# The metaplectic group within the Heisenberg–Weyl ring

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The Heisenberg–Weyl ring contains the metaplectic group of canonical transforms acting unitarily on $\mathscr{L}^2(\mathscr{R})$. These ring elements are characterized through (i) the integral transform kernels, (ii) coset distributions, and (iii) classical functions under any quantization scheme. The isomorphism under group composition leads to several new relations involving twisted products and quantization of Gaussian classical functions. The Wigner inversion operator is a special central group element. It is shown that the only quantization scheme invariant under metaplectic transformations is the Weyl scheme. The structure studied here appears to be relevant to the study of wave optics with aberration.

## I. INTRODUCTION

A *group ring* is the structure composed out of formal linear combinations of the group elements. The group multiplication law induces an operation of multiplication for ring elements; the group unit serves as a ring unit but, since no inverse under multiplication is assured, the structure is not a group, but a *ring*. The Heisenberg–Weyl algebra $w$ and group $W$ were introduced as foundations for quantum mechanics by Heisenberg[1] and Weyl,[2] respectively. The former lead to the representation of canonically conjugate observables (having the real line for its spectrum) by Schrödinger. (The requirement on the spectra of the algebra elements is partially circumvented in Weyl's approach.) The Stone-von Neumann theorem[3] assures us of the existence and uniqueness of the Schrödinger operators representing position and momentum.

The Heisenberg–Weyl group $W$ in $N$ dimensions has $2N+1$ generators, is nilpotent, is an extension of the group of translations of the phase space, and is a non-Abelian group with a nontrivial center. Its early association with quantum mechanics should not hide the fact that it has been most useful recently as a frame to describe wave systems—optical and radar—where a meaningful phase space and geometric (i.e., classical) limit exist.

One of the peculiar features of the Heisenberg–Weyl algebra $w$, is that its isomorphism group is larger than the group $W$ of Heisenberg and Weyl. The endomorphism of the enveloping algebra $\overline{w}$ (factorized by ($\mathbb{H}$-1), where $\mathbb{H}$ is the central generator) of the Heisenberg–Weyl algebra have been studied by Dixmier.[4] The Heisenberg–Weyl algebra can undergo *symplectic* real linear transformations in the position and momentum generators[5]; these are the linear canonical transformations in quantum mechanics studied by Moshinsky and Quesne,[6] who also inquired into the representation of these on the $\mathscr{L}^2(\mathscr{R}^N)$ Hilbert space of wave functions. On this space, the group has a two-valued representation that is faithful for the twofold cover of the symplectic group [Sp($2N,\mathscr{R}$) is infinitely connected], i.e., the *metaplectic* group Mp($2N,\mathscr{R}$) (see Ref. 7). The latter is a subgroup of the universal covering group $\overline{\text{Sp}(2N,\mathscr{R})}$; $\text{Sp}(2N,\mathscr{R}) \simeq \overline{\text{Sp}(2N,\mathscr{R})}/\mathscr{Z}$.

For continuous groups the elements of the group ring may be characterized by a function over the group, which takes the place of generalized linear combination coefficients for the group elements. If, besides functions within some subspace of $\mathscr{L}^2(W)$ over the group manifold, we allow distributions—Dirac $\delta$'s and their derivatives up to arbitrarily high order, then the group ring $\mathscr{W}$ comes to contain the group $W$ itself, its Lie algebra $w$, and its enveloping algebra $\overline{w}$. In this context one of us examined[8] some time ago the question of quantization in physics, using the fact that the Heisenberg–Weyl ring $\mathscr{W}$ contained all operators $\mathbb{A}$ one would wish to quantize, and that these could be described either through their group function $A(g)$, $g \in W$, or through their *integral-kernel* representative $A^{\star}(q,q')$, or through a *classical function* $a_\phi(q,p)$ in some quantization scheme $\phi$. The integral kernels were derivatives of $\delta$'s and Hermiticity of the operators in $\mathscr{L}^2(\mathscr{R})$ was required.

Here we wish to use the rich structure of the Heisenberg–Weyl ring $\mathscr{W}$ to study another object, namely the metaplectic group Mp($2,\mathscr{R}$) of linear canonical transformations,[9] which lies within the ring. The set of these ring elements is characterized by a set of proper functions over the group, by integral transform kernels, and by classical functions. These compose under multiplication of ring elements as a group. The last two realizations, in particular, are interesting even as mathematical relations. For this reason, we work in $N=1$ dimension. The purpose in physics of these will be, in following papers, to treat wave optics with aberration. The Gaussian limit[10] of lens optics is served by the

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results obtained here, which come to describe the results of Nazarathy and Shamir,[11] who used canonical integral transforms. The group-theoretical treatment of wave optics with *aberration*[12] will need further structures within the Heisenberg–Weyl ring, which are under development.

In Sec. II we formally introduce the characteristics freely used above, and Sec. III describes the ring elements in terms of the three functions we have mentioned, particularly under multiplication. The metaplectic group is treated in Sec. IV and shown to contain among its elements the Fourier transform, the free-space propagation, the Gaussian lens transformation, and the Wigner operator (one of the two central elements of the ring). Section V ends showing that the only quantization scheme that is invariant under metaplectic transformations is the Weyl–McCoy rule.[13,14] This fact is probably crucial in the process of quantization (or waveization) of geometrical optics with aberration.

## II. CAST OF CHARACTERS AND ROLES

We shall be dealing with the following mathematical objects, all named after Heisenberg and Weyl (HW): The HW algebra $w$, the HW universal enveloping algebra $\overline{w}$, the HW group $W$, and the HW ring $\mathscr{W}$. Succinct definitions follow.

*The HW algebra* $w$: This is a three-dimensional vector space generated by $\mathbb{Q}$, $\mathbb{P}$, and $\mathbb{H}$, with the commutator Lie bracket

$$[\mathbb{Q},\mathbb{P}] = i\mathbb{H}, \quad [\mathbb{Q},\mathbb{H}] = 0, \quad [\mathbb{P},\mathbb{H}] = 0 . \tag{2.1}$$

It is two-step nilpotent and $\mathbb{H}$ is the central generator. Due to the Stone–von Neumann theorem,[3] the generic Hermitian representation of $w$ is the usual Schrödinger representation on a space of smooth functions:

$$(\mathbb{Q}f)(q) = qf(q), \quad (\mathbb{P}f)(q) = -i\hbar\frac{df(q)}{dq},$$
$$(\mathbb{H}f)(q) = \hbar f(q), \tag{2.2}$$

where $\hbar\in\mathscr{R}$ labels the representation. We write $\hbar = \lambda/2\pi$ in optics and $\hbar = \hbar$ in quantum mechanics.

*The HW enveloping algebra* $\overline{w}$: The generators of $w$ are multiplied (noncommutatively) to form monomials $\mathbb{Q}^m\mathbb{P}^n\mathbb{H}^k$, on which the commutator Lie bracket acts distributively through the Leibnitz identity. These monomials generate an infinite-dimensional algebra $\overline{w}$ under the commutator. Of course, $w\subset\overline{w}$.

Within $\overline{w}$ we also have a *symplectic* sp$(2,\mathscr{R})$ subalgebra, generated by

$$\mathbb{X}^1_{+1} := \mathbb{P}^2, \quad \mathbb{X}^1_0 := \tfrac{1}{2}(\mathbb{P}\mathbb{Q} + \mathbb{Q}\mathbb{P}), \quad \mathbb{X}^1_{-1} := \mathbb{Q}^2, \tag{2.3a}$$

with the well-known commutation relations

$$[\mathbb{X}^1_0,\mathbb{X}^1_{\pm 1}] = \pm 2i\hbar\mathbb{X}^1_{\pm 1}, \quad [\mathbb{X}^1_{+1},\mathbb{X}^1_{-1}] = -4i\hbar\mathbb{X}^1_0 . \tag{2.3b}$$

When writing down the "basic monomials," we may do it (i) in *standard* order, i.e., all $\mathbb{Q}$'s to the left of all $\mathbb{P}$'s as $\mathbb{Q}^m\mathbb{P}^n$; (ii) in *antistandard* order, i.e., all $\mathbb{Q}$'s to the right of all $\mathbb{P}$'s as $\mathbb{P}^n\mathbb{Q}^m$; (iii) in *symmetrized* order;[13] i.e., one-half of (i) plus (ii); (iv) in Weyl order, i.e., the sum of all permutations of the $\mathbb{Q}$'s and $\mathbb{P}$'s considered as individual objects, divided by their total factorial (the explicit expressions furnished by McCoy[14] will be given in the next section); and (v) any of an infinity of orderings, defined by Cohen's[15] ordering function $\phi$, which will be given below.

*The HW group* $W$: The Lie group $W$ generated by $w$ has elements $\omega$, which may be parametrized[8] through

$$\begin{aligned}\omega(x,y,z) :&= \exp i(x\mathbb{Q} + y\mathbb{P} + z\mathbb{H})\\ &= \exp(ix\mathbb{Q})\exp(iy\mathbb{P})\exp(i[z + \tfrac{1}{2}xy]\mathbb{H})\\ &= \exp(iy\mathbb{P})\exp(ix\mathbb{Q})\exp(i[z - \tfrac{1}{2}xy]\mathbb{H}),\end{aligned} \tag{2.4a}$$

and have the composition rule

$$\omega(x_1,y_1,z_1)\omega(x_2,y_2,z_2) = \omega(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \tfrac{1}{2}[y_1x_2 - x_1y_2]) . \tag{2.4b}$$

The group unit is $e = \omega(0,0,0)$ and the inverse $\omega(x,y,z)^{-1} = \omega(-x,-y,-z)$; the Haar measure is $d\omega = dx\,dy\,dz$. The space of self-adjoint irreducible representations of $w,\widehat{w}$ is parametrized by $\hbar\in\mathscr{R}$, with Plancherel measure $d\widehat{\omega}(\hbar) = |\hbar|d\hbar/4\pi^2$.

In the Schrödinger realization, where the algebra generators are (2.2), the unitary group action on functions $f\in\mathscr{L}^2(\mathscr{R})$ is given by

$$[\omega(x,y,z):f](q) = \exp i(qx + \hbar[z + \tfrac{1}{2}xy])f(q + \hbar y) . \tag{2.5}$$

The inner product in $\mathscr{L}^2(\mathscr{R})$ of a function $f_1$ and $\omega{:}f_2$ yields the bilinear functional on $W$ given by

$$\begin{aligned}H(f_1,f_2;x,y,z) :&= (f_1,\omega(x,y,z){:}f_2)\\ &= \int dq\, f_1(q)^*[\omega(x,y,z){:}f_2](q)\\ &= e^{i\hbar z}\int dq\, f_1(q - \tfrac{1}{2}\hbar y)^* e^{iqx} f_2(q + \tfrac{1}{2}\hbar y) .\end{aligned} \tag{2.6}$$

This is the *cross correlation* of $f_1$ and $f_2$; $\hbar y$ is the spatial correlation parameter between the two functions and $x$ the *frequency* correlation parameter. When $f_1$ and $f_2$ are allowed to run over the elements $\{f_\mu\}$ of a *basis* of $\mathscr{L}^2(\mathscr{R})$ (denumerable or generalized), then $D^{\hbar}_{\mu\nu}(\omega) := H(f_\mu,f_\nu;\omega)$ constitute the (matrix or integral kernel) *representations* of $W$. For the generalized eigenfunctions $\delta(q - q_0)$ of $\mathbb{Q}$, this yields the representation *kernel*

$$D^{\hbar}_{qq'}(\omega(x,y,z)) = \delta(\hbar y - [q' - q])\exp i(\hbar z + \tfrac{1}{2}x[q + q']), \tag{2.7}$$

which basically one obtains from (2.5). It is unitary and irreducible. The generalized eigenfunctions of $\mathbb{P}$, $(2\pi)^{-1/2}e^{ipq/\hbar}$, $p\in\mathscr{R}$, may be used to yield [Ref. 8, Eq. (2.24)] $D^{\hbar}_{pp'}(\omega)$. For the harmonic oscillator eigenfunctions of $\tfrac{1}{2}(\mathbb{P}^2 + \mathbb{Q}^2)$ one obtains [Ref. 8, Eq. (2.43) with the exponent sign correction remarked by Dahl, Ref. 16, Eq. (35)] a half-infinite matrix.

*The HW ring* $\mathscr{W}$: The elements of $W$ are taken as the formal basis for a linear vector space $\mathscr{W}$, which thus inherits the multiplication law, and whose elements are in the (Haar, formal) integrals

$$\begin{aligned}\mathbb{A} &= \int_W d\omega\, A(\omega)\omega\\ &= \int dx\,dy\,dz\, A(x,y,z)\exp i(x\mathbb{Q} + y\mathbb{P} + z\mathbb{H}),\end{aligned} \tag{2.8}$$



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with $A(\omega)$ a distribution over $W$ called[8] the *group representative* of $\mathbb{A}\in\mathscr{W}$. The elements of $\mathscr{W}$ may be linearly combined and multiplied but, since no inverse $\mathbb{A}^{-1}$ is assured for every $\mathbb{A}$, the structure of $\mathscr{W}$ is that of a *ring*.

## III. DESCRIPTION OF THE HW RING ELEMENTS

When the elements of $\mathscr{W}$ act on functions in $\mathscr{L}^2(\mathscr{R})$ carrying a definite representation $\hbar$ of $W$, the third generator $\mathbb{H}$ is simply replaced by the real number $\hbar$. The integration over $z$ in (2.8) may be thus performed defining the *coset distribution* over the space $W/Z$, $Z$ being the one-parameter central subgroup generated by $\mathbb{H}$, as

$$\hat{A}^{\hbar}(x,y) := \int_{\mathscr{R}} dz\, A(\omega(x,y,z))e^{i\hbar z}. \tag{3.1}$$

We shall henceforth drop $\hbar$ as an index in the quantities which bear it. We take $\hbar>0$. The ring element (2.8) appears as

$$\begin{aligned}\mathbb{A} &= \int_{\mathscr{R}^2} dx\,dy\,\hat{A}(x,y)\exp i(x\mathbb{Q}+y\mathbb{P})\\ &= \int_{\mathscr{R}^2} dx\,dy\,\hat{A}_{\rm s}(x,y)\exp(ix\mathbb{Q})\exp(iy\mathbb{P})\\ &= \int_{\mathscr{R}^2} dx\,dy\,\hat{A}_a(x,y)\exp(iy\mathbb{P})\exp(ix\mathbb{Q}).\end{aligned} \tag{3.2a}$$

We have also defined the *standard* and *antistandard* coset distributions, $\hat{A}_{\rm s}, \hat{A}_{\rm a}$, over the space $W/Z$ using (2.4a), that is

$$\hat{A}_{\rm s}(x,y) := \hat{A}(x,y)e^{i\hbar xy/2}, \quad \hat{A}_{\rm a}(x,y) := \hat{A}(x,y)e^{-i\hbar xy/z}. \tag{3.2b}$$

We shall associate the coset distribution $\hat{A}(x,y)$ with the name of Weyl and write it as $\hat{A}_{\rm w}(x,y)$ when convenient.

The names of "standard," "antistandard," and "Weyl" should bring to mind the quantization-scheme and operator-ordering problems. In this paper we *start* from a ring element $\mathbb{A}\in\mathscr{W}$ whose action on $\mathscr{L}^2(\mathscr{R})$ is that of an integral transform

$$\begin{aligned}(\mathbb{A}:f)(q) &= \int_{\mathscr{R}^2} dx\,dy\,\hat{A}(x,y)\int_{\mathscr{R}} dq'\, D_{qq'}(\omega(x,y,0))f(q')\\ &=: \int_{\mathscr{R}} dq'\, A(q,q')f(q'),\end{aligned} \tag{3.3}$$

with an *integral kernel* $A(q,q')$, which will be well defined, and in terms of which we shall find the coset distributions in the following way:

$$\begin{aligned}A(q,q') &:= \int_{\mathscr{R}^2} dx\,dy\,\hat{A}(x,y)D_{qq'}(\omega(x,y,0))\\ &= \frac{1}{|\hbar|}\int_{\mathscr{R}} dx\,\hat{A}(x,[q-q']/\hbar)\exp i(\tfrac{1}{2}[q+q']),\end{aligned} \tag{3.4a}$$

$$\hat{A}(x,y) = \frac{|\hbar|}{2\pi}\int_{\mathscr{R}} dq\, A(q-\tfrac{1}{2}\hbar y, q+\tfrac{1}{2}\hbar y)e^{-ixq}. \tag{3.4b}$$

We may now speak of (at least) three *classical functions* $a_c(q,p)$, the Weyl, standard, and antistandard classical functions, denoted by the subindices $c=\mathrm{W}$, s, or a, just as their corresponding coset distributions. The former are defined as the Fourier transforms of the latter:

$$\hat{A}_c(x,y) = \frac{1}{4\pi^2}\int_{\mathscr{R}^2} dq\,dp\,a_c(q,p)\exp(-i[xq+yp]), \tag{3.5}$$

with its well-known inverse (which simply changes the sign of the exponent and removes the $1/4\pi^2$ factor). Since we choose to regard the integral kernel as that which primarily defines the ring element, we write the composition of (3.2b), (3,4), and (3.5) to find the three classical functions as

$$a_{\rm W}(q,p) = \int_{\mathscr{R}} dr\, A(q+\tfrac{1}{2}r, q-\tfrac{1}{2}r)e^{-ipr/\hbar}, \tag{3.6a}$$

$$a_{\rm s}(q,p) = e^{-iqp/\hbar}\int dq'\, A(q,q')e^{iq'p/\hbar}, \tag{3.6b}$$

$$a_{\rm a}(q,p) = e^{iqb/\hbar}\int dq'\, A(q',q)e^{-iq'p/\hbar}. \tag{3.6c}$$

We should remind the reader that to quantize in the standard (antistandard) scheme means to propose functions $a_{\rm s}$ (resp. $a_{\rm a}$) of phase space $(q,p)$ and to replace the monomials (in the Taylor expansion, if need be) by the same functions of the Schrödinger operators $\mathbb{Q}$ and $\mathbb{P}$, all $\mathbb{Q}$'s being left (resp. right) of all $\mathbb{P}$'s. Rather trivially, thus we have a linear mapping $\boldsymbol{\Omega}_{\rm s}$ (resp. $\boldsymbol{\Omega}_{\rm a}$) between functions $\underline{a}(q,p)$ of phase space and elements $\mathbb{A}\in\mathscr{W}$ (which also lie in $\overline{w}$), which effect

$$\boldsymbol{\Omega}_{\rm s}(q^m p^n) = \mathbb{Q}^m\mathbb{P}^n, \tag{3.7a}$$

$$\boldsymbol{\Omega}_{\rm a}(q^m p^n) = \mathbb{P}^n\mathbb{Q}^m. \tag{3.7b}$$

The Weyl quantization scheme $\boldsymbol{\Omega}_{\rm W}$ is not so trivial, but the correspondence

$$\begin{aligned}\boldsymbol{\Omega}_{\rm W}(q^m p^n) &= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\mathbb{Q}^{m-k}\mathbb{P}^n\mathbb{Q}^k\\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\mathbb{P}^{n-k}\mathbb{Q}^m\mathbb{P}^k\end{aligned} \tag{3.7c}$$

has been given by McCoy,[14] as well as the next displayed equation. Basically, the Weyl–McCoy scheme permutes all operators and divides by the factorial of their number.

In what appears to be a characterization of all such schemes $\boldsymbol{\Omega}_\phi$, Cohen[15] introduced an *ordering* function $\phi(u)$, defining $\hat{A}_\phi(x,y) = \hat{A}_{\rm W}(x,y)\phi(\tfrac{1}{2}\hbar xy)$ and, through (3.5), a corresponding coset distribution to be entered in (3.2) to yield the ring element. Written in standard order, this is

$$\begin{aligned}\boldsymbol{\Omega}_\phi(q^m p^n) = &\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}k!\\ &\times\phi_k(-\tfrac{1}{2}i\hbar)^k\mathbb{Q}^{m-k}\mathbb{P}^{n-k},\end{aligned} \tag{3.7d}$$

$$\phi_k := \sum_{l=0}^{k}\binom{k}{l}(-2i)^l\frac{\partial^l\phi(u)}{\partial u^l}\bigg|_{u=0}. \tag{3.7d'}$$

In terms of Cohen functions, the Weyl quantization scheme (3.7d) corresponds to $\phi_{\rm W}(u)=1$, while the standard and antistandard schemes come from $\phi_{\rm s}(u)=e^{iu}$ and $\phi_{\rm a}(u)=e^{-iu}$. The often-used Born–Jordan[17] and symmetrization[13] schemes correspond to $\phi(u)=u^{-1}\sin u$ and $\cos u$, respectively.

There are restrictions on the Cohen function $\phi(u)$, though. If one requires the usual quantization for $q^m$ to $\mathbb{Q}^m$ and $p^n$ to $\mathbb{P}^n$, as demanded by Dirac[18] and von Neumann,[18,19] then $\phi(0)=1$ (so $\phi_0=1$). If one wants $qp$ to have its correspondence with $\tfrac{1}{2}(\mathbb{QP}+\mathbb{PQ})$, as in (2.3), then $\phi'(0)=0$ (so



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$\phi_1 = 1$ also). This last requirement is, of course, violated by the s and a schemes, but it is not a great fault; we shall use it below.

The classical monomials $q^m p^n$, under Fourier transformation inverse to (3.6), yield[8] derivatives of Dirac $\delta$'s and/or powers in $q+q'$ and $q-q'$ for the integral transform kernels (so they become differential operators). In this article, we are mostly interested in a *group* of ring elements whose kernels are proper functions of $q$ and $q'$.

The inner product in $\mathscr{L}^2(\mathscr{R})$ allows us to introduce the *adjoint* of a ring element $\mathbb{A}$, as that $\mathbb{B} = :\mathbb{A}^\dagger$, which satisfies $(\mathbb{B}f,g) = (f,\mathbb{A}g)$ for all $f$ and $g$ in a dense subspace of $\mathscr{L}^2(\mathscr{R})$. The integral kernel of $\mathbb{B}$ is, from (3.3), $\mathbb{B}(q,q') = A(q',q)^*$. The coset distributions of these elements relate, from (3.4b), as $\widehat{B}(x,y) = \widehat{A}(-x,-y)^*$, while, for any Cohen function $\phi$, $\widehat{B}_\phi(x,y) = \widehat{A}_\phi(-x,-y)^*(\phi/\phi^*)$. From (3.6a), thus, $b_\phi(q,p) = a_\phi(q,p)^*$ for all classical functions with a *real* Cohen function $\phi(\frac{1}{2}\hbar xy)$. When the quantization-scheme function is not real, one remains, in general, with a convolution integral relation between $b_\phi(q,p)$ and $a_\phi(q,p)$. Between the standard and antistandard quantization schemes, for which $\phi_s$, $\phi_a$ are not real but $e^{i\hbar xy/2}$, $e^{-i\hbar xy/2}$, one has $b_s(q,p) = a_a(q,p)^*$. Hence, *if* a real classical function is to quantize to a self-adjoint operator, it is *necessary* that the Cohen function $\phi$ be real.

Quantum mechanics is mostly preoccupied with Heisenberg–Weyl ring elements which are self-adjoint. In this work we shall regard *unitary* ring elements, i.e., those which when multiplied by their adjoint—in either order—yield back the identity operator. To detail multiplication we turn now to find explicit forms for the three views we have on ring elements through their integral kernel, coset distribution, and classical function.

Let $\mathbb{C} = \mathbb{A}\mathbb{B}$ be the ring element that is the product of $\mathbb{A}$ and $\mathbb{B}$. From (3.3) it follows that the representing integral kernels compose simply as

$$C(q,q') = \int_{\mathscr{R}} dq''\, A(q,q'')B(q'',q')\,. \tag{3.8}$$

The corresponding functions over the group follow a convolution product[8] and the coset distributions (3.1) then compose, from (3.4), as

$$\widehat{C}(x,y) = \int_{\mathscr{R}^2} dx'\, dy'\, \widehat{A}(x',\tfrac{1}{2}y + y')\widehat{B}(x-x',\tfrac{1}{2}y - y') \times \exp\,[i\tfrac{1}{2}\hbar(xy' - x'y + \tfrac{1}{2}xy)]\,. \tag{3.9}$$

Similarly, from (3.6) and its inversion follows, for the classical function (in the Weyl scheme),

$$\begin{aligned} c_W(q,b) &= \frac{1}{\pi^1|\hbar|^2}\int_{\mathscr{R}^4} dq'\, dq''\, dp'\, dp''\, a_W(q',p')b_W(q'',p'') \\ &\quad\times\exp[2i\{q(p'-p'') + q'(p''-p) + q''(p-p')\}/\hbar] \\ &= \sum_{m=0}^{\infty}\frac{(\hbar/2)^m}{m!}\left(\frac{\partial}{\partial q'}\frac{\partial}{\partial p''} - \frac{\partial}{\partial q''}\frac{\partial}{\partial p'}\right)^m \\ &\quad\times a_W(q',p')b_W(q'',p'')|_{\substack{q'=q''=q\\ p'=p''=p}}\,. \end{aligned} \tag{3.10a}$$

For the classical functions which quantize in the standard scheme, one has

$$\begin{aligned} c_s(q,p) &= \frac{1}{2\pi|\hbar|}\int_{\mathscr{R}^2} dq'\, dp'\, a_s(q,p')b_s(q',p) \\ &\quad\times\exp[-i(q-q')(p-p')/\hbar] \\ &= \sum_{m=0}^{\infty}\frac{(-i\hbar)^m}{m!}\left(\frac{\partial}{\partial q'}\frac{\partial}{\partial p'}\right)^m a_s(q,p')b_s(q',p)|_{\substack{q'=q\\ p'=p}}\,, \end{aligned} \tag{3.10b}$$

and for the classical functions in the antistandard scheme,

$$\begin{aligned} c_a(q,p) &= \frac{1}{2\pi|\hbar|}\int_{\mathscr{R}^2} dq'\, dp'\, a_a(q,p')b_a(q',p) \\ &\quad\times\exp[i(q-q')(p-p')/\hbar] \\ &= \sum_{m=0}^{\infty}\frac{(i\hbar)^m}{m!}\left(\frac{\partial}{\partial q'}\frac{\partial}{\partial p'}\right)^m a_a(q',p)b_a(q,p')|_{\substack{q'=q\\ p'=p}}\,. \end{aligned} \tag{3.10c}$$

These equations between the classical functions define so-called twisted products. Twisted products are known from the theory of operator symbols[20] (pseudodifferential operators[21]) in mathematical literature and phase-space methods in physical literature.[22,23] The second members in the last three equations are simplest to apply to polynomials (and special rational) functions, the case in which the sum is finite. Twisted products have been also applied for calculations in noncommutative algebra by use of computer-algebra systems.[24,25] The integral form we offer seems to be most appropriate for the cases where the factor functions are exponentials or Gaussians.

The unit element $\mathbb{E}$ in the ring $\mathscr{W}$ is described by an integral kernel $E(q,q') = \delta(q-q')$ [see (3.3)], a coset distribution $\widehat{E}(x,y) = \delta(x)\delta(y)$ [see (3.4)], and a classical function $a_\phi(q,p) = 1$ in all quantization schemes $\phi$. This may be used to verify the coefficients in (3.10).

Unitary ring elements are described by unitary integral kernels in the usual $\mathscr{L}^2(\mathscr{R})$ sense.

The HW group $W$ is contained in the ring $\mathscr{W}$; the describing distribution over the group [see (2.8)] of a group element $G_0 = \omega(x_0,y_0,z_0)$ is $G_0(\omega) = \delta_W(\omega_0\omega^{-1}) := \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$; from (3.1), the coset distribution is $\widehat{G}_0(x,y) = \delta(x-x_0)\ \delta(y-y_0)e^{i\hbar z_0}$, from (3.4a) the integral kernel is $G_0(q,q') = D_{qq'}(\omega(x_0,y_0,z_0))$. The Weyl classical function of $G_0$ is $g_{0W}(g,p) = \exp i(x_0q + y_0p + \hbar z_0)$, while any other scheme function $\phi$ yields a well-defined classical function $g_{0\phi}(q,p) = g_{0W}(q,p)\phi(\frac{1}{2}\hbar x_0y_0)$ for this subset of ring elements. Multiplication in $W$ may be followed in the classical functions for the s, a, and W cases through the integral or twisted product composition expressions (3.10).

In the next section we shall introduce the set of ring elements that constitute a metaplectic group. It should be noted that here we have generally a *fixed* ring element (defined through its integral kernel or coset distribution) and a whole $\phi$ range of "classical" functions, the Weyl among them. This is different from the usual stance in quantum mechanics, where one has a classical function to start with, and a $\phi$ range of elements of the ring, among which we try to choose.



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## IV. CANONICAL TRANSFORMS AS RING ELEMENTS

We may be sure the HW ring $\mathscr{W}$ contains other subsets which are groups, besides $W$ itself. The enveloping algebra $\overline{w}$, we noted in Sec. II, contains the symplectic algebra sp(2,$\mathscr{R}$), explicitly given by (2.3). The embeddings of sp(2,$\mathscr{R}$) in $\overline{w}$ [and of sp(2$N$,$\mathscr{R}$) in the generalization of $\overline{w}$ to $N$ canonical pairs $(\mathbb{Q}_1,...,\mathbb{Q}_N,\mathbb{P}_1,...,\mathbb{P}_N)$ and in its quotient division ring] have been systematically studied.[26] Moreover, embeddings (canonical realizations) have been constructed filling the gap between minimal and maximal (with respect to the number $N$ of canonical pairs) canonical (Schurean, anti-Hermitian, nonequivalent) realizations.[26]

Now, integral transform representations[27] of the [universal cover $\overline{\mathrm{Sp}(2,\mathscr{R})}$ of the] symplectic group Sp(2,$\mathscr{R}$) on $\mathscr{L}^2(\mathscr{R})$ are known for some time under the name of canonical transforms.[6,9] They are special in that they are generated by second-order differential operators. If the generators are (2.3a) and the space is $\mathscr{L}^2(\mathscr{R})$, the representation is called the *oscillator* (or metaplectic) representation. It is not irreducible, but consists of a direct sum of the lower-bound discrete series $D^+_{1/4}$ and $D^+_{3/4}$ irreducible representations in the notation of Bargmann.[27,28] The integral transform operators $\mathbb{C}_M$, which act adjointly on $w$ as the linear transformation

$$\mathbb{C}_{\mathbf{M}}\begin{pmatrix}\mathbb{Q}\\ \mathbb{P}\end{pmatrix}\mathbb{C}_{\mathbf{M}}^{-1} = \mathbf{M}^{-1}\begin{pmatrix}\mathbb{Q}\\ \mathbb{P}\end{pmatrix} \tag{4.1a}$$

(using an obvious vector notation and not changing the central element), are specified through their integral kernel,[6,9] which depends on the matrix parameters of

$$\mathbf{M} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}, \quad ad - bc = 1\,. \tag{4.1b}$$

The integral kernels are given by

$$C_{\mathbf{M}}(q,q') = \begin{cases} \dfrac{e^{-i\pi/4}}{\sqrt{2\pi b \hbar}} \exp\left[\dfrac{i(dq^2 - 2qq' + aq'^2)}{2b\hbar}\right], & b \neq 0,\\ (1/\sqrt{a}) e^{icq^2/2a\hbar}\delta(q - q'/a)\,, & b = 0\,. \end{cases} \tag{4.1c}$$

For lower-triangular matrices ($b = 0$), the integral kernels become one of the sequences leading to a Dirac $\delta$, and the integral transform becomes a Lie transformation (with multiplier factor). This subgroup is generated by the first-order differential operators in (2.3a), $\mathbb{X}^1_0$ generating the scale transformations and $\mathbb{X}^1_{-1}$ generating the multiplier.

Through the integral kernel composition (3.8), it follows that the ring elements (4.1) have the composition property

$$\mathbb{C}_{\mathbf{M}_1}\mathbb{C}_{\mathbf{M}_2} = \sigma(\mathbf{M}_1,\mathbf{M}_2)\mathbb{C}_{\mathbf{M}}, \quad \mathbf{M} = \mathbf{M}_1\mathbf{M}_2\,, \tag{4.2a}$$

where $\sigma$ is a *sign* given by

$$\sigma(\mathbf{M}_1,\mathbf{M}_2) = \exp\left[\frac{i\pi}{4}\left(\operatorname{sgn} b - \operatorname{sgn} b_1 - \operatorname{sgn} b_2 + \operatorname{sgn}\frac{b}{b_1 b_2}\right)\right]. \tag{4.2b}$$

This sign is quite fundamental and it may be observable. It may be a wave-mechanics counterpart of spin, for here it is the symplectic group Sp(2,$\mathscr{R}$), which is doubly covered. To uncover its significance, consider the ($\hbar = 1$) harmonic oscillator Hamiltonian $\mathbb{H} = \frac{1}{2}(\mathbb{P}^2 + \mathbb{Q}^2) + \frac{1}{2}(\mathbb{X}^1_{+1} + \mathbb{X}^1_{-1})$, which lies in $\overline{w}$ and exponentiate it. This yields a line of integral transforms

$$e^{i\alpha\mathbb{H}} = \mathbb{C}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix},$$

where for $\alpha = \pi/2$ we have the inverse Fourier transform *times* $e^{+i\pi/4}$. For $\alpha = \pi$ we have the square of this, which is $e^{i\pi/2}$ times the inversion operator; $\alpha = 3\pi/2$ corresponds to $e^{3i\pi/4}$ times the Fourier transform, and for $\alpha = 2\pi$ we have the operator $-\mathbb{1}$, which is the unit ($\mathbb{1} = \mathbb{C}_1$) operator times $-1$. When $\mathbb{C}_{\mathbf{M}}$ acts *adjointly* on $w$, this yields the transformation $(-\mathbb{1})\mathbb{X}(-\mathbb{1})^{-1} = \mathbb{X}$, which is an identity transformation of the algebra, but not for functions in $\mathscr{L}^2(\mathscr{R})$, where $-\mathbb{1}$ acts. It is only when we let $\alpha = 4\pi$ that we obtain back the unit operator in $\mathscr{L}^2(\mathscr{R})$. On $\mathscr{L}^2(\mathscr{R})$ we thus have a group of operators that is the double cover of the circle matrices

$$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}.$$

When Bargmann[27] described the connectivity properties of the Lorentz group (in three dimensions) SO(2,1), he also introduced a proper parametrization of the metaplectic group, which for fourth-integer $k$, $D^+_k$ representations are faithful. For our purposes here we shall not need the composition formula (4.2a), and it suffices to consider group elements near the identity of the metaplectic group, corresponding to elements near the symplectic group identity element. In this regard it is helpful to note that $\mathbb{C}_{\mathbf{M}}$ is a Hilbert–Schmidt operator when its integral kernel has parameters $a$, $b$, $c$, $d$ with small imaginary parts such that the integral kernel is a decreasing (rather than increasing) Gaussian, i.e., for Im $(a/b) > 0$. For real $a > 0$ we can ascribe to $b$ a small negative imaginary part, so that the limit of real parameters $a$, $b$, $c$, $d$ can be approached from $b$'s in the *lower-half complex plane*. In that case, the argument of positive $b$ is zero and that of negative $b$ is $-\pi$. When $b$ vanishes from negative values (as when we followed the Fourier circle above), then $a$ can be thought to be constrained to the lower-half plane (so $a < 0$ means $\arg a = -\pi$); when $b$ vanishes from positive values, $a$ is contrained to the upper-half plane (so $a < 0$ means $\arg a = +\pi$).

In order to investigate the properties of the coset distributions and classical functions of the ring elements $\mathbb{C}_{\mathbf{M}}$ we use integral kernels we have defined above. The following integrals (regularized for real values of the parameters) are useful:

$$I(R,S) := \int_{\mathscr{R}} dx \exp i(Rx^2 + Sx) = e^{i\pi/4}\sqrt{\frac{\pi}{R}} \exp(-iS^2/4R), \quad \operatorname{Im} R \geqslant 0\,, \tag{4.3a}$$



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$$I(A,B,C,D,E):=\int_{\mathscr{R}^2} dx\,dy \exp i(Ax^2+Bxy+Cy^2+Dx+Ey)=\frac{2\pi i}{\sqrt{4AC-B^2}}\exp\left(-i\frac{D^2C+E^2A-EBD}{4AC-B^2}\right),$$

$$\operatorname{Im} A\geqslant 0,\quad \operatorname{Im}(C-B^2/4A)\geqslant 0,\quad \operatorname{Im} C\geqslant 0,\quad \operatorname{Im}(A-B^2/4C)\geqslant 0. \tag{4.3b}$$

From (3.4b) we find the coset distribution of $\mathbb{C}_{\mathrm{M}}$:

$$\hat{C}_{\mathrm{M}}(x,y) = \begin{cases} \dfrac{1}{2\pi\hbar\sqrt{a+d-2}} \times\exp\left(i\hbar\dfrac{-bx^2+(a-d)xy+cy^2}{2(a+d-2)}\right), & b\neq 0,\\ \dfrac{1}{2\pi\hbar}\dfrac{\sqrt{a}}{a-1} \times\exp\left(i\hbar\left[\dfrac{cay^2}{2(a-1)^2}+\dfrac{1}{2}\dfrac{a+1}{a-1}xy\right]\right), & b=0. \end{cases} \tag{4.4}$$

The Weyl, standard, and antistandard classical functions are obtained as

$$C_{\mathrm{M,w}}(q,p) = \frac{2}{\sqrt{a+d+2}} \times\exp\left(\frac{2i}{\hbar(a+d+2)}[-bp^2+(d-a)pq+cq^2]\right), \tag{4.5a}$$

$$C_{\mathrm{M,s}}(q,p) = \frac{1}{\sqrt{a}}\exp\left(\frac{i}{2a\hbar}[-bp^2+2(1-a)pq+cq^2]\right), \tag{4.5b}$$

$$C_{\mathrm{M,a}}(q,p) = \frac{1}{\sqrt{d}}\exp\left(\frac{i}{2d\hbar}[-bp^2+2(d-1)pq+cq^2]\right). \tag{4.5c}$$

The Weyl classical function as presented above reproduces correctly the results of Combe *et al.*[29] and Burdet and Perrin[30] contained in the metaplectic group. The product composition is (3.10a), (3.10b), and (3.10c), respectively.

The Lie algebra corresponding to the integral kernel action of $\mathbb{C}_{\mathrm{M}}$ on $\mathscr{L}^2(\mathscr{R})$ yields up-to-second-order differential operators, and $\mathbb{C}_{\mathrm{M}}$ may be written directly as the exponential of second-order operators in various alternative forms using the product (4.2); similarly, as an integral over exponentials of first-order operators we may use (3.2)–(4.4). We thus obtain

$$\begin{aligned}
&\mathbb{C}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\
&= \exp\left(-\frac{ib}{2d\hbar}\mathbb{P}^2\right)\exp\left(i\ln d\cdot\frac{1}{2\hbar}[\mathbb{PQ}+\mathbb{QP}]\right) \times\exp\left(\frac{ic}{2d\hbar}\mathbb{Q}^2\right)\\
&= \exp\left(\frac{ic}{2a\hbar}\mathbb{Q}^2\right)\exp\left(-i\ln a\cdot\frac{1}{2\hbar}[\mathbb{PQ}+\mathbb{QP}]\right) \times\exp\left(\frac{-ib}{2d\hbar}\mathbb{P}^2\right)\\
&= \exp\left(i\frac{\operatorname{arccosh}\frac{1}{2}(a+d)}{\hbar\sqrt{(a+d)^2-4}} \times\left[b\mathbb{P}^2+\frac{1}{2}(a-d)(\mathbb{PQ}+\mathbb{QP})-c\mathbb{Q}^2\right]\right)\\
&= \frac{\hbar}{2\pi\sqrt{a+d-2}}\int_{\mathscr{R}^2} dx\,dy \times\exp\left(i\frac{-bx^2+[a-d]xy+cy^2}{2(a+d-2)}\right) \times\exp i(x\mathbb{Q}+y\mathbb{P}),
\end{aligned} \tag{4.6}$$

as the Weyl, standard, and antistandard quantization of (4.5a), (4.5b), and (4.5c), respectively.

Particularizing, we may obtain several interesting relations. The exponential function of $qp$ in different quantization schemes, expressed in terms of the anticommutator $\frac{1}{2}(\mathbb{PQ}+\mathbb{QP})$, is

$$\begin{aligned}
&\boldsymbol{\Omega}_{\mathrm{w}}(e^{ixqp/\hbar}) = \frac{1}{\sqrt{1-(x/2)^2}}\exp\left[2i\arctan\frac{x}{2}\cdot\frac{1}{2}(\mathbb{PQ}+\mathbb{QP})\right],\\
&\boldsymbol{\Omega}_{\mathrm{s}}(e^{1xqb/\hbar}) = \frac{1}{\sqrt{1+x}}\exp\left[i\ln(1+x)\cdot\frac{1}{2}(\mathbb{PQ}+\mathbb{QP})\right],\\
&\boldsymbol{\Omega}_{\mathrm{a}}(e^{ixqp/\hbar}) = \frac{1}{\sqrt{1-x}}\exp\left[-i\ln(1-x)\cdot\frac{1}{2}(\mathbb{PQ}+\mathbb{QP})\right].
\end{aligned} \tag{4.7}$$

The (normalized) Fourier transform operator may be written as the ring element

$$\mathbb{C}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}_{\hbar=1}$$

applied to a smooth function (multiplied by $e^{-i\pi/4}$):



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$$\tilde{f}(q) = \left[\mathbb{C}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right](q)$$

$$= \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_{\mathscr{R}} dq'\, e^{-iqq'} f(q')$$

$$= \frac{-i}{2\pi\sqrt{2}} \int_{\mathscr{R}^2} dx\, dy \exp\left[\frac{i(x+y)^2}{4}\right] e^{ixq} f(q+y) . \qquad (4.8)$$

The correctness of the last result may be verified reducing the twofold integral to a single one by means of (4.3a).

Corresponding to the matrix $-\mathbf{1}$ one has one of the central elements in the group,

$$\mathbb{C}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the Wigner inversion operator studied by Grossmann,[31] Huguenin,[32] Royer,[33] and Dahl.[16] Indeed, following the one-parameter "harmonic oscillator" subgroup

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

we arrive, for $\alpha \to \pi^-$, at the ring element given through (4.7) by

$$\mathbb{C}_{-1} = \frac{i\hbar}{4\pi} \int_{\mathscr{R}^2} dx\, dy \exp(i[x\mathbb{Q} + y\mathbb{P}]) . \qquad (4.9)$$

This corresponds to a coset distribution constant over the manifold, an integral kernel $\delta(q+q')$, and the Weyl quantization of a classical Dirac $\delta$ at the origin of phase space.

In Gaussian geometrical optics,[5] the Hamiltonian formulation introduces a momentum canonically conjugate to the position $q$ of a ray at a plane perpendicular to the optical axis $z$. This momentum is $p = n \sin\theta$, where $n$ is the refraction index of the medium and $\theta$ is the angle between the optical axis and the light ray.

For lens systems,[11] the subsets of $\mathrm{Mp}(2,\mathscr{R})$ that are of interest are the parabolic subgroup of upper-triangular matrices

$$\mathbb{F}_z := \mathbb{C}\begin{pmatrix} 1 & -z/n \\ 0 & 1 \end{pmatrix} = \exp\left(-i\frac{z}{n}\mathbb{P}^2\right), \quad z \geqslant 0 , \qquad (4.10a)$$

corresponding to free propagation through a length $z$ in a medium with refraction index $n$, and the group elements

$$\mathbb{S}_P := \mathbb{C}\begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix}, \quad P = 2\beta(n'-n) , \qquad (4.10b)$$

which corresponds to the action on optical phase space of a refracting surface $z = \zeta(q) = \beta q^2$, where light passes from a medium $n$ to a medium $n'$. If standing for a lens, $P$ is the Gaussian power.

From these optical elements we may build as a limit any $\mathrm{Mp}(2,R)$ transformation of optical phase space.[34] The corresponding canonical transform is the Huygens–Fresnel integral cut to the quadratic exponential term. This acts on the object phase function to yield the image.

One further development, which is immediate but not of central import to this paper, is the consideration of the semidirect product between the Heisenberg–Weyl group $W$ and the metaplectic group explored above, $W\mathrm{Mp}(2,\mathscr{R}) = W \wedge \mathrm{Mp}(2,\mathscr{R})$, with $W$ a normal ideal. In terms of the results of Combe *et al.*,[29] Burdet and Perrin,[30] this allows the inclusion of the linear potential (free-fall) Hamiltonian. The group of translations and inversions of Dahl[16] is simply (4.9) in semidirect product with $W$.

## V. SYMPLECTIC IDEALS AND WEYL QUANTIZATION

As we stated in the Introduction, our eventual aim is to apply the structure of the Heisenberg–Weyl ring $\mathscr{W}$ to the description of wave optics in aberration. The metaplectic group within $\mathscr{W}$ describes Gaussian optics, its elements acting on the object phase function to yield the image phase function.

The study of geometrical optics of aligned systems with third aberration order has been done using the "classical" Poisson-bracket Lie algebra of observables quadratic and quartic in the phase-space variables[12,35] *modulo* higher-order terms. Preliminary results suggest that the relevant algebraic structure is $\nu \wedge \mathrm{sp}(2,\mathscr{R})$, where $\nu$ is a nilpotent ideal under the Gaussian algebra $\mathrm{sp}(2,\mathscr{R})$. This ideal *decomposes* into $N$th-order aberration ideals[36] under $\mathrm{sp}(2,\mathscr{R})$. Concretely (in one dimension), if we denote

$$\chi^j_m := p^{j+m} q^{j-m} ,$$
$$m = j, j-1, \ldots, -j, \quad 2j = 0,1,2,\ldots, \qquad (5.1)$$

then the Poisson bracket $\{\cdot,\cdot\}$ between these elements is given by

$$\{\chi^j_m, \chi^{j'}_{m'}\} = 2(jm' - j'm)\chi^{j+j'-1}_{m+m'} . \qquad (5.2)$$

Now, it is a well-known fact that the quantities (5.1) with Lie bracket given by the Poisson bracket form an infinite-dimensional Lie algebra with a grading. If we now consider $\chi^j_m$ as classical functions to be quantized according to some scheme, it is also a well-known fact that the algebra whose Lie bracket is the commutator will *not* be isomorphic to the previous one. Classical Poisson brackets and quantum commutators do *not* follow each other, except[8] for (i) up-to-quadratic expressions in the basic Heisenberg–Weyl constituents, (ii) classical functions of the form $pf(q) + g(q)$ and their corresponding quantum operators, and (iii) classical functions $qf(p) + g(p)$ and their quantized operators. [We have not included some finite-dimensional Lie subalgebras of $\overline{w}$ $\{(1,\mathbb{Q}\mathbb{P},\mathbb{Q}^k\mathbb{P}^l,)k,\ l \text{ fixed},\ k \neq l,\ k,l \leqslant 2\}$ and Abelian infinite-dimensional Lie subalgebras (polynomials in one element, e.g., $\mathbb{Q}^k\mathbb{P}^k$, $k = 0,1,2,\ldots$), which are also isomorphic to the corresponding Lie algebras of their classical functions under Poisson bracket.] The algebraic span of any two of the above classes is outside the span of each class by itself.

In Gaussian optics (with prisms) the geometrical and wave treatments follow each other since the generating operators all belong to class (i). Optics with aberration lacks this isomorphism due to the fact that it uses quantities (5.1) with $j > 1$.

In following papers we intend to show that a quantization analog of wave optics out of geometrical optics—in an approximation still to be explored—*is* achieved if we retain only the requirement

$$\{\chi^1_m, \chi^j_{m'}\} = 2(m' - jm)\chi^j_{m+m'} , \qquad (5.3a)$$

so that the quantized version of this relation,

$$[\mathbb{X}^1_m, \mathbb{X}^j_{m'}] = 2i\hbar(m' - jm)\mathbb{X}^j_{m+m'} , \qquad (5.3b)$$

holds. We show here that this is possible *if and only if* $\mathbb{X}^j_m$ is the Heisenberg–Weyl ring element corresponding to the classical function $\chi^j_m$ *in the Weyl quantization scheme*. What



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we are asking for is that $\{\mathbb{X}^j_m\}^j_{m=-j}$ transform under the symplectic group in the same way as the classical quantities $\{\chi^j_m\}^j_{m=-j}$ do. We shall detail elsewhere the association between the *multiplet* $\chi^j$ and $Sp(2,R)$*-classified* aberrations of order $A = 2j-1$. In the following we prove that (5.3b) follows from (5.3a) in (and only in) Weyl quantization.

We refer to (3.7) for the general-scheme quantization of the basic monomial $q^m p^n$, and calculate its commutator with $\mathbb{X}^1_{-1} = \mathbb{Q}^2$, the sp(2,$R$) lowering operator in (2.3):

$$\begin{aligned}[\mathbb{Q}^2,\boldsymbol{\Omega}_\phi(q^m p^n)] &= \sum_k \binom{m}{n}\binom{n}{k} k!\phi_k \\ &\quad\times\left(-\frac{1}{2}i\hbar\right)^k \mathbb{Q}^{m-k}[\mathbb{Q}^2,\mathbb{P}^{n-k}] \\ &= 2i\hbar n \sum_k \binom{m+1}{k}\binom{n-1}{k} k! \\ &\quad\times\left\{\phi_k - \frac{k}{m+1}(\phi_k - \phi_{k-1})\right\} \\ &\quad\times\left(-\frac{1}{2}i\hbar\right)^k \mathbb{Q}^{m+1-k}\mathbb{P}^{n-1-k}. \qquad (5.4)\end{aligned}$$

If the last member is to be an operator quantized from a classical monomial function in the same scheme, the quantity in curly brackets must be again $\phi_k$ and independent of $m$. This is possible if and only if $\phi_k = \text{const}$. This is the trademark of the Weyl scheme. The same derivation applies for the commutator between $\mathbb{X}^1_1 = \mathbb{P}^2$ and $\boldsymbol{\Omega}_\phi(q^m p^n)$ and thus for any sp(2,$R$) element and its exponential to the group. The Weyl rule is thus invariant under metaplectic transformations. The proof given here is algebraic; a geometric proof would be of interest.


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