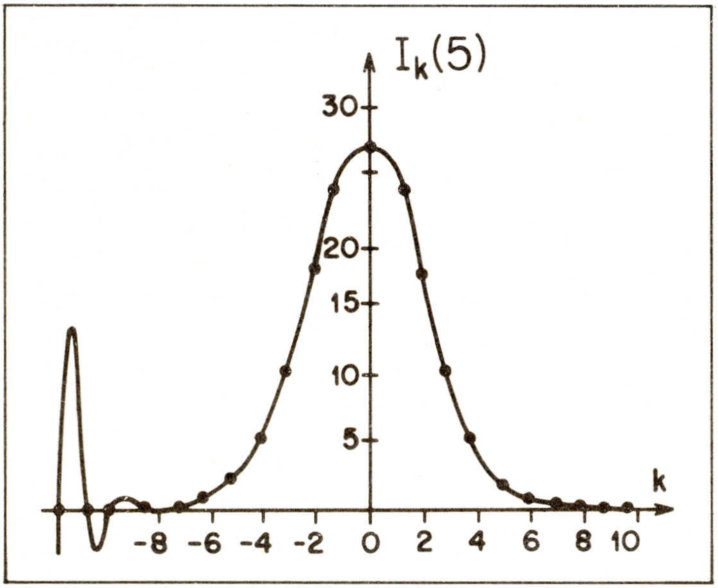

Figure 2 The modified Bessel function for constant argument outlines the shape of a Gaussian beam at the object plane. Black dots are the position coordinate. (From Ref. 23.)

The direction (momentum) distribution of a thus defined Gaussian beam, (23), is, due to (10) and (22b),

$$\widetilde{G}_w\,(\theta) = \exp\,[wn\cos\theta] \tag{25}$$

which is indeed preferentially in the forward direction for large width w. Rotated and translated Gaussian beams are obtained acting with $W(ZS^2)$.

IX. DEFORMATION OF THE EUCLIDEAN GROUP

Still within the framework of the Euclidean algebra we can build enveloping algebras. Indeed, since in this representation the lenght of the Euclidean translation vector is constant ($\mathbf{P}^2 + \mathbf{H}^2 = n^2\,1$), we can use it to deform [24, 25] the Euclidean group to $SO(2, 1)$, locally isomorphic to $Sp(2, R)$. Specifically, Bargmann's realization [24] of this group on the circle is generated by

$$\mathbf{L}_1 := \frac{1}{2}(\mathbf{QH} + \mathbf{HQ}) + c\mathbf{P} \rightarrow -in\lambda\cos\theta\,\frac{d}{d\theta} + n(c + \frac{i}{2}\lambda)\sin\theta\ , \tag{26a}$$

$$\mathbf{L}_2 := \frac{1}{2}(\mathbf{QP}+\mathbf{PQ}) - c\mathbf{H} \to in\,\lambda \sin\theta \frac{d}{d\theta} + n\,(c + \frac{i}{2}\lambda)\cos\theta, \tag{26b}$$

$$\mathbf{L}_0 := \mathbf{Q} \quad \to i\lambda \frac{d}{d\theta}. \tag{26c}$$

Quite interestingly, the first generator is the classical Snell invariant [14], generator of phase-space transformations corresponding to rotations in the $z-q$ plane around the point $z=c$. In $L^2(S_1)$ we have the principal series C_c^0 of representations of $SO(2,1)$. Fittingly, also, the compact subgroup reduction diagonalizes $\mathbf{L}_0$, which is the compact generator having a discrete spectrum. The Bargmann d- and D-functions [24] appear when (26) are exponentiated to the group $SO(2,1)$.

X. SOME CONCLUDING REMARKS

Any discussion of a model in optics would be incomplete if it did not include Gaussian –linear– optics as a limiting case for small angles. This is an $Sp(2,R)$ group with well-defined properties [3, 12] whose generators are –as Schrödinger-Heisenberg-Weyl symbols–

$$\mathbf{P}^2, \frac{1}{2}(\mathbf{PQ}+\mathbf{QP}), \mathbf{Q}^2, \quad R^3 - (\text{Heisenberg-Weyl}) \tag{27}$$

The role of the first one is taken very naturally to be $\mathbf{H}$ in (18). The last one generates Gaussian refracting-surface transformations which do not change the position of an object, but bend its rays through $\mathbf{P} \to \mathbf{P} + \gamma\,\mathbf{Q}$, where γ is the Gaussian power of the lens. Hence, $\mathbf{Q}^2$ has also a clear counterpart in the self-adjoint operator represented by $-\lambda^2 d^2/d\theta^2$ in $L^2(S_1)$ and $q^2 = \lambda^2 k^2$ in ℓ^2.

Although the basic algebra is the Euclidean one, the commutators of $\mathbf{H}$, $\mathbf{Q}^2$ and the resulting expressions are

$$[\mathbf{H}, \mathbf{Q}^2] = -i\lambda(\mathbf{QP}+\mathbf{PQ}), \tag{28a}$$

$$[\mathbf{H}, \mathbf{QP}+\mathbf{PQ}] = 2i\lambda\,\mathbf{P}^2, \tag{28b}$$

$$[\mathbf{Q}^2, \mathbf{QP}+\mathbf{PQ}] = -i\lambda\,(\mathbf{Q}^2\,\mathbf{H} + 2\mathbf{QHQ} + \mathbf{HQ}^2). \tag{28c}$$

In the paraxial approximation $\mathbf{H} \to n1 - (1/2n)\,\mathbf{P}^2$, and disregarding all operators of order higher than second, we have indeed $Sp(2, R)$. In the model presented here, (28) do *not* close into a finite-dimensional algebra for all orders. The $W(SZ^2)$ model of optics, therefore, does have a Gaussian approximation corresponding to the well-studied metaplectic transformations of $L^2(R)$. The only deformation of the Euclidean algebra, however, is the $so(2, 1)$ algebra (26), which still has to be studied before an enveloping algebra can be given to account for aberrations.

A basic objection to Schrödinger aberration Lie optics is contained in the statement that not all operators associated to observables $p^m q^n$ may be exponentiated to unitary transformations [26]. We may expect, though, that by having a compact momentum space (*i.e.* a circle), this objection may be circumvented. The main taks in this regard is to represent the action of arbitrary refracting surfaces on the spaces ℓ^2 and $L^2(S_1)$ as in (8) and (21). To this purpose we have at our disposition the results on classical systems in Ref. 27. A refracting surface $z = \zeta(q)$ produces a transformation $S(n, n'; \zeta)$ which is shown to *factorize* as

$$S(n, n'; \zeta) = R(n; \zeta)\, R(n'; \zeta)^{-1} \quad , \tag{29}$$

where the "root" transformation is locally canonical (q, $p = n \sin\theta$), and acts as

$$R(n; \zeta) : q =: \bar{q} = q + \zeta(\bar{q}) \tan\theta \quad , \tag{30a}$$

$$R(n; \zeta) : \sin\theta =: \sin\bar{\theta} = \sin\theta + \cos\theta \, \frac{d\zeta(\bar{q})}{d\bar{q}} \quad . \tag{30b}$$

Note that this is an *implicitly*-defined canonical transformations, which *must* have singularity caustics. The appropriate way to embed this transformation into the $W(ZS^2)$ model is being sought.

Finally, this model generalizes easily to an N-dimensional space where angles run over a torus Yet what is needed is some nontrivial phase-space topology to have them run over a sphere. What is discrete position then?

Acknowledgements. I would like to thank for enlightening discussions on the topic of this article to Vladimir I Man'ko (PN Lebedev Institute of Physics, USSR Academy of Sciences), Octavio Castaños (Centro de Estudios Nucleares, UNAM), Miguel Navarro-Saad (IIMAS, UNAM), and, from the symplectic seminar at the Universidad Autónoma Metropolitana, Ernesto Lacomba and Diego Rapoport.

REFERENCES

1. H Razillier and W Schempp, *Fourier optics from the perspective of the Heisenberg group*. Preprint Univ. Bonn HE 84-34, to appear in: *Lie methods in optics*, J Sánchez-Mondragón and KB Wolf (Eds.) (Proceedings of the CIFMO-CIO workshop on Jan 7-10, 1985, León, Mexico), *Lecture Notes in Physics*, (Springer Verlag). M Schmidt, *Die reele Heisenberg Gruppe und einige ihrer Anwendungen in Radarortung un Physik*. (Diplomarbeit Universität-Hochschule Siegen, DBR, April 1985).
2. KB Wolf, The Heisenberg-Weyl ring in quantum mechanics, in: *Group theory and its applications*, Vol. 3, EM Loebl (Ed.) (Academic Press, 1975).
3. M Nazarathy and J Shamir, *J Opt Soc Am* **70**, 150 (1980); *ibid. J Opt Soc Am* **72**, 356 (1982).
4. H Bacry and M Cadilhac, *Phys Rev* **A23**, 2533 (1981).
5. VI Man'ko and KB Wolf, "The influence of aberrations in the optics of Gaussian beam propagation". *Preprint Univ. Metropolitana*, May 1985, to appear in shortened versión in: *Lie methods in optics, op. cit.*
6. AJ Dragt, *Lectures on nonlinear orbit dynamics*. AIP Conference Proceedings Vol. **87**, 1982, Sect.
7. M García-Bullé, W Lassner, and KB Wolf, "The metaplectic group within the Heisenberg-Weyl ring". *Preprint Univ. Metropolitana* **2**, No. 20, Jan 1985.
8. Reference 2, Section VI.
9. AJ Dragt, *J Opt Soc Am* **72**, 372 (1982).
10. V Guillemin and S Sternberg, *Symplectic techniques in physics*. (Cambridge Univ. Press, 1984)
11. M Navarro-Saad and KB Wolf, "The group-theoretical treatment of aberrating systems. I Aligned lens svstems in third aberration order". *Comunicaciones Técnicas IIMAS*, preprint No. 363 (1984); KB Wolf, *id*. II. "Axis-symmetric inhomogeneous systems and fiber optics in third aberration order". *CT IIMAS* preprint No. 366 (1984).
12. O Castaños, E López Moreno, and KB Wolf, "The group-theoretical description of geometric and wave Gaussian optics". Preprint in elaboration.
13. KB Wolf, *Integral transforms in science and engineering*. (Plenum Publ. Corp., 1979.)
14. KB Wolf, "Symmetry in Lie optics". Preprint Universidad Metropolitana, May, 1985.
15. H Weyl, *Z Phys* **46**, 1 (1928); *ibid. The theory of groups and quantum mechanics*. (2nd Ed., Dover, 1930.)
16. Reference 1a., $(q, p, \lambda)_b$ there is $\omega(p, q, \lambda)$ here.
17. JW Goodman, *Introduction to Fourier optics*, (Mc Graw-Hill, 1968), Sect. 2.3.
18. PAM Dirac, *The principles of quantum mechanics*, (4th Ed. Oxford Univ. Press, 1958); J von Neumann, *Math Ann* **104**, 570 (19831).
19. Reference 13, Sect. 5.3.
20. Reference 13, Fig. 5.9 on page 217.
21. M Abramowitz and I Stegun, Eds., *Hadbook, of mathematical functions*, (Natl. Bureau of Standards, Applied mathematics series, vol. 55, 1st Ed. 1964), Eq. 9.1.21.
22. Ref. 21, Eqs. 9.6.3 and 9.6.19.
23. From Ref. 21, Fig. 9.9.
24. V Bargmann, *Ann Math* **48**, 568 (1947).
25. CP Boyer and KB Wolf, The algebra and group deformations $I^m[SO(n) \otimes SO(m)] \Rightarrow SO(n, m)$, $I^m[U(n) \otimes U(m)] \Rightarrow SU(n, m)$, and $I^m[Sp(n) \otimes Sp(m)] \Rightarrow Sp\ (n, m)$, for $1 \leqslant m \leqslant n$. *J Math Phys* **15**, 2096 (1974); See also, *ibid., J Math Phys* **14**, 1853 (1973).
26. JR Klauder, "Wave theory of imaging systems". To appear in: *Lie methods in optics, op. cit.*
27. M Navarro-Saad and KB Wolf, *Preprint CINVESTAV*, March 1984. To appear in: *J Opt Soc America*.

RESUMEN

El espacio fase de la óptica tiene una coordenada de momento que toma valores sobre un círculo; su inmersión en un grupo de Heisenberg-Weyl mixto require de una coordenada de posición que toma valores sobre una retícula de puntos "de muestreo". Sobre el álgebra de Lie del grupo mixto, construimos un álgebra euclideana cuyos generadores son los operadores de posición, momento y

el Hamiltoniano del sistema óptico. El propagador libre se encuentra, e introduce haces "Gaussianos" mediante la continuación difusiva del tiempo de propagación a valores imaginarios del ancho, tal como se hace con los haces Gaussianos exponenciales en la óptica de Lie paraxial. Cuando se consideran haces "fuertemente enfocados hacia adelante", se recobran los resultados de la óptica de Lie paraxial en su formulación metapléctica.