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Canonical transformations to phase variables in quantum oscillator systems. A group theoretic solution

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We consider the problem of finding the unitary transformation which intertwines a self-adjoint quantum hamiltonian operator $\frac{1}{2}\mathbf{P}^2 + V(\mathbf{Q})$ in L^2-type function spaces with a local weight function, to its phase-variable form $-id/d\zeta$, self-adjoint in a second Hilbert space. This can be done in the context of the Heisenberg-Weyl algebra only for the free fall case. Within the $sl(2, R)$ algebra one may consider potentials $V(\mathbf{Q}) = \beta\mathbf{Q}^2 + \gamma\mathbf{Q}^{-2}$ for all real β, γ, corresponding to harmonic or repulsive oscillators, or a free particle, with a centrifugal barrier or centripetal well. This treatment exhaust the class of hamiltonians for which the problem of quantum canonical transformations to phase variables may be solved entirely within the framework of group theory. The results naturaly agree with the ones obtained from the Mello-Moshinsky equations, except that the spectra are here automatically matched through the (in general nonlocal) measure in the second Hilbert space. The present treatment does not require, thus, the introduction of the Moshinsky-Seligman ambiguity group. The algebra $sl(2, R)$ replaces the quantum phase space associated to the Heisenberg-Weyl algebra. Furthermore, we examine in detail a range of singular potentials –those which include a strong centripetal well– where the hamiltonian has a one –parameter family of self-adjoint extensions, and where the discrete spectra are neither unique nor lower bound. Finally we find a set of new generating relations for Whittaker functions.

I. HISTORICAL PERSPECTIVE AND INTRODUCTION

The hamiltonian formulation of classical mechanics [1] affords a deeper understanding of the structure of the theory through according equal status to various selections of coordinate and momentum variables (q_i, p_j), $(\bar{q}_i, \bar{p}_j)$, ... $i, j = 1, 2, \ldots, N$, describing a system. If one such variable set is understood to be canonical under the Poisson bracket $\{q_i, p_j\} = \delta_{i,j}$ the transformation $q_i \to \bar{q}_i = \bar{q}_i\ (q_j, p_k), p_j \to \bar{p}_j = \bar{p}_j\ (q_i, p_k)$ is termed *canonical* if the brackets are preserved as $\{\bar{q}_i\ (q, p), \bar{p}_j\ (q, p)\} = \delta_{i,j}$ so that $(\bar{q}_i, \bar{p}_j)$ constitute a second canonical variable set. In one-dimensional systems, if $\bar{p}(q, p)$ is the hamiltonian, the time variable is $\bar{q}(q, p)$. When the transformation is inverted to $q = q(\bar{q}, \bar{p}), p = p(\bar{q}, \bar{p})$ the motion of the

* Dedicated to Professor Marcos Moshinsky on his 60th birthday.

system is explicitly obtained as the initial conditions $\bar{q}(q_0, p_0) = 0$, $\bar{p}(q_0, p_0) = E$ are substituted.

Quantum mechanics, in its Dirac-von Neumann presentation [2-4], replaces observables by self-adjoint operators in a suitable Hilbert space $L^2\ (R^N)$ and Poisson brackets by commutators. This scheme applies to the coordinate **Q** and momentum **P** operators, which in conjuction with their commutator **I** –a third operator– close into the three-dimensional nilpotent Heisenberg-Weyl algebra

$$[\mathbf{Q}, \mathbf{P}] = i\mathbf{I}, \ [\mathbf{Q}, \mathbf{I}] = 0 \ , \quad [\mathbf{P}, \mathbf{I}] = 0 \tag{1.1}$$

In $L^2(R)$, **I** is a multiple $\hbar$ of the identity operator **1** (we set $\hbar = i$), while **Q** and **P** may be always brought by means of a unitary transformation to their Schrödinger realization [4, 5]

$$(\mathbf{Q}f)(q) = qf(q) \ , \tag{1.2a}$$

$$(\mathbf{P}f)(q) = -i \frac{df(q)}{dq} \ , \tag{1.2b}$$

on a domain dense in $L^2(R)$.

The quantum mechanical anologue of the hamiltonian formulation would at this point seem reasonably straightforward. Indeed, within a couple of years of the beginning of the new theory [6, 7], Jordan and London had published papers in this direction [8, 9]. But the canonical transformation to hamiltonian and time observables remained elusive. The quantization-scheme problems associated with $\bar{q}(q, p)$ and $\bar{p}(q, p)$ had not been fully explored; Born and Jordan [6] had given one practical rule, Dirac's rule [2] was not self-consistent, and shortly thereafter Weyl [7] and Mc Coy [10] proposed theirs. Furthermore, the Stone-von Neumann theorem and a simple argument [11] on the possible spectra of **Q** and **P** showed that (1.2) is essentially the only realization of the Heisenberg-Weyl algebra of use, and that it covers the real line R. As we shall show in section 2, unitary canonical transformations within $L^2(R)$ thus allow only for the solution of the free-fall system. We thus exhaust this class of transformations.

The interest in canonical transormations in quantum mechanics waned when it was realized that these were unnecessary for the solution of the many systems studied in the following decades; a satisfactory historical account of this change of focus is still to be given. Dirac's classic book [3], to be sure, includes a section on this problem, but it is geared towards finding the unitary operator in $L^2(R)$ such that

$$\mathrm{Q} \overset{\mathrm{F}}{\rightarrow} \bar{\mathrm{Q}} = \mathrm{FQF}^{-1} = -\mathrm{P}, \quad \mathrm{P} \overset{\mathrm{F}}{\rightarrow} \bar{\mathrm{P}} = \mathrm{FPF}^{-1} = \mathrm{Q} , \qquad (1.3)$$

and leads to the momentum realization of quantum mechanics. This operator **F** is shown to be the Fourier integral transform on $L^2(R)$:

$$(\mathrm{F}f)(q) = \int_{-\infty}^{\infty} dq \, [(2\pi)^{-1/2} e^{-ipq}] \, f(q) \qquad (1.4)$$

The work of Marcos Moshinsky on canonical transformations in quantum mechanics started in 1970 in collaboration with Christiane Quesne. The first results were presented at the XV Solvay Conference on Physics [12] and were published in two contiguous articles [13] in 1971. They analyzed the question of the symplectic group of transformations generated by the dynamical algebra of the N-dimensional harmonic oscillator from the point of view that this is a canonical transformation of phase space. The problem of unitary operators generating linear transformations between Q and **P** and their integral transform kernels, to be sure, had been discussed previously by Infeld and Plebański [14] for the problem of coherent states, by Weil [15] from the point of view of representations of the *metaplectic* group, two-fold covering of $Sp(2N, R)$, and as an example in a short article by Itzykson [16]. These authors seem to have been unaware of each other's work. Weil's article has generated overlapping lines of research among mathematicians [17-19]. Moshinsky's formulation, on the other hand, has had impact among physicists working with him in group theoretical methods in quantum mechanics and in nuclear physics.

The interest in canonical transformations to action-and-angle variables (or *phase* variables, as we shall call them here) was present early in the program [20], but the basic problem –hamiltonians having spectra in general different [21] from that of **P** in (1.2b)– prevented further progress in this respect. Various treatments for phase-and-angle [22] and phase-and time [23] operators had been given in the literature showing that these were not simple reductible to canonical transformations from canonical pairs (1.2).

In the teeth of these precedents, Moshinsky, Seligman [24-26] and Deenen [27] re-examined the problem of action and angle variables, noting that even classically this mapping is not globally bijective. Classical phase space motion may be subject to a discrete group A of transformations called the *ambiguity group* (which in some cases may be only the identity) such that $q \xrightarrow{g} q_g, p \xrightarrow{g} p_g, g \in A$, leaves the hamiltonian and time variables invariant: $\bar{q}(q,p) = \bar{q}(q_g, p_g), \bar{p}(q,p) = \bar{p}(q_g, p_g)$. Another discrete ambiguity group $\bar{A}$ may exist for the second phase space, such that under $\bar{q} \xrightarrow{\bar{g}} \bar{q}_{\bar{g}}, \bar{p} \xrightarrow{\bar{g}} \bar{p}_{\bar{g}}, \bar{g} \in \bar{A}$, ordinary phase space is left invariant: $q(\bar{q}, \bar{p}) = q(\bar{q}_{\bar{g}}, \bar{p}_{\bar{g}}), p(\bar{q}, \bar{p}) = p(\bar{q}_{\bar{g}}, \bar{p}_{\bar{g}})$. A bijective mapping is nevertheless established between the manifolds $\Pi = R^2 \times \bar{A}$ and $\Pi = \overline{R^2} \times A$, R^2 being the ordinary phase-space plane (q, p) and $\overline{R^2}$ the region classically covered by the energy and time variables $(\bar{q}, \bar{p})$, the latter connected in a way such that $\overline{R^2} \times A$ covers one or more times the full plane.

In reference [24], the first of a series of three articles, this construction was undertaken for the repulsive oscillator ($A = Z_2$, $\bar{A} = id.$, Z_N being the group of integer translations modulo N), the free particle ($A = Z_2$, $\bar{A} = Z_2$), and the harmonic oscillator ($A = id.$, $\bar{A} = Z_2 \wedge Z_\infty = D_\infty$, the dihedral group of integer translations and invertions). The authors then implement the Mello-Moshinsky [28] unitary transformations between Π and $\bar{\Pi}$, *i.e.* between L^2 -spaces of $|\bar{A}|$ –component functions (in configuration q or momentum p realization) and $|A|$-component functions (in time $\bar{q}$ or energy $\bar{p}$ realizations) where the hamiltonian or the action (or their absolute values) are diagonal. A second conjugate time or phase operator may be constructed such that in closes into a Heisenberg-Weyl algebra with the first. The basic problem when the hamiltonian spectrum is discrete and lower-bound (as for the harmonic oscillator) is here solved restricting functions of Π to those belonging to an irreducible representations of $\bar{A}$. The second article of the series [25] deals with the three-dimensional Coulomb problem, whose spectrum is mixed, and where the bound and free orbits are associated with different ambiguity groups. The last article [27] gives an elegant review of the definition and general solution of the problem for essentially arbitrary potentials whose spectrum may be discrete, continuous or mixed. The examples included are the free-fall and the Morse potentials.

In the present paper, the approach to the problem of defining canonical transformations to hamiltonian and 'time' operators is different. It stresses the dynamical algebra of the system instead of the Heisenberg-Weyl algebra associated to quantum mechanical phase space. The three "oscillator systems" named in the title share $sl(2, R)$ as their dynamical algebra. They are the systems with potentials

$$V(r) = \frac{1}{2}\,[\beta r^2 + \frac{\gamma}{r^2}]\,, \quad r > 0\,, \tag{1.5}$$

for all real values of β and γ. Through changes of scale β can be brough to the values +1 (harmonic oscillator), –1 (repulsive oscillator) or 0 (free particle). The γ / r^2 term represents a centrifugal barrier ($\gamma > 0$) or a centripetal well ($\gamma > 0$) at the origin. These systems are restricted to the positive half-axis $R^+ : r > 0$. The case of lower-bound spectra for the oscillator case was considered in [29]; in [30] the unbounded spectrum case was given, but the concomitant phase problems and features associated with the multivalued and exceptional representation series were not satisfactorily analyzed.

In section III, here, we pose the problem in the more precise Hilbert space terms which are necessary when the hamiltonians have a one-parameter family of self-adjoint extensions (for $\gamma < 3/4$). Finding a quantum canonical transformation to phase variables means to find a unitary transformation to a new Hilbert space such that the hamiltonian operator is realized as $-id / d\zeta$, ζ being the phase variable. The new Hilbert space must be such that this operator have the spectrum of the hamiltonian of the original system. A phase operator "$\hat{\zeta}$" is not assumed to exist. Rather, the full $sl(2, R)$ dynamical algebra of the system is mapped onto the Bargmann realization of the same algebra on the circle. This is detailed in section IV so as to place emphasis on the non-local inner product measure which is required so that the representations be self-adjoint.

Sections V, VI, and VII contain the intertwining canonical transform integral kernels for the harmonic oscillator $+\gamma / r^2$ potential case, the pure γ / r^2 potential, and the repulsive oscillator $-\gamma / r^2$ case. All $\gamma \in R$ in (1.5) are considered here, requiring all self-adjoint representations of the algebra. We particularize the results to those formerly obtained [29] and to the ordinary ($\gamma = 0$) oscillator systems. We refer to the Mello-Moshinsky method [29] as an alternative method of solution, pointing out the advantages of using generating functions for the unambiguous determination of phases. This needs a generating relation for Whittaker functions which is given in the appendix, and which appears to be new.

The formulation given here to the problem of canonical transformation to phase variables is plainly group theoretical, and is intended to avoid the use of the Heisenberg-Weyl algebra in the treatment of oscillator systems. We explicitly disclaim that this method be applicable to all systems subject to quantization, since most of them do not possess a natural dynamical algebra. For the class of one-dimensional oscillator systems, however, we pose a concrete task and give what appears to be its complete solution.

II. THE (ONLY) HEISENBERG-WEYL ALGEBRA CASE: THE FREE-FALL SYSTEM

Consider the free-fall [*i.e.* linear potential $V(q) = q$] quantum hamiltonian operator $\overline{\mathbf{P}}$ given by [31]

$$\overline{\mathbf{P}} = \frac{1}{2}\,\mathbf{P}^2 + \mathbf{Q} = \mathbf{C}\mathbf{P}\mathbf{C}^{-1} \quad , \quad i.e.\ \overline{\mathbf{P}}\mathbf{C} = \mathbf{C}\mathbf{P} \tag{2.1}$$

defined on a domain dense in $L^2(R)$, and written as the similarity transform $\mathbf{C}$ of $\mathbf{P}$ in (1.2b). It is immediate to verify that the only operator with which $\overline{\mathbf{P}}$ can close into a Heisenberg-Weyl algebra (1.1) is

$$\overline{\mathbf{Q}} = -\mathbf{P} = \mathbf{C}\mathbf{Q}\mathbf{C}^{-1} \quad , \quad i.e.\ \overline{\mathbf{Q}}\mathbf{C} = \mathbf{C}\mathbf{Q} \ , \tag{2.2}$$

so that $\overline{\mathbf{I}} = \mathbf{I}$. The map $\mathbf{C}$ is said to be canonical. since it maps the Heisenberg-Weyl algebra basis $\mathbf{Q}$, $\mathbf{P}$ and $\mathbf{I}$ into a similar basis for the same algebra.

Making use of the relation

$$\exp\,[if(\mathbf{Q})]\,\mathbf{P}\exp\,[-if(\mathbf{Q})] = \mathbf{P} - f'(\mathbf{Q}) \tag{2.3}$$

and (1.3), we may write the canonical transformation operator $\mathbf{C}$ as

$$\mathbf{C} = \exp\left(\frac{i\mathbf{P}^3}{6}\right)\mathbf{F} = \mathbf{F}\exp\left(\frac{-i\mathbf{Q}^3}{6}\right) \ . \tag{2.4}$$

The action of $\mathbf{C}$ on $L^2(R)$ functions is the (*canonical*) integral transform

$$f(q) \xrightarrow{\mathbf{C}} \overline{f}(\overline{q}) = (\mathbf{C}f)(\overline{q}) = \int_{-\infty}^{\infty} dq\ C(\overline{q},q) f(q) \ , \tag{2.5a}$$

with a kernel which can be found from (1.2)–(1.4) to be

$$C(\overline{q},q) = (2\pi)^{-1/2}\exp\,(-i[q\overline{q} + q^3/6]) \ . \tag{2.5b}$$

The transform inverse to (2.5) may be found to be

$$\bar{f}(q)\xrightarrow{\mathbf{C}^{-1}} f(q) = (\mathbf{C}^{-1}\bar{f})(q) = \int_{-\infty}^{\infty} d\bar{q}\, C^{-1}(q, \bar{q})\, \bar{f}(\bar{q}) \tag{2.6a}$$

with kernel

$$C^{-1}(x, y) = C(y, x)^* \ , \tag{2.6b}$$

confirming that **C** is a unitary operator in $L^2(R)$, consequence of the unitarity of **F** and self-adjointness of **Q** and **P** on that space.

The generalized eigenfunctions of $\mathbf{P} = -id/dq$ are

$$f_\lambda(\bar{q}) = (2\pi)^{-1/2} \exp(i\lambda q) \ , \quad \lambda \in R \ , \tag{2.7}$$

with Dirac normalization. From these we may find the generalized eigenfunctions of $\bar{\mathbf{P}}$ using (2.1b):

$$\bar{f}_\lambda(\bar{q}) = (\mathbf{C}f_\lambda)(\bar{q}) = \int_{-\infty}^{\infty} dq\, C(\bar{q}, q) f_\lambda(q)$$

$$= 2^{1/3} \mathrm{A}_i(2^{1/3}[\bar{q} - \lambda]) \ , \quad \lambda \in R \ , \tag{2.8}$$

which are also Dirac-normalized in $L^2(R)$.

We have been able to find the canonical transform (2.5) and its inverse (2.6) due to the ease with which we were able to write **C** in (2.4) in terms of known operators. This is an exceptional case.

The Mello-Moshinsky differential equation method [28] finds the integral kernel of the canonical transform when only (2.1) and (2.2) are prescribed. Since we are within a Heisenberg-Weyl algebra in $L^2(R)$ we may allways resort to the Schrödinger realization where **Q** and **P** are given by (1.2). Application of these equations on (2.5a)—with $C(\bar{q}, q)$ to be found—and integration by parts when derivatives appear under the integral sign, yields a pair of coupled partial differential equations for the kernel within an integral, in company with two arbitrary $L^2(R)$ functions. As only the null function is orthogonal to all of $L^2(R)$, these expressions are valid for the integrand itself, *i.e.*

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial \bar{q}^2} + \bar{q}\right] C(\bar{q}, q) = i\frac{\partial}{\partial q} C(\bar{q}, q) \ , \tag{2.9a}$$

$$i\frac{\partial}{\partial \bar{q}} C(\bar{q}, q) = qC(\bar{q}, q) \ . \tag{2.9b}$$

The second equation applied twice (to yield $\partial^2 C / \partial \bar{q}^2$) replaced in the first one yields a first-order ordinary differential equation whose solution is (2.5b), up to a factor function of $\bar{q}$, which (2.9b) determines to be a constant. This constant, finally, is found through the unitarity condition $\mathbf{C}^\dagger \mathbf{C} = 1$ up to a phase.

A third method to find the integral kernel of canonical transformations taking $\mathbf{P}$ to $\overline{\mathbf{P}}$ can be used if we know the eigenfunctions of both operators, $f_\lambda(q)$ and $\bar{f}_\lambda(\bar{q})$. The kernel is built as a generalized generating function between (2.7) and (2.8):

$$C(\bar{q}, q) = \int_{-\infty}^{\infty} d\lambda \, \omega(\lambda) \overline{f_\lambda} (\bar{q}) f_\lambda (q)^* \,, \tag{2.10}$$

where $|\omega(\lambda)| = 1$ is a phase function of λ. Because of (2.8) and the invertibility of the Fourier transform, we recover (2.5b) when $\omega(\lambda) = 1$. Equation (2.10), by definition, satisfies (2.9a) for general $\omega(\lambda)$. Eigenfunctions, even when normalized, are defined up to a phase, and the functions $f_\lambda(q)$ and $\bar{f}_\lambda(q)$, as given, already imply a choice of phase. Other choices allow the introduction of an arbitrary unimodular function $\omega(\lambda)$ into the generating function representing an operator intertwining $\mathbf{P}$ and $\overline{\mathbf{P}}$ as in (2.1). This would not necessarily intertwine $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ as in (2.2) unless (2.9b) is satisfied. This is a second condition on (2.10) and sets $\omega(\lambda) = 1$. Emphasizing these points about (2.9) *vs.* (2.10) will serve us in the following sections.

Lastly, it should be noted that, whatever other $\mathbf{C}$ we take, (2.1)-(2.2) is basically the *only* quantum canonical transformation bijectively [32] leading to a hamiltonian-type operator $\overline{\mathbf{P}} = 1/2\, \mathbf{P}^2 + V(\mathbf{Q})$ element of a Heisenberg-Weyl algebra. The $1/2\, \mathbf{P}^2$ term is produced by the similarity transformation (2.3) followed by a Fourier transform, and fixes $f(\mathbf{Q})$ to be $-\mathbf{Q}^3 / 6$; the potential $\mathbf{Q}$ arises from the original Heisenberg-Weyl generators, and is thus also fixed. The free-fall potential seen in this section is thus the *only* bona fide quantum system whose transformation to energy-time coordinates may be based on the Heisenberg-Weyl algebra. It has been given by Deenen, Moshinsky and Seligman as an example [27, appendix]. Having exhausted this class of systems and collected some notation, we now turn to oscillator systems.

III. CLASSICAL AND QUANTUM SYSTEMS WITH $sl(2, R)$ DINAMICAL ALGEBRA

Consider the following classical quantities

$$I_0 = \frac{1}{2}\left(p^2 + r^2 + \frac{\gamma}{r^2}\right) \tag{3.1}$$

$$I_1 = \frac{1}{4}\left(p^2 - r^2 + \frac{\gamma}{r^2}\right), \tag{3.1b}$$

$$I_2 = \frac{1}{2}\, rp\,, \tag{3.1c}$$

and, associated to the first two ones,

$$I_+ = I_0 + I_1 = \frac{1}{2}\left(p^2 + \frac{\gamma}{r^2}\right), \tag{3.1d}$$

$$I_- = I_0 - I_1 = \frac{1}{2}\, r^2\,. \tag{3.1e}$$

Among these, we recognize the hamiltonians of systems with a γ / r^2 centrifugal ($\gamma > 0$) barrier or centripetal ($\gamma < 0$) well at the origin, I_+ in (3.1d), plus a harmonic or repulsive oscillator potential, $2\, I_0$ in (3.1a) or $2\, I_1$ in (3.1b), respectively. Under the Poisson bracket [1], (3.1) constitute a Lie algebra:

$$\{I_1, I_2\} = -I_0,\ \{I_2, I_0\} = I_1,\ \{I_0, I_1\} = I_2, \tag{3.2}$$

which we recognize [33] as $so(2, 1) \simeq su(1, 1) \simeq sl(2, R) \simeq sp(2, R)$. The value of the Casimir invariant is $I^2 = I_1^2 + I_2^2 - I_0^2 = -\gamma / 4$, but no Hilbert space structure has been introduced.

The classical quantities (3.1) may be turned into first-order differential operators on phase space through associating [34, 35 Sec. V.A.2]

$$Z(q, p) \longrightarrow Z_{\mathrm{op}} = \frac{\partial z}{\partial r}\frac{\partial}{\partial p} - \frac{\partial z}{\partial p}\frac{\partial}{\partial r}, \tag{3.3a}$$

so that

$$[x_{\mathrm{op}}, y_{\mathrm{op}}] = \{x, y\}_{\mathrm{op}}\,. \tag{3.3b}$$

The action of $\exp(t\, h_{\mathrm{op}})$ on a function $f(r, p)$ yields a new function

$f(r_t(r,p), p_t(r,p))$ where r_t and p_t trace the classical trajectory of r and p under a hamiltonian $h(r,p)$. Alternatively,we may proceed constructing the quantities canonically conjugate to the I's:

$$\zeta_0 = \arctan \frac{I_2}{I_1}, \tag{3.4a}$$

$$\zeta_1 = \operatorname{arctanh} \frac{I_2}{I_0}, \tag{3.4b}$$

$$\zeta_+ = \frac{I_2}{I_+}, \tag{3.4c}$$

satisfying

$$\{\zeta_\alpha, I_\alpha\} = 1, \quad \alpha = 0, 1, + \Leftrightarrow \quad \text{[Eqs. (3.2)]}, \tag{3.4d}$$

and hence qualifying for time or phase variables [1] in the system governed by a hamiltonian I_α. The explicit solution for $r = r(\zeta_\alpha, I_\alpha)$, $p = p(\zeta_\alpha, I_\alpha)$ yields the classical transformation to phase variables.

We now turn to quantum mechanics. The Dirac-von Neumann prescription [2-4] allows us to associate unique self-adjoint operators $x(\mathbf{Q})$ and $y(\mathbf{P})$ on a domain dense in $L^2(R)$ [*c.f.* (1.2)] to the classical functions $x(q)$ and $y(p)$ on the full phase-space plane $q, p \in R$. The classical expressions written above, however, present several difficulties in this respect. (*i*) When $\gamma \neq 0$, the hamiltonians $2I_0$, $2I_1$ and I_+ in (3.1) exhibit a singularity at $r = 0$. Related to this, (*ii*) the straightforward replacement $p^2 \to \mathbf{P}^2$, $r^{\pm 2} \to \mathbf{Q}^{\pm 2}$ yields operators whose self-adjunction properties on $L^2(R)$ or $L^2(R^+)$ are quite nontrivial, as will be seen below. Finally, (*iii*) the consistent quantization of the ζ_α in (3.3), at least for $\alpha = 0, +$, with (3.4d) holding, is impossible due to the Jordan argument [11], since the (generalized) spectra of the quantum conjugate hamiltonians for $\alpha = 0$ and $+$ are not R. As stated in the introduction, we choose to take point (*iii*) as an interdiction, and abandon the effort to produce quantum operators of phase. The first two difficulties can be met through defining the operator domain carefully [36].

We consider the formal differential operators

$$\mathbf{J}_0 = \frac{\sigma}{4}\left[-\frac{d^2}{dr^2} + r^2 + \frac{\gamma}{r^2}\right], \tag{3.5a}$$

$$\mathbf{J}_1 = \frac{\sigma}{4}\left[-\frac{d^2}{dr^2} + r^2 + \frac{\gamma}{r^2}\right] , \tag{3.5b}$$

$$\mathbf{J}_2 = -\frac{i}{2}\left[r\frac{d}{dr} + \frac{1}{2}\right] , \tag{3.5c}$$

and their linear combinations

$$\mathbf{J}_+ = \mathbf{J}_0 + \mathbf{J}_1 = \frac{\sigma}{2}\left[-\frac{d^2}{dr^2} + \frac{\gamma}{r^2}\right], \tag{3.5d}$$

$$\mathbf{J}_- = \mathbf{J}_0 - \mathbf{J}_1 = \frac{\sigma}{2} r^2 , \tag{3.5e}$$

acting on the space F of functions $f(\sigma, r)$ of an underlying space

$$S = \{\sigma, r\}, \ \sigma \in \{-1, +1\} , \quad r \in R^+ = (0, \infty) , \tag{3.6}$$

twice-differentiable in r. On this space F, (3.5) close under commutation into a Lie algebra $sl(2, R)$:

$$[\mathbf{J}_1, \mathbf{J}_2] = -i\mathbf{J}_0, \ [\mathbf{J}_2, \mathbf{J}_0] = i\mathbf{J}_1, [\mathbf{J}_0, \mathbf{J}_1] = i\mathbf{J}_2 , \tag{3.7}$$

and the Casimir operator is a multiple of the identity

$$\mathbf{J}^2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 - \mathbf{J}_0^2 = c1 , \tag{3.8a}$$

$$c = \frac{-\gamma}{4} + \frac{3}{16} = k(1 - k) . \tag{3.8b}$$

On F we define a sesquilinear inner product

$$(f, g)_S = \sum_{\sigma = \pm 1} \int_0^\infty dr f(\sigma, r)^* g(\sigma, r) , \tag{3.9}$$

restricting first F to those functions with finite norm under (3.9) and completing then to define the Hilbert space which we call $L^2(S)$.

We shall now recount the nature of the self-adjoint extensions of the operators (3.5) with domain $L^2(S)$ [37, 38] with particular emphasis on the spectrum of $\mathbf{J}_0$, since this allows us to readily recognize the Self-Adjoint Irreducible Representations (SAIRs) of $sl(2, R)$ [39, 40] which the operators realize, and the domain restrictions which make these representations irreducible.

The eigenfunction of $\mathbf{J}_0$ in $L^2(S)$ corresponding to the real eigenvalue μ is ([38, 40] Eqs. (3.5) and (4.5))

$$\Psi_\mu^{k,\varepsilon}(\sigma, r) = (-\sigma)^{\mu-\varepsilon}\left[\frac{1}{2}\,\Gamma(k+\sigma\mu)\Gamma(1-k+\sigma\mu)\right]^{-1/2}$$

$$\times\; r^{-1/2}\, W_{\sigma\mu,\,k-1/2}(r^2)\;, \qquad (3.10)$$

with k related to the Casimir operator eigenvalue c through (3.8b), ε to be defined below, and $W_{u,\kappa}(x)$ being the Whittaker function ([41], Eq. 13.1.33).

(i) When $\gamma \geqslant 3/4$ ($c \leqslant 0, k \geqslant 1$), $\mathbf{J}_0$ has two self-adjoint extensions, distinguished by $\varepsilon = +k$ and $\varepsilon = -k$. The spectrum of $\mathbf{J}_0$ in the first case is $\mu \in \{k, k+1, k+2, \ldots\}$ with $\Psi_\mu^{k,k}(\sigma, r)$ having support only on $\sigma = +1$, and in the second case it is $\mu \in \{-k, -k-1, -k-2, \ldots\}$ with $\Psi_\mu^{k,-k}(\sigma, r)$ having support only on $\sigma = -1$. Since the raising and lowering operators ($\mathbf{J}_\uparrow = \mathbf{J}_1 \pm i\mathbf{J}_2$) map this space onto itself irreducibly, functions in $L^2(S)$ with support on $\sigma = +1$ —denote them by $L_+^2(S) = L^2(R^+)$— define a common domain where (3.5) realize the D_k^+ "discrete"-series SAIRs of $sl(2, R)$. Similarly, the space $L_-^2(S) = L^2(R^+)$ defined as those functions in $L^2(S)$ with support on $\sigma = -1$ serves as a common domain where (3.5) realize the D_k^- SAIRS of $sl(2, R)$. When $\gamma < 3/4$, there exists a one-parameter family of self-adjoint extensions [36] in Hilbert spaces which are common invariant domains parametrized by ε as follows.

(ii) In the *exceptional interval* $-1/4 \leqslant \gamma < 3/4$ ($1/4 \geqslant c > 0$, $1/2 \leqslant k > 1$) we have for $\varepsilon = \pm k \in \pm[1/2, 1]$ the coninuation of the (i)-cases: the domains $L_\pm^2(S)$ leading to the $D_k^\pm$ SAIRs, where $\mathbf{j}_0$ has spectra $\mu \in \{\pm k, \pm(k+1), \pm(k+2), \ldots\}$ and the functions have support on $\sigma = \pm 1$. This self-adjoint extension is the Friedrichs extension [36]. For $\varepsilon = \pm(1-k) \in \pm(0, 1/2)$, the spectrum of $\mathbf{J}_0$ is $\mu \in \{\pm(1-k), \pm(2-k), \ldots\}$ and $L_\pm^2(S)$ also provides a common domain where (3.5) realize the $D_k^\pm$ SAIRs, formally for $k \in (0, 1/2)$, again with the property that the support of $\Psi_\mu^{k,\pm k}(\sigma, r)$ is $\sigma = \pm 1$. The function domains for these extensions are thus also $L_\pm^2(S) = L^2(R^+)$. Up to this point, we have all and only the discrete series

SAIRs. In particular D_k^+ would be exhaustively realized [42] on a $L^2(R^+)$ function space, where $\sigma = +1$, and where (3.10) is given by Laguerre polynomials ([38] Eqs. (4.8), [41] Eq. 13.6.9).

The continuous series C_c^ε, however, requires both $\sigma = +1$ and $\sigma = -1$ values, and here appears for $|\varepsilon| < 1 - k$ ($1/2 < k < 1$, *i.e.* specifically excluding $k = 1/2$). Each value of ε contained within a symmetric subinterval of $(-1/2, 1/2)$ determines one common self-adjoint extension of (3.5), where the spectrum of $\mathbf{J}_0$ is $\mu \in \}\varepsilon + n, n \in Z\}$ (Z is the set of integers), so (3.5) realize the exceptional continuous series SAIRs C_c^ε ([38], Sect. 5). The function domain $L_\varepsilon^2(S)$ is defined as the vector sum of the subspace of functions $f \in L^2(S)$ such that $(\Psi_\varepsilon^{k,\varepsilon}, f)_S = 0$, with the one-dimensional space of multiples of $\Psi_\varepsilon^{k,\varepsilon}$ itself. It is a Hilbert space. (The cases $k = 1/2$, $\varepsilon \neq \pm k$ are included in the non-exceptional continuous series, below.) The structure of the operators in the exceptional interval is thus completely described.

(*iii*) When $\gamma < 1/4$ ($c > 1/4$, $k = (1 + i\kappa/2)$, $\kappa \in [0, \infty)$) we define the Hilbert spaces $L_\varepsilon^2(S)$ as the vector sum of the subspace of functions $f \in L^2(S)$ such that $(\Psi_\varepsilon^{k,\varepsilon}, f)_S = 0$ with $\Psi_\varepsilon^{k,\varepsilon}$ itself, for $\varepsilon i(-1/2, 1/2]$, excepting only $\pm\varepsilon = k = 1/2$ which serves as a domain for the realizations of the $D_{1/2}^\pm$ SAIRs. In this domain, the spectrum of $\mathbf{J}_0$ is $\mu \in \{\varepsilon + n, n \in Z\}$, *i.e.* equally-spaced ([38], Sect. 5) and hence invariant and irreducible under all the operators (3.5) which realize the nonexceptional continuous series C_c^ε SAIRs of $sl(2, R)$.

The spaces $L_\varepsilon^2(S)$ in (*ii*) and (*iii*) leading to C_c^ε SAIRs require functions with support on both values of the dichotomic variable σ. If only $L^2(R^+)$ were used for any of them, the spectrum of $\mathbf{J}_0$ in any self-adjoint extension (other than the cases leading to the discrete SAIRs) would *not* be equally spaced ([38], Sect. 4) and hence this domain would not be invariant under the rest of the $sl(2, R)$ generators.

Having thus specified the possible domains $L_\varepsilon^2(S)$ of (3.5) (the lablel ε standing for the continuous SAIR C_c^ε or for + or − in the discrete SAIR $D_k^\pm$), the properties of the functions (3.10) may be restated to be

$$\mathbf{J}_0 \Psi_\mu^{k,\varepsilon} = \mu \Psi_\mu^{k,\varepsilon} , \quad \mu \in \Sigma(\mathbf{J}_0, k, L_\varepsilon^2) , \tag{3.11}$$

i.e. for μ in the spectrum of $\mathbf{J}_0$, for definite values of k and ε. These functions were built so that they are orthonormal

$$(\Psi_\mu^{k,\varepsilon}, \Psi_\nu^{k,\varepsilon})_S = \delta_{\mu,\nu}, \quad \mu,\nu \in \Sigma(\mathbf{J}_0, k, L_\varepsilon^2) , \tag{3.12}$$

and are complete on $L^2_\varepsilon(S)$. Furthermore, the phase of $\Psi^{k,\varepsilon}_\mu(\sigma, r)$ was chosen according to Bargmann's convention ([39], Eqs. (6.22)-(6.26), (7.10)-(7.11) and (8.10)-(9.15)) *i.e.* such that the matrix elements of the raising and lowering operators be positive:

$$\mathbf{J}_\updownarrow \Psi^{k,\varepsilon}_\mu = \gamma^k_\mu \Psi^{k,\varepsilon}_{\mu \pm 1}, \quad \mathbf{J}_\updownarrow = \mathbf{J}_1 \pm i\mathbf{J}_2 , \tag{3.13a}$$

$$\gamma^k_\mu = +[(k \pm \mu)(1 - k \pm \mu)]^{1/2} . \tag{3.13b}$$

To sum up this section, we may say that the formal operators (3.5) in the domains $L^2_\pm(S) = L^2(R^+)$ and $L^2_\varepsilon(S)$ as specified above constitute the *proper* quantization of the classical observables (3.1) in configuration space. For $\gamma \in R$ they realize all SAIRs of $sl(2, R)$. Such quantization "problems" are seldom encountered in physics due to he fact that the actual energy spectra are allways bounded from below, normally precluding the C^ε_c SAIR series from appearing as such. Ours is a group-theoretic problem, however, and we aim for the complete solution. In the next section we present the phase-variable realization of the same algebra, and then proceed to intertwine the two.

IV. REALIZATIONS OF $sl(2, R)$ ON THE CIRCLE

The realization of the group $SO(2, 1)$ as a group of multiplier transformations on the space of functions on the unit circle leads to the following realization [43] for the generators of the Lie algebra $so(2, 1) \cong sl(2, R)$ on the space of differentiable functions on the unit circle $\phi \in S_1$:

$$\mathbf{K}_0 = e^{-i\varepsilon\phi}\left(-i\frac{d}{d\phi}\right) e^{i\varepsilon\phi} = -i\frac{d}{d\phi} + \varepsilon , \tag{4.1a}$$

$$\mathbf{K}_1 = ie^{-i\varepsilon\phi}\left(\cos\phi\frac{d}{d\phi} - k\sin\phi\right)e^{i\varepsilon\phi} , \tag{4.1b}$$

$$\mathbf{K}_2 = ie^{-i\varepsilon\phi}\left(\sin\phi\frac{d}{d\phi} + k\cos\phi\right)e^{i\varepsilon\phi} , \tag{4.1c}$$

whose Lie brackets under commutation are identical to (3.7) with $\mathbf{K}_\alpha$ replacing $\mathbf{J}_\alpha$ and whose Casimir operator $\mathbf{K}^2 = \mathbf{K}_1^2 + \mathbf{K}_2^2 - \mathbf{K}_0^2$ is a multiple $c = k(1 - k)$ of the identity operator as in (3.8). Parallel to (3.5d)-(3.5e) we construct

$$\mathbf{K}_+ = \mathbf{K}_0 + \mathbf{K}_1 = -ie^{-i\epsilon\phi}[(1-\cos\phi)\frac{d}{d\phi} + k\sin\phi]e^{i\epsilon\phi} , \tag{4.1d}$$

$$\mathbf{K}_- = \mathbf{K}_0 - \mathbf{K}_1 = -ie^{-i\epsilon\phi}[(1+\cos\phi)\frac{d}{d\phi} - k\sin\phi]e^{i\epsilon\phi} . \tag{4.1e}$$

As will be brought out in sections 6 and 7, changes of variables will allow, alternatively, for $\mathbf{K}_+$ or $\mathbf{K}_1$ to be realized as $-id\,/\,d\xi$ or $-id\,/\,d\eta$, respectively.

As stated in the introduction, the aim of this paper is to construct the intertwining operator $\mathbf{C}$ such that

$$\mathbf{J}_\alpha = \mathbf{C}\mathbf{K}_\alpha\mathbf{C}^{-1} , \quad \alpha = 0, 1, 2, +, - , \tag{4.2}$$

which will realize the quantum canonical transformation between a hamiltonian operator $\mathbf{J}_\alpha$ ($\alpha = 0, +, 1$) and translation operators $-id\,/\,d\phi$, $-id\,/\,d\xi$ or $-id\,/\,d\eta$. This is the direct generalization of (2.1). In this section we present the $\alpha = 0$ oscillator case, with the set of $\mathbf{K}_\alpha$ realized as (4.1). We shall ask this transformation $\mathbf{C}$ to be unitary, and so we must specify the inner product and function domain under which (4.1) are to be self-adjoint. In particular the spectrum of $\mathbf{K}_0$ in Eq. (4.1a) must be equal to the spectrum of $\mathbf{J}_0$ as described in the last section.

On the space of square-integrable functions on the circle, $L^2(S_1)$, defined through completion with respect to the usual inner product

$$(f, g)_{S_1} = \int_{-\pi}^{\pi} d\phi\, f(\phi)^*\, g(\phi) \tag{4.3}$$

the operator $\mathbf{K}_0$ is self-adjoint, and its spectrum is $\{m + \epsilon, m \in Z\}$. This space can thus only accomodate the continuous series representations C_c^ϵ. Further, since it can be verified that $\mathbf{K}_1$ and $\mathbf{K}_2$ are symmetric under (4.3) only for $k = (1 + i\kappa)\,/\,2$, $\kappa \in R$, only the nonexceptional continuous series may be obtained. In fact, Bargmann [39] used $L^2(S_1)$ for this only purpose. In [29] we constructed a k-parametrized family of non-local inner products on S_1 such that in the Hilbert spaces thus defined [44], $\mathbf{K}_0$ had the lower-bound spectrum characteristic of the discrete series D_k^+. The nonlocal measure was found asking for $(\mathbf{K}_1 \pm i\mathbf{K}_2)^\dagger = (\mathbf{K}_1 \mp i\mathbf{K}_2)$ to hold. This led to an inhomogeneous differential equation of which only the special (non-homogeneous) solution is retained ([29] Eqs. (2.21)-(2.22)). Here we present a parallel method (indicated in [29] and developed in

([45], Sec. 3.16) valid for all representation series of $sl(2, R)$. To this purpose, let us first construct representation bases for (4.1) where $\mathbf{K}_0$ is diagonal [*i.e* of functions $\chi^{k,\varepsilon}_{m+\varepsilon}(\phi) \sim e^{im\phi}$, $m \in Z$, for all representation series of the algebra, and then build appropriate inner products where these be orthonormal, as a necessary condition for the $\mathbf{K}_\alpha$ to be self-adjoint.

With the values of ε as specified below Eq. (3.10), it is clear that all $\chi^{k,\varepsilon}_{m+\varepsilon}(\phi)$ may be reached, through application of the raising or lowering operators

$$\mathbf{K}_{\updownarrow} = \mathbf{K}_1 \pm i\mathbf{K}_2 = ie^{-i\varepsilon\phi}e^{\pm i\phi}\left[\frac{d}{d\phi} \pm ik\right] e^{i\varepsilon\phi} , \tag{4.4}$$

acting on the $m = 0$ state, which is a constant conveniently chosen as

$$\chi^{k,\varepsilon}_{\varepsilon}(\phi) = (2\pi)^{-1/2} . \tag{4.5}$$

Indeed,

$$\mathbf{K}_{\updownarrow}\chi^{k,\varepsilon}_{\mu}(\phi) = \gamma^k_\mu \, \chi^{k,\varepsilon}_{\mu+1}(\phi) , \tag{4.6}$$

with γ^k_μ given by (3.13b) leads through iteration to

$$\chi^{k,\varepsilon}_{\varepsilon\pm m}(\phi) = \left[\prod_{\nu=k}^{k+m-1} (\varepsilon \pm \nu) \prod_{\nu'=1-k}^{m-k} (\varepsilon \pm \nu')\right]^{-1/2} (\mathbf{K}_{\updownarrow})^m \chi^{k,\varepsilon}_{\varepsilon}(\phi) , \tag{4.7a}$$

where for complex k the running index in the product is meant to take integer-spaced values, *i.e.*

$$\prod_{\nu=\kappa}^{\kappa+m-1} (\varepsilon \pm \nu) = (\varepsilon \pm \kappa)(\varepsilon \pm [\kappa + 1]) \ldots (\varepsilon \pm [\kappa + m - 1])$$

$$= (\pm 1)^m (\kappa \pm \varepsilon)_m = \frac{(\pm 1)^m \Gamma(\kappa + m \pm \varepsilon)}{\Gamma(\kappa \pm \varepsilon)} , \tag{4.7b}$$

where $(a)_m = a(a + \nu) \ldots (a + m - 1) = \Gamma(a + m) / \Gamma(a)$ is the Pochhammer symbol, and κ is k or $1 - k$.

Now (4.4) and (4.5) yield the functions

$$\chi^{k,\varepsilon}_{\varepsilon\pm m}(\phi) = \alpha^{k,\varepsilon}_{\pm,m} e^{\pm im\phi}, \quad m = 0, 1, 2, \ldots \tag{4.8a}$$

$$\alpha^{k,\varepsilon}_{\pm,m} = (-1)^m (2\pi)^{-1/2} \left[\frac{(k \pm \varepsilon)_m}{(1 - k \pm \varepsilon)_m} \right]^{1/2}, \tag{4.8b}$$

and in particular $\alpha^{k,\varepsilon}_{\pm,0} = (2\pi)^{-1/2}$ from (4.5). It is important to underline two properties of the purported bases $\{\chi^{k,\varepsilon}_{\varepsilon\pm m}\}$. First, they should provide orthonormal bases for *all* SAIRs of the algebra. For the continuous non-exceptional series $k^* = 1 - k$ so $\alpha^{k,\varepsilon}_{\pm,m}$ is a phase: $|\alpha^{k,\varepsilon}_{\pm,m}| = 1$; for real k, $\alpha^{k,\varepsilon}_{\pm,m}$ is real. In the exceptional interval, the D^+_k representations are considered for $\varepsilon = +k$, $0 < k < 1$ and similarly for the D^-_k representations. For $D^{\pm}_k$, $\alpha^{k,\varepsilon}_{\mp,m} = 0$. Second, the phase of the functions (4.8) is the necessary one for $\mathbf{J}_\updownarrow$ in (3.13) to be mapped onto $\mathbf{K}_\updownarrow$ in (4.6). It is basically irrelevant that we use the Bargmann phase convection or any other one, as long as the *same* convention is used for both. This guarantees that the canonical transformation (4.2) will intertwine the *three* algebra generators properly.

We now construct an inner product where, for each (k, ε), *i.e.* for each $sl(2, R)$ representation, (4.8) constitute an orthonormal basis. This is easiest to do in the Fourier coefficient basis for functions on the circle S_1, where

$$f_m = (F_m, f)_{S_1}, \quad F_m(\phi) = (2\pi)^{-1/2} e^{im\phi}, \tag{4.9}$$

proposing

$$(f, g)_{k,\varepsilon} = \sum_{m \in Z(k,\varepsilon)} f^*_m \, \omega^{k,\varepsilon}_m \, g_m, \tag{4.10a}$$

with $Z(k, \pm k) = \{0, \pm 1, \pm 2, \ldots\}$ for $(k, \varepsilon = \pm k)$ a discrete series $D^{\pm}_k$ representation, $Z(k, \varepsilon) = Z$ for (k, ε) a continuous series C^{ε}_c representation. The coefficients $\omega^{k,\varepsilon}_{\sigma m}$, $\sigma = \pm 1$, $m \in \{0, 1, 2, \ldots\}$, may be found from (4.8b) to be

$$\omega^{k,\varepsilon}_{\sigma m} = (2\pi)^{-1} |\alpha^{k,\varepsilon}_{\sigma,m}|^{-2} = \left| \frac{\prod_{\nu=1-k}^{m-k} (\varepsilon + \sigma\nu)}{\prod_{\nu'=k}^{k+m-1} (\varepsilon + \sigma\nu')} \right| = \left| \frac{(1-k+\sigma\varepsilon)_m}{(k+\sigma\varepsilon)_m} \right|. \tag{4.10b}$$

In particular, $\omega_0^{k,\varepsilon} = 1$.

Making use of Fourier series of (generalized) functions on the circle, we may also write $(f, g)_{k,\varepsilon}$ as a non-local inner product on the circle:

$$(f, g)_{k,\varepsilon} = \int_{S_1} d\phi \int_{S_1} d\phi' f(\phi)^* \Omega_{k,\varepsilon}(\phi - \phi') g(\phi')$$

$$= (f, \Omega_{k,\varepsilon} g)_{S_1} \quad , \tag{4.11a}$$

where $\Omega_{k,\varepsilon}$ is a convolution operator with integral kernel

$$\Omega_{k,\varepsilon}(\theta) = (2\pi)^{-1} \sum_{m \in Z(k,\varepsilon)} \omega_m^{k,\varepsilon} e^{im\theta} \quad . \tag{4.11b}$$

We shall now give the explicit forms of the weight factors (4.10b) and kernel (4.11b) for all representation series, thereby defining the (Hilbert) spaces where (4.1) are self-adjoint.

For the discrete series $D_k^{\pm}$, $\varepsilon = \pm k$, $k > 0$, the weight function in the Fourier-coefficient inner product form is

$$\omega_{\pm m}^{k,\pm k} = \frac{(1)_m}{(2k)_m} = \frac{m!\, \Gamma(2k)}{\Gamma(2k+m)} \quad , \quad m = 0, 1, 2, \ldots . \tag{4.12}$$

Note that for $k > 1/2$ (resp. $k = 1/2$ and $0 < k < 1/2$), $\omega_{\pm m}^{k,\pm k} > 1$ (resp. $= 1$ and < 1). Spaces of functions on the circle with only nonnegative Fourier coefficients are boundary values of functions analytic in the unit disc $|z| < 1$ for $z = \rho e^{i\phi}$. Their $L^2(S_1)$-norm (the Hardy-Lebesgue norm) majorizes their $(k, \varepsilon = k)$-norm (4.10) for $k > 1/2$, while for $k = 1/2$ the latter *is* the $L^2(S_1)$-norm. When $0 < k < 1/2$, (4.12a) is bounded by $m / 2k$ so the $(k, \varepsilon = k)$-norm of a function with Fourier coeffients f_m is finite provided $f_m \lesssim m^{-(1+\delta)}$ for $\delta > 0$. Norms of this type were discussed by Sally [46]. We shall call $L^2_{(k,\pm k)}(S_1)$ the Hilbert spaces obtained through completion with respect to the inner product (4.10). The non-local integral form of the inner products (4.11) for $L^2_{(k,\pm k)}(S_1)$ has a (nonlocal) weight function given by the Gauss hypergeometric function

$$\Omega_{k,\pm k}(\theta) = (2\pi)^{-1} {}_2F_1(1, 1\,; 2k\,; e^{\pm i\theta}) \quad . \tag{4.13}$$

This series is absolutely convergent for $k > 1$. When $k = 1$ then ([41], Eq. 15.1.3) (4.13) is $(2\pi)^{-1}\ln(1 - e^{\pm i\theta})/e^{\pm i\theta}$, displaying a logarithmic singularity at $\theta \to 0$. In fact, for $0 < k \leqslant 1$, (4.13) behaves singularly near $\theta = 0$ as ([41], Eq. 15.3.6) $\sim \theta^{2k-2}$. This singularity is integrable for $1/2 < k \leqslant 1$; for $k = 1/2$ it is the Cauchy representation ([47], Secs. 7.4.5-7.4.6) of a Dirac δ under integration with a continuous function in the Hardy-Lebesgue norm, *i.e.* $L^2_{(1/2,\,\pm 1/2)}(S_1) = L^2(S_1)$ for functions with nonnegative or nonpositive Fourier coefficient support. For $0 < k < 1/2$ additional smoothness conditions are required which are imposed on the Fourier coefficients as $|f_m|^2 \omega^{k,k}_{\pm\, m} \lesssim m^{-(1+\delta)}$, $\delta > 0$. In ([29], Eqs. (A.3)-(A.5)) we have shown that the nonlocal inner product on the circle with weight function (4.13) is identical with Bargmann's local inner product on the unit complex disc ([39], Eq. (9.9)) with $z = \zeta e^{i\phi}$, $\zeta < 1$.

For the continuous series C^{ε}_{c} there are two cases to consider, the exceptional $[1/2 < k < 1,\ |\varepsilon| < 1 - k]$ and the nonexceptional $[k = (1 + i\kappa)/2,\ \kappa \geqslant 0,\ \varepsilon \in (-1/2, 1/2]]$ representations. For the former, $\omega^{k,\varepsilon}_{\sigma\, m}$ in (4.10b) cannot be simplified beyond its last expression in that formula; since $k > 1 - k$ in this interval, however, it is clear that $\omega^{k,\varepsilon}_{\sigma\, m} < 1$ and in fact these coefficients behave asymptotically as ([41], Eq. 6.1.47) $\sim |m|^{1-2k}$ with $-1 < 1 - 2k < 0$. The (k, ε)-norm on the circle is majorized by the $L^2(S_1)$ norm, and the Hilbert space thus defined by completion will be denoted by $L^2_{(k,\,\varepsilon)}(S_1)$ in the appropiate (k, ε) range. The nonlocal integral form (4.11) in the exceptional continuous series has thus a weight function ([45], Eq. (3.46))

$$\Omega_{k,\,\varepsilon}(\theta) = (2\pi)^{-1}[{}_2F_1(1, 1-k+\varepsilon; k+\varepsilon; e^{i\theta}) + {}_2F_1(1, 1-k-\varepsilon; k-\varepsilon; e^{-i\theta}) - 1]\,, \tag{4.14a}$$

with an innocuous integrable singularity at $\theta = 0$. For the special case of the single-valued representations of the $SO(2, 1)$ group, $\varepsilon = 0$, calculated by Bargmann ([39], Eqs. (8.7), (8.9), and (8.11)) the series (4.14) may be summed ([48], Eq. 3.631.8) and yields Bargmann's result

$$\Omega_{k,\,0}(\theta) = 2^{-k}\pi^{-1/2}\,\Gamma(k)\,[\Gamma(k - 1/2)]^{-1}(1 - \cos\theta)^{k-1}\,. \tag{4.14b}$$

Finally, for the nonexceptional continuous representation series, $\omega^{k,\,\varepsilon}_{m} = 1$ as we remarked before, and hence

$$\Omega_{k,\,\varepsilon}(\theta) = \delta(\theta)\,, \quad k = (1 + i\kappa)/2\,, \quad \kappa \in [0, \infty)\,, \tag{4.14c}$$

so that for this representation series $L^2_{(k,\varepsilon)}(S_1) = L^2(S_1)$.

Two more particular cases which are of interest are the two weight functions for the reducible oscillator representation irreducible components, namely $D^+_{1/4}$ and $D^+_{3/4}$. These are given by (4.13) which for $\varepsilon = k = 1/4$ and $3/4$ yield trigonometric functions. In fact, ([41], Eq. 15.1.6)

$$\Omega_{3/4}(\theta) = \frac{1}{2\pi} \frac{\arcsin e^{i\theta/2}}{e^{i\theta/2}[1 - e^{i\theta}]^{1/2}} , \tag{4.15a}$$

and [41, Eqs. 15.1.6 and 15.2.4]

$$\Omega_{1/4}(\theta) = \frac{1}{2\pi} \frac{1}{1 - e^{i\theta}} \left[1 + e^{i\theta/2} \frac{\arcsin e^{i\theta/2}}{[1 - e^{i\theta}]^{1/2}} \right]$$

$$= \frac{1}{1 - e^{i\theta}} \left[\frac{1}{2\pi} + e^{i\theta/2}\Omega_{3/4}(\theta) \right] . \tag{4.15b}$$

These expressions will be used in the next sections.

Reproducing kernels for $L^2_{(k,\varepsilon)}(S_1)$ may be constructed as linear functionals $K_{k,\varepsilon}(\phi, \phi')$ with the property

$$(K_{k,\varepsilon}(\cdot\, , \phi'), f)_{k,\varepsilon} = f(\phi') \tag{4.16}$$

for all $f \in L^2_{(k,\varepsilon)}(S_1)$ continuous at ϕ'. Indeed, written in terms of its Fourier coefficient inner product (4.10a), the following series has the required property:

$$K_{k,\varepsilon}(\phi, \phi') = \sum_{m \in Z(k,\varepsilon)} \chi^{k,\varepsilon}_{m+\varepsilon}(\phi)\, \chi^{k,\varepsilon}_{m+\varepsilon}(\phi')^*$$

$$= (2\pi)^{-1} \sum_{m \in Z(k,\varepsilon)} (\omega^{k,\varepsilon}_m)^{-1} e^{im(\phi-\phi')} = K_{k,\varepsilon}(\phi - \phi') . \tag{4.17}$$

This expression is analogous to $\Omega_{k,\varepsilon}(\phi - \phi')$ with $(\omega^{k,\varepsilon}_m)^{-1}$ in place of $\omega^{k,\varepsilon}_m$, which allows an abbreviated derivation of results. For the discrete series $D^\pm_k$, $\varepsilon = \pm k$, the reproducing kernel for $L^2_{(k,\pm k)}(S_1)$ is simplified:

$$K_{k,+k}(\theta) = (2\pi)^{-1}\, {}_1F_0(2k; e^{\pm i\theta}) = (2\pi)^{-1}(1 - e^{\pm i\theta})^{-2k} , \tag{4.18}$$

as found in ([29], Eq. (2.27)). For the continuous series we need only exchange $k \leftrightarrow 1-k$ to relate (4.17) through (4.10b) to (4.11b), *i.e.*

$$K_{k,\varepsilon}(\theta) = \Omega_{1-k,\varepsilon}(\theta) \tag{4.19}$$

in the exceptional and nonexceptional interval cases. As expected, for the latter we have the Dirac delta in $L^2(S_1)$. In all other cases the reproducing kernel has a singularity $\sim \theta^{-2k}$ at $\theta = 0$.

We thus end our description of the inner products and domains under which the $sl(2, R)$ algebra realization (4.1) is self-adjoint, leading to all SAIRs.

V. UNITARY CANONICAL TRANSFORMATIONS TO PHASE VARIABLES: THE HARMONIC OSCILLATOR + γ/r^2 POTENTIAL CASE

Consider a given SAIR (k, ε) of $sl(2, R)$, its realization by self-adjoint operators $\mathbf{J}_\alpha$, (3.5), in the corresponding domain $L^2_\varepsilon(S)$ as described in Sect. III, and by its realization by similar operators $\mathbf{K}_\alpha$, (4.1), in the domain $L^2_{(k,\varepsilon)}(S_1)$ as described in Sect. IV. Consider an arbitrary function $\mathring{f}(\phi) \in L^2_{(k,\varepsilon)}(S_1)$ and its C-*transform* function $\bar{f}(\sigma, r) \in L^2_\varepsilon(S)$, obtained from the former through an integral transform with a kernel $C_{k,\varepsilon}(\sigma, r; \phi)$,

$$\begin{aligned}\bar{f}(\sigma, r) &= (\mathbf{C}\mathring{f})(\sigma, r) = (C_{k,\varepsilon}(\sigma, r; \cdot)^*, \mathring{f})_{k,\varepsilon} \\ &= (C_{k,\varepsilon}(\sigma, r; \cdot)^*, \Omega_{k,\varepsilon}\mathring{f})_{S_1} \\ &= \int_{S_1} d\phi \int_{S_1} d\phi' \Omega_{k,\varepsilon}(\phi - \phi') C_{k,\varepsilon}(\sigma, r; \phi) \mathring{f}(\phi') \ .\end{aligned} \tag{5.1a}$$

The operator $\mathbf{C}$ is to intertwine the two Hilbert spaces, and should be such that

$$\mathbf{C}\mathbf{K}_\alpha = \mathbf{J}_\alpha \mathbf{C} \tag{5.1b}$$

hold on the appropiate domains dense in the above Hilbert spaces, and finally, the statement of unitarity of $\mathbf{C}$ is to be made through the Parseval equality

$$(\bar{f}, \bar{g})_S = (\mathring{f}, g)_{k,\varepsilon} \ . \tag{5.1c}$$

The integral form (5.1a) of the C-transform implies that C is a linear operator. The conditions (5.1b) tell us that if $\bar{f}$ is the C-transform of $\mathring{f}$, then $\mathbf{J}\bar{f}$ is the C-transform of $\mathbf{K}\mathring{f}$ for all elements of the algebra $\mathbf{J} = \Sigma c_\alpha \mathbf{J}_\alpha$ and $\mathbf{K} = \Sigma c_\alpha \mathbf{K}_\alpha$. Finally, (5.1c) implies that the C-transform maps an orthonormal basis in one space onto an orthonormal basis in the other.

While Eqs. (5.1b) lead to the determination of the canonical transform kernel $C_{k,\varepsilon}(\sigma, r; \phi)$ as the analogue of (2.9) *i.e.* two Mello-Moshinsky-type simultaneous partial differential equations, we shall see that Eq. (5.1c) leads to its alternative determination as a generating function, in analogy with (2.10). The results we presented in the last two sections were geared to allow us to follow the second procedure. Indeed, given that we know the two normalized eigenbases of $\mathbf{J}_0$ and $\mathbf{K}_0$, $\{\Psi_\mu^{k,\varepsilon}\}$ in (3.10) and $\{\chi_\mu^{k,\varepsilon}\}$ in (4.8), we may use the generating series for Whittaker functions obtained in the appendix to find

$$C_{k,\varepsilon}(\sigma, r; \phi) = \sum_{m \in Z(k,\varepsilon)} \Psi^{k,\varepsilon}_{m+\varepsilon}(\sigma, r)\chi^{k,\varepsilon}_{m+\varepsilon}(\phi)^*$$

$$= \left[\pi^{-1/2}\left(\frac{\Gamma(1-k+\sigma\varepsilon)}{\Gamma(k+\sigma\varepsilon)}\right)^{1/2} r^{-1/2} \sum_{m \in Z(k,\varepsilon)} \frac{\sigma^m W_{\sigma(\varepsilon+m),\, k-1/2}(r^2)e^{im\phi}}{\Gamma(1-k+\sigma[\varepsilon+m])}\right]^*$$

$$= \left[\left(\frac{\Gamma(1-k+\sigma\varepsilon)}{2\pi\Gamma(k+\sigma\varepsilon)}\right)^{1/2}\left(\frac{r}{2}\right)^{2k-1/2} e^{-i\varepsilon\phi}\Phi_\sigma^\varepsilon(\phi)\,|\operatorname{trg}_\sigma \tfrac{\phi}{2}|^{-2k}\exp\left(i\tfrac{r}{2}\tan^\sigma \tfrac{\phi}{2}\right)\right]^*$$

$$= \left[\left(\frac{\Gamma(1-k+\sigma\varepsilon)}{\pi\Gamma(k+\sigma\varepsilon)}\right)^{1/2} r^{2k-1/2} e^{i(k-\varepsilon)\phi}\Phi_\sigma^{\varepsilon-k}(\phi)(1+\sigma e^{i\phi})^{-2k}\right.$$

$$\left.\times \exp\left(-\frac{r^2}{2}\,\frac{\sigma - e^{i\phi}}{1+\sigma e^{ie}}\right)\right]^*, \tag{5.2a}$$

where we have abbreviated $\operatorname{trg}_{+1} = \cos$ and $\operatorname{trg}_{-1} = \sin$, $\tan^\sigma = \tan$ or $\cot$ according to whether $\sigma = +1$ or -1, and used the phase function

$$\Phi_\sigma^\alpha(\phi) = \begin{cases} 1 & \text{for } \sigma = +1 \\ e^{i\alpha\pi\,\operatorname{sgn}\phi} & \text{for } \sigma = -1 \end{cases} \tag{5.2b}$$

The general result (5.2) simplifies considerably in two cases: $D_k^\pm$ and the monexceptional C_c^{ε}. For the lower-bound D_k^+ SAIR series ($\varepsilon = k > 0$) we have a single component $\sigma = +1$ to consider, where we may write

$$C_{k,k}(+1, r; \phi)$$

$$= [\pi\,\Gamma(2k)]^{-1/2} r^{2k-1/2} (1 + e^{-i\phi})^{-2k} \exp\left(\frac{r^2}{2}\,\frac{1 - e^{i\phi}}{1 + e^{i\phi}}\right), \qquad (5.3)$$

and compare with previous results [49]. For the nonexceptional continuous C_c^{ε} SAIRs ($k = [1 + i\kappa]/2$, $\kappa \in R^+$, $\varepsilon \in (-1/2, 1/2]$), the ratio of Gamma functions in (5.2a) is an overall phase which may be discarded. We write

$$C_{k,\varepsilon}(\sigma, r; \phi) = \pi^{1/2} r^{1/2 - i\kappa} \exp[-(\kappa + i\{1 - 2\varepsilon\})\phi/2]\, \Phi_\sigma^{\varepsilon - (1 - i\kappa)}(\phi)$$

$$\times (1 + \sigma e^{-i\phi})^{-1 + i\kappa} \exp\left(\frac{r^2}{2}\,\frac{\sigma - e^{i\phi}}{1 + \sigma e^{i\phi}}\right), \qquad (5.4)$$

and also compare with previous results [50]. For the exceptional continuous series the result (5.2) is new and covers all cases with a single expression.

The ordinary harmonic oscillator subcase has interest by itself: Eq. (3.5) with $\gamma = 0$ and r ranging over R. This case is a direct sum of $D_{1/4}^+$ and $D_{3/4}^+$ SAIRs, which on R may be distinguished by parity under space reflections $r \leftrightarrow -r$ in the single $\sigma = +1$ component of the configuration-space wavefunctions (3.10). The latter reduce to even- and odd-order Hermite polynomials. We must therefore double the phase variable space to two separate circles, one for each irreducible representation, and consider functions $\mathring{f} = \{\mathring{f}_e, \mathring{f}_o\}$ on $Z_2 \times S_1$, which on the first circle are $\mathring{f}_e(\phi)$, transformed through the $k = 1/4$ -transform kernel (5.3) to even functions on r, and on the second circle are $\mathring{f}_o(\phi)$ which through the $k = 3/4$ -transform kernel map on odd functions of r. The $\sigma = +1$ component of (5.1a) becomes

$$\bar{f}(r) = (C\mathring{f})(r) = (C_e(r; \cdot)^*, \mathring{f}_e)_{1/4} + (C_o(r; \cdot)^*, \mathring{f}_o)_{3/4} \qquad (5.5a)$$

where we use the inner product (4.11) denoting $(\cdot\,,\cdot)_{k,\varepsilon}$ by $(\cdot\,,\cdot)_k$ with nonlocal measures (4.15) for $k = 1/4$ and $3/4$. The canonical transform kernels are

$$C_e(r;\phi) = \frac{1}{2}\,[C_{1/4,1/4}(+1,r;\phi) + C_{1/4,1/4}(+1,-r;\phi)]$$

$$= \pi^{-3/4}(1+e^{-i\phi})^{-1/2}\exp\left(-\frac{r^2}{2}\,\frac{1-e^{-i\phi}}{1+e^{-i\phi}}\right), \qquad (5.5b)$$

$$C_o(r;\phi) = \frac{1}{2}\,[C_{3/4,3/4}(+1,r;\phi) - C_{3/4,3/4}(+1,-r;\phi)]$$

$$= 2^{1/2}r(1+e^{-i\phi})^{-1}C_e(r;\phi) \qquad (5.5c)$$

The C-transform maps, thus, two-component functions on the circle [with an inner product constructible as a sum of $L^2_{(1/4,1/4)}(S_1)$ and $L^2_{(3/4,3/4)}(S_1)$ inner products] onto the space of $L^2(R)$ functions on the real line.

In this paper we have determined the transform kernel (5.2) as a generating function. We would like to examine briefly the construction problems which must be encountered if we follow the Mello-Moshinsky method [28]. For each of the generators $\mathbf{J}_\alpha$ in (3.5) and $\mathbf{K}_\alpha$ in (4.1), $\alpha = 0, 1, 2$, the canonical transform (5.1b) must have the property, acting on any function $\mathring{f} \in L^2_{(k,\varepsilon)}(S_1)$,

$$(\mathbf{C}\mathbf{K}_\alpha\mathring{f})(\sigma, r) = (\mathbf{J}_\alpha\mathbf{C}\mathring{f})(\sigma, r)\ , \quad \alpha = 0, 1, 2\ . \qquad (5.6a)$$

Due to (5.1a) we may write this as a (k, ε)-inner product on the circle. Now, while $\mathbf{J}_\alpha$ as an operator on S acts on the transform kernel, $\mathbf{K}_\alpha$ is self-adjoint in $L^2_{(k,\varepsilon)}(S_1)$ so

$$(\mathbf{K}_\alpha C_{k,\varepsilon}(\sigma, r;\cdot)^*, \mathring{f})_{k,\varepsilon} = (\mathbf{J}_\alpha C_{k,\varepsilon}(\sigma, r;\cdot)^*, \mathring{f})_{k,\varepsilon} \qquad (5.6b)$$

for arbitrary $\mathring{f}$ in this space. A sufficient condition for this to hold are the set of partial differential equations

$$\mathbf{K}_\alpha(\phi)C_{k,\varepsilon}(\sigma, r;\phi)^* = \mathbf{J}_\alpha(\sigma, r)C_{k,\varepsilon}(\sigma, r;\phi)^*\ , \qquad (5.6c)$$

which are the Mello-Moshinsky equations for the oscillator systems. Of the three equations, two are algebraically independent. The $\mathbf{K}_\alpha$ are all first-order differential operators [*c.f.* (4.1)], while out of the $\mathbf{J}_\alpha$ we have second-order ones ($\alpha = 0, 1, +$), first order ($\alpha = 2$) and zeroth order ($\alpha = -$). The latter is the first-order ordinary differential equation

$$[(1+\cos\phi)(-i\frac{d}{d\phi}+\varepsilon)+ik\sin\phi-\frac{1}{2}\sigma r^2]C_{k,\varepsilon}(\sigma,r;\phi)^* = 0\ , \quad (5.7a)$$

whose solution is of the general from

$$C_{k,\varepsilon}(\sigma,r;\phi)^* = K\exp[i\,\Theta(\varepsilon,k,\sigma r^2/2,\sigma)]\,r^{1/2}\cos^{-2k}\frac{1}{2}\phi\,e^{-i\varepsilon\phi}$$

$$\times\ \exp(i\frac{1}{2}\sigma r^2\tan\frac{1}{2}\phi)\ . \quad (5.7b)$$

The normalization coefficient $K > 0$ is $(4\pi)^{-1/2}$ for $L^2(S_1)$, but is more difficult to determine for the other $L^2_{(k,\varepsilon)}(S_1)$ spaces. Most important, there is a nontrivial phase function $\Theta(\varepsilon, k, \sigma r^2/2, \sigma)$ which must be determined from an equation (5.6c) for another $\alpha \neq -$. In [30] for the nonexceptional continuous series, we chose this second equation for $\alpha = 0$, expressing the kernel as the Fourier series in (5.2a) and adjusting the phase of the Fourier coefficients $\Psi^{k,\varepsilon}_{m+\varepsilon}(\sigma, r)$ to follow Bargmann's convention. The singularity of the equation (5.6d) at $\phi = \pm\pi$ must be paid special attention, and it should be borne in mind that the kernel is *not* to be multivalued in ϕ. The final result in (5.2), obtained here through the alternative generating-function method, indeed solves (5.6d) and manifestly keeps track of the correct normalization and phase for all representation series.

It is interesting to note that the Mello-Moshinsky equation (5.6) for $\alpha = 0$ is formally the time-dependent Schrödinger equation for the system with hamiltonian $\mathbf{J}_0$. One solution is the Green's function for that system, which in turn is the (k, ε)-canonical transform kernel ([40], Sec. 2) $C^{k,\varepsilon}_{g(\phi)}(\sigma, r; +1, 0)$ of the $SL(2, R)$ subgroup generated by $\mathbf{J}_0$ and parametrized by time ϕ. We have to be careful about the initial conditions, though. Whereas the (k, ε)-canonical transform kernel is built such that at $\phi = 0$ it be $\delta_{\sigma\sigma'}\,\delta(r - r')$, the C-transform kernel to phase variables (5.2a), solution to the Mello-Moshinsky equations, has no otherwise meaningful property as a function of r at the point $\phi = 0$, except that derived from the generating function (5.2) itself: that it behave as $\sim r^{2k^* - 1/2}$. In the particular case of the ordinary $(\gamma = 0)$ harmonic oscillator, $k = 1/4$ and $3/4$, the transforms (5.5) correspond to initial conditions $\sim r^0$ and r^1. Except for the point $\phi = 0$, thus, (5.5a) is proportional, as regards r-dependence to the canonical transform kernel for the $\mathbf{J}_0$-generated elliptic orbit [51] and (5.5b) to a linear combination involving its r-derivative.

The inversion of the C-transform (5.1)-(5.2) is

$$\mathring{f}(\phi) = (\mathbf{C}^{-1}\overline{f})(\phi) = (C_{k,\varepsilon}(\cdot,\cdot\,;\phi), \overline{f})_S$$

$$= \sum_{\sigma=\pm 1} \int_0^\infty dr\, C_{k,\varepsilon}(\sigma, r; \phi)^* \overline{f}(\sigma, r) \ , \tag{5.8}$$

and the Parseval identity (5.1c) holds. Unitarity of the transform implies the relations

$$\Big(C_{k,\varepsilon}(\sigma, r; \cdot), C_{k,\varepsilon}(\sigma', r'; \cdot)\Big)_{k,\varepsilon}$$

$$= \int_{S_1} d\phi \int_{S_1} d\phi' \Omega_{k,\varepsilon}(\phi - \phi') \times C_{k,\varepsilon}(\sigma, r; \phi)^* C_{k,\varepsilon}(\sigma', r'; \phi')$$

$$= \delta_{\sigma,\sigma'}\delta(r - r') \ , \tag{5.9a}$$

$$\Big(C_{k,\varepsilon}(\cdot\,,\cdot\,;\phi), C_{k,\varepsilon}(\cdot\,,\cdot\,;\phi'\Big)_S$$

$$= \sum_{\sigma=\pm 1} \int_0^\infty dr \times C_{k,\varepsilon}(\sigma, r; \phi)^* C_{k,\varepsilon}(\sigma, r; \phi') = K_{k,\varepsilon}(\phi, \phi')^* \ , \tag{5.9b}$$

where $K_{k,\varepsilon}$ is the reproducing kernel in $L^2_{(k,\varepsilon)}(S_1)$ given by (4.17). In the Mello-Moshinsky approach, the unitarity conditions in r, (5.9b), fix the normalization constant K in (5.7) but yield no information on the phase. Morevover, the unitarity conditions in ϕ, Eq. (5.9a), will fail unless the proper inner product is taken with the in general nonlocal measure. The ordinary $L^2(S_1)$ measure is correct for the nonexceptional continuous series only.

We would like to close this section with a comment on the way in which the Moshinsky-Seligman ambiguity group [24-28] re-appears in our group-theoretical framework. The group $SL(2, R)$ has a Z_∞-homotopy group, *i.e.* it is infinitely connected. This fact is displayed upon expressing real 2×2 matrices in terms of pseudo-unitary ones ([39], 45 Sec. 3.4) as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} (1 - |\rho|^2)^{-1/2} \begin{pmatrix} e^{i\phi} & \rho^* e^{-i\phi} \\ \rho e^{i\phi} & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}^{-1} , \tag{5.10a}$$

$$\rho \in C, \quad |\rho| < 1 \ , \qquad \phi \equiv \phi \bmod N\pi \ , \tag{5.10b}$$

where $N=1$ for $SO(2,1)=SL(2,R)/Z_2$, $N=2$ for $SL(2,R)=SU(1,1)=Sp(2,R)$, $N=4$ for the metaplectic group $Mp(2,R)$ which covers $Sp(2,R)$ twice, and $N=\infty$ for the universal covering group $\overline{SL(2,R)}$. When the spectrum of $\mathbf{J}_0$ modulo unity is M/N (M and N relatively prime integers) the representation is single-valued for a group G such that $G/SO(2,1)=Z_N$. For the harmonic oscillator $+\gamma/r^2$ potential systems, the time-evolution canonical transform kernel (Green's function) is [40] $C^{k;\,\varepsilon}_{g(\phi)}(r,0)$ along a line parametrized by ϕ and given by (5.10a) with $\rho=0$, with the same multivaluation properties in ϕ as the C-transform kernel to phase variables (5.2), which as remarked above, differs from the former only in the initial conditions. For all $\phi=K\pi$, K integer, $C^{k;\,\varepsilon}_{g(\phi)}(r,0)$ is a phase multiple $\exp(2\pi iK/N)$ of the unit operator, and this characterizes the motion as cyclic. It is a 'weak' type of cyclicity, however, meant in the sense of having a non-unity phase for $K\neq N$.

Time inversion, the normal subgroup of the ambiguity group, is described through the outer automorphism of $SL(2,R)$ given by ([39], Eqs. (9.16) and (9.21))

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{t} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad \text{i.e.} \quad \rho \xrightarrow{t} \rho^*,\ \phi \xrightarrow{t} -\phi, \tag{5.11}$$

which effects $\mathbf{J}_\alpha \xrightarrow{t} -\mathbf{J}_\alpha$, $\alpha=0, 1$, and $\mathbf{J}_2 \xrightarrow{t} \mathbf{J}_2$, and thus inter twines the D_k^+ and D_k^- representations (inverting the spectrum of $\mathbf{J}_0$), leaving invariant the C_c^ε representations. Since Moshinsky and Seligman identify the action with the absolute value of the position operator ([27], Eqs. (3.4)), t is an element of the D_∞ ambiguity group. In the present paper, the ambiguity group does not play a role as central as it does in the Moshinsky-Seligman approach: it only describes the origin of the apparent multivaluation in ϕ of the C-transform integral kernel (5.2), and does not include the time-reversal operation t in (5.11) which would invert the spectrum.

VI. THE PURE γ/r^2-POTENTIAL SYSTEMS

The realization (4.1) of $sl(2,R)$ suits the harmonic-oscillator since the elliptic operator (4.1a), $\mathbf{K}_0$, is the operator expressed as a single derivative. We shall obtain a similar form for the parabolic operator $\mathbf{K}_+$ corresponding through $\mathbf{J}_+$ to the pure centrifugal/centripetal potential. In the $D_k^\pm$ series, the generalized spectrum of this operator is $R^\pm$, and is R for C_c^ε. This can be manifestly seen in the form $\sigma r^2 / 2$ for $\mathbf{J}_-$ in (3.5e), and the latter is obtained through rotation from $\mathbf{J}_+$.

Indeed, a change of variables in (4.1) given analytically by

$$e^{i\phi} = \frac{\xi - i}{\xi + i}\,, \quad \xi = i\,\frac{1 + e^{i\phi}}{1 - e^{i\phi}} = -\cot\frac{1}{2}\phi\,, \tag{6.1}$$

and a further similarity transformation of the algebra generators and Hilbert space

$$\mathbf{L}_\alpha = \mathbf{U}_{k,\varepsilon}\mathbf{K}_\alpha\mathbf{U}_{k,\varepsilon}^{-1}\,, \quad \alpha = 0, 1, 2\,; \tag{6.2a}$$

$$(\mathbf{U}_{k,\varepsilon}f)(\xi) = U_{k,\varepsilon}(\xi)f(\xi) = (\xi^2 + 1)^{-k}\left(\frac{\xi - i}{\xi + i}\right)^{\varepsilon} f(\xi)$$

$$= |\sin\frac{\phi}{2}|^{2k} e^{i\varepsilon\phi} f(\xi(\phi))\,, \tag{6.2b}$$

yields the $sl(2, R)$ realization equivalent to the former one in (4.1) suited to our purpose. Explicitly, it gives

$$\mathbf{L}_0 = -i\,\frac{1}{2}\,(\xi^2 + 1)\frac{d}{d\xi} - ik\xi\,, \tag{6.3a}$$

$$\mathbf{L}_1 = i\,\frac{1}{2}\,(\xi^2 - 1)\frac{d}{d\xi} + ik\xi\,, \tag{6.3b}$$

$$\mathbf{L}_2 = -i\xi\frac{d}{d\xi} - ik\,, \tag{6.3c}$$

$$\mathbf{L}_+ = \mathbf{L}_0 + \mathbf{L}_1 = -i\frac{d}{d\xi}\,, \tag{6.3d}$$

$$\mathbf{L}_- = \mathbf{L}_0 - \mathbf{L}_1 = -i\xi^2\frac{d}{d\xi} - 2ik\xi\,, \tag{6.3e}$$

where $\mathbf{L}_+$ in (6.3d) has the required form. This realization exponentiates to the conformal action of $SL(2, R)$ on the line with a unitarity-preserving multiplier [52].

The same transformation (6.1)-(6.2) applied to the functions in the inner product (4.11) tells us that (6.3) belong to a definite SAIR (k, ε) in a Hilbert space $L^2_{(k,\ \varepsilon)}(R)$. This is defined by an inner product $(\cdot\ ,\ \cdot)^{(+)}_{k,\ \varepsilon}$ given in terms of $(\cdot\ ,\ \cdot)_R$, the ordinary $L^2(R)$-inner product of $(\cdot\ ,\ \cdot)^{(0)}_{k,\ \varepsilon}$, the inner product on the circle as

$$(\overset{+}{f}, \overset{+}{g})^{(+)}_{k,\varepsilon} = (\mathbf{U}^{-1}_{k,\varepsilon}\overset{+}{f}, \mathbf{U}^{-1}_{k,\varepsilon}\overset{+}{g})^{(0)}_{k,\varepsilon} = (\overset{\circ}{f}, \overset{\circ}{g})^{(0)}_{k,\varepsilon}$$

$$= (\overset{+}{f}, \mathbf{U}^{-1*}_{k,\varepsilon}\Omega_{k,\varepsilon}\mathbf{U}^{-1}_{k,\varepsilon}\overset{+}{g})_{S_1} = (\overset{+}{f}, \Upsilon_{k,\varepsilon}\overset{+}{g})_R$$

$$= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \Upsilon_{k,\varepsilon}(\xi, \xi')\overset{+}{f}(\xi)^*\overset{+}{g}(\xi')\ , \tag{6.4a}$$

where functions of $\phi \in S_1$ transform to functions of $\xi \in R$ as

$$\overset{+}{f}(\xi) = (\mathbf{U}_{k,\ \varepsilon}\overset{\circ}{f})(\xi) = U_{k,\varepsilon}(\xi)\overset{\circ}{f}(\phi(\xi))\ . \tag{6.4b}$$

The metric operator $\Upsilon_{k,\ \varepsilon} = \mathbf{U}^{-1\ *}_{k,\ \varepsilon}\Omega_{k,\ \varepsilon}\mathbf{U}^{-1}_{k,\ \varepsilon}$ has for its kernel

$$\Upsilon_{k,\varepsilon}(\xi, \xi') = \frac{d\phi(\xi)}{d\xi}\frac{d\phi'(\xi')}{d\xi'}\ U_{k,\varepsilon}(\xi)^{-1*}\Omega_{k,\varepsilon}(\phi(\xi) - \phi'(\xi'))\,U_{k,\varepsilon}(\xi')^{-1}$$

$$= 4(\xi^2 + 1)^{k^*-1}(\xi'^2 + 1)^{k-1}\left(\frac{\xi - i}{\xi + i}\ \frac{\xi' + i}{\xi' - i}\right)^{\varepsilon}$$

$$\times\ \Omega_{k,\varepsilon}\ e^{i[\phi(\xi) - \phi'(\xi')]} = \left(\frac{\xi - i}{\xi + i}\ \frac{\xi' + i}{\xi' - i}\right)\ , \tag{6.4c}$$

where $\Omega_{k,\ \varepsilon}$, depending on the angles ϕ, ϕ' through $\exp[i(\phi - \phi')]$ replaces this argument as shown. In particular, for the nonexceptional continuous series (4.14c), $\Upsilon_{k,\ \varepsilon}(\xi, \xi') = 2\delta(\xi - \xi')$ and the inner product (6.4a) becomes local, so $L^2_{(k,\ \varepsilon)}(R) = L^2(R)$ in this case.

In $L^2_{(k,\ \varepsilon)}(R)$, thus, $-id/d\xi$ is self-adjoint and has a simple generalized spectrum covering $R^{\pm}$ for $D^{\pm}_k$, the usual R for the local inner product for the nonexceptional C^{ε}_c series, and also for the nonlocal exceptional series inner product. In ([29], Eqs. (A.6)-(A.7)) we showed for the $D^{\pm}_k$ series that this inner product is identical to Gel'fand's local inner product in the complex upper half-plane ([52], Vol. 5, Chapter VII), as will be shown below.

The quantum canonical transformation to pure centrifugal/centripetal phase variables is thus shown to be, from (5.1) and (6.4), the integral transform

$$\bar{f}(\sigma, r) = (\mathbf{D}_{k,\varepsilon}\overset{+}{f})(\sigma, r) = (D_{k,\varepsilon}(\sigma, r; \cdot)^*, \overset{+}{f})^{(+)}_{k,\varepsilon}$$

$$= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \Upsilon_{k,\varepsilon}(\xi, \xi') D_{k,\varepsilon}(\sigma, r; \xi) \overset{+}{f}(\xi') , \qquad (6.5a)$$

with integral kernel given in terms of (5.2)

$$D_{k,\varepsilon}(\sigma, r; \xi) = U_{k,\varepsilon}(\xi)^* C_{k,\varepsilon}(\sigma, r; \phi(\xi)) . \qquad (6.5b)$$

which may be further particularized to (5.3) or (5.4). We note that, when $C_{k,\varepsilon}$ depends on $e^{i\phi}$, (6.1) is used, and sgn ϕ = sgn ξ. The transformation is obviously such that

$$\mathbf{D}_{k,\varepsilon} \mathbf{L}_\alpha = \mathbf{J}_\alpha \mathbf{D}_{k,\varepsilon} , \quad \alpha = 0, 1, 2, +, - . \qquad (6.5c)$$

The inverse transform follows from (5.8) and is written

$$\overset{+}{f}(\xi) = (\mathbf{D}^{-1}\bar{f})(\xi) = (D_{k,\varepsilon}(\cdot\,,\cdot\,; \xi), \bar{f})_S . \qquad (6.6)$$

The unitarity conditions (5.9) are similarly verified with the appropiate reproducing kernel in $L^2_{(k,\varepsilon)}(R)$.

Although we will not rederive the kernel (6.5b) as a generating function as done for the D_k^+ series in ([29], Sec. 2.c), it is worthwhile to look at the information contained in such a construction. In the D_k^+ series the generalized eigenfunctions of $\mathbf{J}_+$, Eq. (3.5d), are basically Bessel functions of order $2k - 1$ in the variable ρr ([40], Eq. (3.1)) corresponding to eigenvalue $\rho \in R^+$. The generalized eigenfunctions of $\mathbf{L}_+$, Eq. (6.3d), are $\sim \exp[i\rho\xi]$ again for eigenvalues $\rho \in R^+$. Finally, the transform kernel (6.5b) is the generating function built as the product of these two eigenfunction sets (the second complex conjugate) integrated over ρ, *i.e.* the Fourier transform of the former in ρ. Now, Fourier transforms of functions with support on R^+ are entire analytic functions in the lower complex ξ-half-plane, and there its growth is bounded by a decreasing exponential ([47], Sec. 7.4.2), and hence so is the transform kernel in the D_k^+ case. The D_k^- series kernel will be analytic in the upper ξ-half-plane. The functions in the space

$L^2_{(k,\varepsilon)}(R)$ are $\mathbf{U}_{k,\varepsilon}$-transforms (6.4b) of functions in $L^2_{(k,\varepsilon)}(S_1)$, which we remarked are boundary values of analytic functions on the disc $z = \zeta e^{i\phi}$ $\zeta < 1$. The change of variables (6.1), setting $e^{i\phi} \to z$, maps the disc $|z| < 1$ onto the half-plane Im $\xi > 0$. The space $L^2_{(k,\varepsilon)}(R)$ for the D_k^+ SAIR series consists hence of entire analytic functions in the upper half-plane, with growth bounded by a decreasing exponential in Im ξ, as in Gel'fand's construction [52] quoted above. For the D_k^- series anlogous remarks apply under $z \leftrightarrow -z$, $\xi \leftrightarrow -\xi$.

In the C_c^ε SAIR series, the eigenfunctions of $\mathbf{J}_+$ are expressible as cylinder or Whittaker functions of complex argument ([40], Eq. (4.2)), with eigenvalues $\rho \in R$. The C-transform kernel to phase variables will be the complete Fourier transform of these functions $\sim \exp[i\rho\xi]$, since the $\mathbf{L}_+$ eigenfunctions are again the Fourier kernel functions for $\rho \in R$.

The main difficulty in both the direct generating-function method for deriving the canonical transform kernel to phase variables, as well as for the Mello-Moshinsky differential-equation method to the same purpose, is a problem of correct choice of phases. The former method has no way of assuring what the ρ-dependent phase factors for the $\mathbf{J}_+$ and for the $\mathbf{K}_+$ eigenfunctions may be so that they are consistent with the rest of the algebra, *i.e.* that $\mathbf{D}_{k,\varepsilon}\mathbf{K}_\alpha = \mathbf{J}_\alpha \mathbf{D}_{k,\varepsilon}$ for the $\alpha \neq +$ indices as well. This precaution was not taken in ([29], Sec. 2.B) since the phase liberty may only rotate the $sl(2, R)$ algebra (3.1) around the $\alpha = +$ parabolic axis, and such equivalences were deemed unimportant there. The Mello-Moshinsky differential equation (5.6) with $C \to D$ for $\alpha = +$ is the Schrödinger equation for the hamiltonian (3.5d). It gives the solution up to a multiplicative, complex constant, as indicated in (5.6)-(6.7). A second equation (say, for $\alpha = 2$) yields the phase and, lastly, unitarity (5.9b) must be brought into the picture to fix the absolute value of the constant. Unitarity (5.9a) under integration on ξ is not present, and does not hold (except for the nonexceptional C_c^ε) unless the appropiate nonlocal or Gel'fand measures are used.

The phase problems seem to be the least severe when we follow the generating-function method for the $\mathbf{J}_0$ and $\mathbf{K}_0$ eigenfunctions, as done in the last section, and then implement the similarity transform $\mathbf{U}_{k,\varepsilon}$. The explicit results for the oscillator case are, following their discussion in (5.5),

$$D_e(r;\xi) = 2^{-1/2}\pi^{-3/4}\xi^{-1/2}\exp(ir^2/2\xi) , \tag{6.7a}$$

$$D_o(r;\xi) = \frac{1}{2}\pi^{-3/4} r\xi^{-3/2}\exp(ir^2/2\xi) , \tag{6.7b}$$

where for $\xi < 0$ we must take $\arg \xi = -\pi$ since, as we mentioned above, the transform kernel $D_{k,\varepsilon}(\sigma, r; \xi)$ is an analytic function in the lower complex ξ-half-plane. The Mello-Moshinsky differential equation for $\alpha = +$ is the free-particle Schrödinger equation and has (6.7) for its solutions: (6.7a) is the Green's function for time ξ and (6.7b) its r-derivative, or its Green's function for odd solutions. However, being a parabolic differential equation, it does not specify the phase of ξ when $\xi < 0$. Moreover, while the Schrödinger Green's function (6.7a) is unitary under integration over configuration space (the r variable), and yields a Dirac δ in the time variable ξ, no unitarity property is ascribed to this function under time ξ-integration. This situation is analogous to heat diffusion, which is not unitary for time translations under the L^2-inner product, but may be made unitary if a nonlocal inner product, or local product over the complex plane, is proposed [53]. Here, the unitarity in the phase variable ξ is obtained under the nonlocal inner product (6.4) and the results of Sect. V, especially (5.5a).

VII. THE REPULSIVE OSCILLATOR $+\gamma/r^2$ POTENTIAL CASE

We now search for a realization of $sl(2, R)$ where the hamiltonian $2\mathbf{J}_1$ in (3.5b) is mapped to the simple translation operator $-id/d\eta$. The required change of variables is between $\phi \in S_1$ and a pair of real lines $\{\tau, \eta\} \in Z_2 \times R$ where $\tau \in \{-1, +1\}$ and $\eta \in R$:

$$e^{i\phi} = i\,\frac{1 - i\tau e^{\eta}}{1 + i\tau e^{\eta}}\;, \qquad \tau e^{\eta} = -i\,\frac{1 + ie^{i\phi}}{1 - ie^{i\phi}}\;. \tag{7.1}$$

Here $\tau = +1$ for the half-circle $|\phi| < \pi/2$ and $\tau = -1$ for $|\phi| > \pi/2$; $\phi \to (\pi/2)^{\mp}$ is mapped on $\eta \to -\infty$, $\tau = \pm 1$, and $\phi \to (-\pi/2)^{\pm}$ on $\eta \to +\infty$, $\tau = \pm 1$. We define a similarity transform $\mathbf{V}_{k,\varepsilon}$ given by

$$\mathbf{M}_\alpha = \mathbf{V}_{k,\varepsilon}\mathbf{K}_\alpha \mathbf{V}_{k,\varepsilon}^{-1} \tag{7.2a}$$

$$\begin{aligned}(\mathbf{V}_{k,\varepsilon} f)(\tau, \eta) &= V(\tau, \eta) f(\tau, \eta)\\ &= \operatorname{sech}^k \eta \left(\frac{\tau e^{\eta} + i}{1 + i\tau e^{\eta}}\right)^{\varepsilon} f(\tau, \eta) = |\cos\phi|^k e^{i\varepsilon k} f(\tau, \eta(\phi))\,,\end{aligned} \tag{7.2b}$$

whereupon we obtain the $sl(2, R)$ algebra in the realization

$$\mathbf{M}_0 = i\tau\left(\cosh \eta \frac{d}{d\eta} + k \sinh \eta\right)\,, \tag{7.3a}$$

$$\mathbf{M}_1 = -i\frac{d}{d\eta} . \tag{7.3b}$$

$$\mathbf{M}_2 = i\tau(\sinh\eta\frac{d}{d\eta} + k\cosh\eta) . \tag{7.3c}$$

as well as $\mathbf{M}_{\pm}$ which can be obtained from them, but are not of particular interest here.

We define the Hilbert spaces $L^2_{(k,\varepsilon)}(Z_2 \times R)$ through the inner product $(\cdot,\cdot)^{(1)}_{k,\varepsilon}$ under which (7.3) are self-adjoint. This is obtained from the results in Sect. IV, and appears as

$$(\overset{1}{f},\overset{1}{g})^{(1)}_{k,\mu} = (\mathbf{V}^{-1}_{k,\varepsilon}\overset{1}{f}, \mathbf{V}^{-1}_{k,\varepsilon}\overset{1}{g})^{(0)}_{k,\varepsilon} = (\overset{\circ}{f},\overset{\circ}{g})^{(0)}_{k,\varepsilon}$$

$$= (\overset{1}{f}, \mathbf{V}^{-1*}_{k,\varepsilon}\Omega_{k,\varepsilon}\mathbf{V}^{-1}_{k,\varepsilon}\overset{1}{g})_{S_1} = (\overset{1}{f}, \Phi_{k,\varepsilon}\overset{1}{g})_{Z_2\times R}$$

$$= \sum_{\tau=\pm1}\int_{-\infty}^{\infty} d\eta \sum_{\tau=\pm1}\int_{-\infty}^{\infty} d\eta'\Phi_{k,\varepsilon}(\tau,\eta;\tau',\eta')\overset{1}{f}(\tau,\eta)^*\overset{1}{g}(\tau',\eta') . \tag{7.4a}$$

Here again $(\cdot,\cdot)^{(0)}_{k,\varepsilon}$ is the inner product of Sect. IV, $(\cdot,\cdot)_{S_1}$ the $L^2(S_1)$-inner product, and $(\cdot,\cdot)_{Z_2\times R}$ is the ordinary $L^2(R)$-inner product of two-component functions. As in last section,

$$\overset{1}{f}(\tau,\eta) = (\mathbf{V}_{k,\varepsilon}\overset{\circ}{f})(\tau,\eta) = V_{k,\varepsilon}(\tau,\eta)\overset{\circ}{f}(\phi(\tau,\eta)) \tag{7.4b}$$

are the transformed function on $Z_2 \times R$, and the metric operator $\Phi_{k,\varepsilon} = \mathbf{V}^{-1*}_{k,\varepsilon}\Omega_{k,\varepsilon}\mathbf{V}^{-1}_{k,\varepsilon}$ has for its kernel in $L^2_{(k,\varepsilon)}(Z_2 \times R)$

$$\Phi_{k,\varepsilon}(\tau,\eta;\tau',\eta') = \frac{d\phi(\tau,\eta)}{d\eta}\frac{d\phi'(\tau',\eta')}{d\eta'} V_{k,\varepsilon}(\tau,\eta)^{-1*}$$

$$\times\ \Omega_{k,\varepsilon}(\phi(\tau,\eta) - \phi'(\tau',\eta'))\quad V_{k,\varepsilon}(\tau',\eta)^{-1}$$

$$= \tau\tau'\ \mathrm{sech}^{1-k^*}\eta' \left(\frac{1-i\tau e^{\eta}}{\tau e^{\eta}-i}\ \frac{1+i\tau' e^{\eta'}}{\tau' e^{\eta'}+i}\right)^{\varepsilon}$$

$$\times\ \Omega_{k,\varepsilon}\left(\exp\{i[\phi(\tau,\eta)-\phi'(\tau',\eta')]\} = \frac{1-i\tau e^{\eta}}{1+i\tau e^{\eta}}\ \frac{1+i\tau' e^{\eta'}}{1-i\tau' e^{\eta'}}\right). \tag{7.4c}$$

The cases (4.13) and (4.14) for the discrete and continuous series give corresponding Hilbert spaces whose inner product is in general nonlocal over $Z_2 \times R$; in particular for the nonexceptional continuous series $\Phi_{k,\varepsilon}(\tau,\eta;\tau',\eta') = \delta_{\tau,\tau'}\, \delta(\eta-\eta')$ so in this case $L^2_{(k,\varepsilon)}(Z_2 \times R) = L^2(R) + L^2(R)$.

The differentiation operator $\mathbf{M}_1$ in (7.3b) has R for its generalized spectrum in D_k^+, and twice R in C_c^ε. In the latter case it is easy to see that the generalized eigenfunctions of $\mathbf{M}_1$ are $\sim \exp[i\lambda\eta]$ with eigenvalue λ, once for $\tau = +1$, and once for $\tau = -1$, so each eigenvalue is doubly degenerate.

In D_k^+ the spectrum of $\mathbf{M}_1$ is nondegenerate, however, and hence a relation should exist between the $\tau = +1$ and $\tau = -1$ values of the eigenfunction. In order to find this recall that for D_k^+, $L^2_{(k,\varepsilon)}(S_1)$ is the space of boundary values of analytic functions in the unit disc $z = \zeta e^{i\phi}$, $\zeta < 1$. If we replace in (7.1) z for $e^{i\phi}$ and e^η for $\tau\, e^\eta$, η will be an analytic function of z where the unit disc $|z| < 1$ is mapped onto the strip $0 > \operatorname{Im}\eta > \pi$, the right-half circle $|\pi| < \pi/2$ onto the real line corresponding to $\tau = +1$, and the left half-circle $|\phi| > \pi/2$ onto the line $R - i\pi$, corresponding to $\tau = -1$. The spaces $L^2_{(k,\varepsilon)}(Z_2 \times R)$ for D_k^+ must consist thus of analytic functions in that strip, and hence $\overset{1}{f}(\tau=-1,\eta) = \overset{1}{f}(\tau=+1,\eta-i\pi)$. In particular, the generalized $\mathbf{M}_1$ eigenbasis functions are $e^{i\lambda\eta}$ for $\tau = +1$ and $e^{-\lambda\pi} e^{i\lambda\eta}$ for $\tau = -1$, with no multiplicity.

Finally, the quantum canonical transformation to the phase variables of the repulsive oscillator $+\gamma/r^2$ potential system is found from (5.1) and (7.4) to be the integral transform

$$\bar{f}(\sigma, r) = (\mathbf{E}_{k,\varepsilon}\overset{1}{f})(\sigma, r) = (E_{k,\varepsilon}(\sigma, r; \cdot\,,\cdot)^*, \overset{1}{f})^{(1)}_{k,\varepsilon}$$

$$= \sum_{\tau=\pm 1} \int_{-\infty}^{\infty} d\eta' \Phi_{k,\varepsilon}(\tau,\eta;\tau',\eta')\, E(\sigma, r; \tau,\eta) \overset{1}{f}(\tau',\eta')\,, \qquad (7.5a)$$

with kernel

$$E_{k,\varepsilon}(\sigma, r; \tau, \eta) = V_{k,\varepsilon}(\tau,\eta)^* C_{k,\varepsilon}(\sigma, r; \phi(\tau,\eta))\,, \qquad (7.5b)$$

obtained from (5.2). It is such that

$$\mathbf{E}_{k,\varepsilon}\mathbf{M}_\alpha = \mathbf{J}_\alpha \mathbf{E}_{k,\varepsilon}\,, \qquad (7.5c)$$

and its inverse follows as

$$\overset{1}{f}(\tau,\eta) = (\mathbf{E}_{k,\varepsilon}^{-1}\bar{f})(\tau,\eta) = (E_{k,\varepsilon}(\cdot\,,\cdot\,;\tau,\eta),\bar{f})_{\mathcal{S}}\,. \tag{7.6}$$

The unitarity conditions (5.9) are verified, and we can make the same remarks as we did in the last section following (6.6). The generating-function approach is here more difficult since the eigenfunctions of $\mathbf{J}_1$ are given in terms of Whittaker functions of the first kind along the imaginary axis ([40] Eqs. (3.4) for D_k^+ and (4.4) for C_c^ε). The analiticity properties of the transform kernel in the strip $0 < \operatorname{Im}\eta < \pi$ are hidden here. However, a slightly different realization of $sl(2,R)$, related to (7.3) by a rotation of $\pi/2$ around the $\mathbf{M}_0$ axis, puts $\mathbf{M}_2$ into a $-id/d\eta$ form which is more amenable to the construction of a generating function. This is so since the $\mathbf{J}_2$ generalized eigenbasis is the Mellin transform kernel $\sim q_\pm^{-1/2+2i\lambda}$ ([40], Eq. (3.3) for D_k^+ and (4.3) for C_c^ε). The Fourier transform (*i.e.* integration with $e^{-i\lambda\eta}$ over λ) of the latter should yield (7.5b). Again, however, the phase problem is serious and one should adjust a λ-dependent phase to the integral transform (7.5a) so as to map the two other generators of the algebra appropiately.

For the ordinary repulsive oscillator ($\gamma = 0$) we may follow (5.5), (7.1) and (7.5b) to write

$$E_e(r;\tau,\eta) = \pi^{-3/4}\left(\frac{e^{\eta/2}}{\tau e^\eta + 1}\right)^{1/2} \exp\left(\frac{ir^2}{2}\,\frac{\tau e^\eta - 1}{\tau e^\eta + 1}\right), \tag{7.7a}$$

$$E_o(r;\tau,\eta) = 2^{1/2}\pi^{-3/4}\left(\frac{e^{\eta/2}}{\tau e^\eta + 1}\right)^{3/2} r\exp\left(\frac{ir^2}{2}\,\frac{\tau e^\eta - 1}{\tau e^\eta + 1}\right). \tag{7.7b}$$

We note that in these expressions the terms in brackets reduce to hyperbolic functions sech or $-$csch, tanh or coth in $\eta/2$, according to whether $\tau = +1$ or -1. This substitution would make the phase of the first parenthesis indeterminate for $\tau = -1$, $\eta > 0$. Keeping the term τe^η explicit and following the analiticity arguments above tells us that we should take functions of η analytic in the stip $0 < \operatorname{Im}\eta < \pi$ and hence the phase of $\tau = -1$ is taken as $e^{+i\pi}$.

The results of this section can be derived from those of last section through the change of variables

$$\tau e^\eta = \frac{\xi + 1}{\xi - 1}\,, \quad \xi = \frac{\tau e^\eta + 1}{\tau e^\eta - 1}\,, \tag{7.8}$$

and a similarity transformation replacing $\mathbf{V}_{k,\varepsilon}$ in (7.2) by $\mathbf{W}_{k,\varepsilon} = \mathbf{V}_{k,\varepsilon}\mathbf{U}_{k,\varepsilon}^{-1}$ where

$$W_{k,\varepsilon}(\tau,\eta) = |\tfrac{1}{2}[e^{\eta/2} - \tau e^{-\eta/2}]|^{-2k} = |\xi^2 - 1|^k . \tag{7.9}$$

In particular, the Mello-Moshinsky differential equation method meets with the same normalization and phase difficulties. Indeed, it was with the aim of surmounting these mostly technical problems that we chose the harmonic oscillator phase-variable case to develop the canonical transform intertwining kernel as a generating function, in Sect. V. In retrospect, we see that the phase problems for the D_k^+ series are not serious —witness Eq. (5.3)— and only for the generally unphysical continuous series C_c^ε is extra care required.

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APPENDIX

GENERATING SERIES FOR WHITTAKER FUNCTIONS

Generating series for Whittaker functions of the type

$$\sum_{m\in Z} A_m W_{m+\varepsilon,\, k-1/2}(r^2) z^m \tag{A.1}$$

seem to be absent form the better-known tables of special functions [54] except for the cases $\varepsilon = \pm k, \pm(1-k)$, where the Whittaker function reduces to an expression in terms of Laguerre polynomials ([41], Eqs. 13.1.33 and 13.6.27)

$$W_{k+m,\, k-1/2}(r^2) = (-1)^m m!\; r^{2k} e^{-r^2/2} L_m^{(2k-1)}(r^2) . \tag{A.2}$$

The case $\varepsilon = 1-k$ in $1/2 < k < 1$ follows from (A.2) setting $k \leftrightarrow 1-k$ and, since $(2k-1) \leftrightarrow -(2k-1,)$recalling that $W_{\mu,\kappa}(x) = W_{\mu,-\kappa}(x)$. ($L_m^{(\alpha)}(x)$ is defined for $\alpha > -1$). The generating function for the Laguerre polynomials is well known ([40], Eq. 22.9.15):

$$\sum_{m=0}^{\infty} \frac{W_{m+k,\, k-1/2}(r^2)\, z^m}{\Gamma(m+1)} = r^{2k}(1+z)^{-2k} \exp\left(-\frac{r^2}{2}\,\frac{1-z}{1+z}\right)$$

$$, \quad |z| < 1 . \tag{A.3}$$

This relation leads to the canonical transform kernel to phase variables for the lower-bound discrete series ([29], Eq. (2.32)).

We shall find here a generating series for the more general case (A.1) where ε is free, the summation extends over all $m \in Z$ and z is evaluated on the unit circle for $z = e^{i\phi}$. In fact for k, r^2 fixed and $\mu \to \infty$ the asymptotic behaviour of the Whittaker function is ([41], Eqs. 13.1.33-13.5.16):

$$r^{-1/2} W_{\mu,\, k-1/2}(r^2) \simeq \pi^{-1/2} \Gamma(\mu + 1/4) \cos(\pi\mu - 2r\mu^{1/2}) \; . \tag{A.4}$$

We may expect summability only if $A_m \lesssim [\Gamma(m + \varepsilon + 1/4)]^{-1}$. This much is assured by the form of (5.2). Due to the presence of an infinite set of negative powers in the series, however, the series may converge only for points on the unit circle.

An integral representation of the Whittaker functions may be found in ([48], Eq. 3.718.6, 64 Sect. 6.11.2, Eq. (13)),

$$\frac{W_{\mu,\pm(\nu+1)/2}(r^2)}{\Gamma(1 + \mu + \nu/2)} = \frac{1}{\pi}\left(\frac{r}{2}\right)^{-\nu} \int_0^\pi d\theta \cos^\nu \frac{1}{2}\theta \cos(\frac{1}{2} r^2 \tan \frac{1}{2}\theta - \mu\theta) \; . \tag{A.5}$$

This integral has the restrictions $r^2 > 0$ and $\mu + \nu / 2 \neq -1, -2, \ldots$. It is also restricted to Re $\nu > -1$ due to the behaviour δ^ν of the first cosine factor at $\theta = (\pi - \delta), \delta \to 0^+$; the second cosine factor oscillates with period $\sim \delta^{-1}$ there, and summation over $\mu = m + \varepsilon$ will produce sign cancellations in the neighborhood of $\theta = \pi$. We may thus surmise that (A.5) can be manipulated formally and set $\nu = -2k$. We next note that since the integrand in (A.5) is even in θ, we may double the integration interval to $(-\pi, \pi)$; a similar (null) integral with the second cosine factor replaced by a sine, times $-i$, summed with (A.5) yields

$$\frac{W_{\mu,\, k-1/2}(r^2)}{\Gamma(1 + k + \mu)} = \frac{1}{2\pi}\left(\frac{r}{2}\right)^{2k} \int_{-\pi}^{\pi} d\theta \cos^{-2k}\frac{1}{2}\theta \exp(i\,\frac{1}{2} r^2 \tan \frac{1}{2}\theta) e^{-i\mu\theta} \; . \tag{A.6}$$

Writing now $\mu = m + \varepsilon$, the left hand side is displayed as the mth Fourier coefficient of the factor of $e^{-im\theta}$ in the integrand, *i.e.* we have the generating function

$$\sum_{m\in Z} \frac{W_{m+\varepsilon,\,k-1/2}(r^2)}{\Gamma(1-k+m+\varepsilon)} e^{im\phi}$$

$$=\left(\frac{1}{2}r\right)^{2k} e^{-i\varepsilon\phi}\cos^{-2k}\frac{1}{2}\phi\,\exp(i\,\frac{1}{2}r^2\tan\frac{1}{2}\phi)$$

$$= r^{2k}e^{i\phi(k-\varepsilon)}[1+e^{i\phi}]^{-2k}\exp\left(-\frac{r^2}{2}\,\frac{1-e^{i\phi}}{1+e^{i\phi}}\right), \qquad (A.7)$$

which is used in (5.2) to compute the canonical transform kernel to phase variables. When $\varepsilon = k$, Eq. (A.7) reduces to a sum of nonnegative-m terms, and reproduces neatly the generating function of the Laguerre polynomials, Eq. (A.7) for $z = e^{i\phi}$.

As (A.3), (A.7) exhibits an essential singularity in $z = e^{i\phi}$ at $\phi = \pm\pi$, $(z = -1)$. For $\varepsilon \neq k$, moreover, a multivaluated factor $e^{i\phi(k-\varepsilon)}$ appears; our derivation, however, makes it clear that it is the segment $(-\pi, \pi)$ containing $\phi = 0$ which must be considered. This has to be borne in mind when we inquire into the generating function needed in the text of section V, which requires a sum of the type (A.7) with a extra factor $(-1)^m$, and which is a simple translate $\phi \to \phi \pm \pi$ of the left-hand. side.

In the right-hand side this "sheet-preserving" translation of multivalued functions on the two open half-circles $\phi \in (-\pi, 0) \cup (0, \pi)$ is $F(\phi) \to F(\phi - \pi\,\mathrm{sgn}\,\phi)$ so that the value of the former at $\pi - \delta$ and at $-\pi + \delta$ $(\delta \to 0^+)$ becomes the value of the latter at $-\delta$ and $+\delta$, respectively. Hence $\tan\phi/2 \to -\cot\phi/2$, but $\cos\phi/2 \to |\sin\phi/2|$ (this factor is positive over the circle), and $e^{-i\varepsilon\phi} \to \exp[-i\varepsilon(\phi - \pi\,\mathrm{sgn}\,\phi)]$.

When the functions are of bounded variation we may asign the value at the midpoint of the discontinuity –if any– as is the case with Fourier series due to the Dirichlet theorem ([47], Sec. 4.2). With the extra factor $(-1)^m$ (A.7) thus becomes

$$\sum_{m\in Z} \frac{(-1)^m W_{m+\varepsilon,\,k-1/2}(r^2)}{\Gamma(1-k+m+\varepsilon)} e^{im\phi}$$

$$= \left(\frac{1}{2}r\right)^{2k} \exp(-i\varepsilon[\phi - \pi\mathrm{sgn}\phi])\,\left|\sin\frac{1}{2}\phi\right|^{-2k}\exp(-i\,\frac{1}{2}r^2\cot\frac{1}{2}\phi)$$

$$= r^{2k}\exp(i[k-\varepsilon][\phi - \pi\mathrm{sgn}\phi])(1-e^{i\phi})^{-2k}\exp\left(-\frac{r^2}{2}\,\frac{1+e^{i\phi}}{1-e^{i\phi}}\right). \qquad (A.8)$$

For $\varepsilon = k$ this agrees again with (A.3) after the transformation $z \to -z$.

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RESUMEN*

Consideramos el problema de encontrar la transformación unitaria que entrelaza un operador hamiltoniano cuántico autoadjunto $1/2\ \mathbf{P}^2 + V(Q)$ en espacios de funciones del tipo L^2 con una función de peso local, a su forma de variable fase $-\ i\ d/d\zeta$, autoadjunto en un segundo espacio de Hilbert. Esto puede hacerse en el contexto del álgebra de Heisenberg-Weyl sólo para el caso de caída libre. Dentro del álgebra $sl(2,R)$ pueden considerarse potenciales $V(Q) = \beta Q^2 + \gamma Q^{-2}$ para toda γ real correspondiente a osciladores armónicos o repulsivos, o una partícula libre con barrera centrífuga o pozo centrípeto. Este tratamiento agota la clase de hamiltonianos para los cuáles el problema de transformaciones canónicas cuánticas a variables fase puede ser resuelto enteramente dentro del marco de la teoría de grupos. Los resultados concuerdan con aquellos obtenidos de las ecuaciones de Mello-Moshinsky, excepto que los espectros están conjuntados automáticamente a través de la (en general no local) medida en el segundo espacio de Hilbert. El presente tratamiento no requiere, por lo tanto, la introducción del grupo de ambigüedad de Moshinsky-Seligman. El álgebra $sl(2,R)$ reemplaza el espacio de fase cuántico asociado al álgebra de Heisenberg-Weyl. Además, examinamos en detalle un conjunto de potenciales singulares —aquéllos que incluyen un fuerte pozo centrípeto— donde el hamiltoniano tiene una familia uniparamétrica de extensiones autodajuntas, y donde los espectros discretos no son ni únicos ni acotados por debajo. Finalmente encontramos un conjunto de nuevas relaciones generadores para funciones de Whittaker.

* Traducido del inglés por la Redacción.