The unitary irreducible representations of SL(2, R) in all subgroup reductions [a)]

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We use the canonical transform realization of SL(2, R) in order to find all matrix elements and integral kernels for the unitary irreducible representations of this group. Explicit results are given for all mixed bases and subgroup reductions. These provide the full multiparameter set of integral transforms and series expansions associated to SL(2, R).

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1. INTRODUCTION

The complete classification of the Unitary Irreducible representations (UIRs) of the three-dimensional Lorentz group SO(2, 1) and of its twofold covering group SL(2, R) were given by Bargmann in his classic 1947 article,[1] where one can find the UIR matrix elements—rows and columns classified by the UIRs of the compact subgroup SO(2)—in explicit form. This group, its covering groups $\mathrm{SO}(2,1) \overset{1:2}{\simeq} \mathrm{SU}(1,1) \simeq \mathrm{Sp}(2,R) \approx \mathrm{SL}(2,R) \overset{1:\infty}{\simeq} \overline{\mathrm{SL}(2,R)}$ and its representations were further studied by Barut and Fronsdal,[2] Pukański,[3] Sally, Jr.,[4] and in a book by Lang.[5]

The study of group representations in different bases is of interest both from the mathematical and the physical point of view. The intimate connections between the representations of Lie groups and the special functions of mathematical physics have long been recognized and treated in textbooks.[6] In physics, subgroup reductions corresponding to different bases of the Lorentz and other groups lead to various ways to correlate or interpret data, as in the description of the high-energy scattering dynamics,[7] which requires the reduction $\mathrm{SO}(2,1)\supset\mathrm{SO}(1,1)$ among others. This interest coincided with the investigations of Mukunda,[8–11] Barut,[2,12] Lindblad and Nagel,[13] and others, who analyzed this chain in some detail and computed the generalized representation matrices (or integral kernels) of one-parameter subgroups and found the coupling coefficients.

In the study of the role of canonical transformations in quantum mechanics, the work of Moshinsky and Quesne[14,15] started from linear transformations between coordinate and momentum observables and lead to the oscillator (metaplectic) representation of Sp(2, R). In contrast to the realizations given by Bargmann[1] and by Gel'fand *et al.*,[16] in which the group acts as a Lie transformation group on functions of a coset manifold, the group actions in the constructions of Moshinsky,[14,15,17] Seligman, Wolf,[18–23] Burdet, Perrin and Perroud,[24] and present in the work of others,[25–27] is an integral transform realization of SL(2, R) on $\mathscr{L}^2(R)$ Hilbert spaces. This group of integral transforms has been called *canonical* transforms.[18,28] It is unique in that the associated Lie algebra is an algebra of second-order differential operators on a dense common domain in these Hilbert spaces. The action is thus distinct from—although unitarily equivalent[20,21] to—the SL(2, R) action as a Lie transformation group on coset spaces, of the Lie–Bargmann multiplier representations[29] on the unit circle or disk.

The canonical transform realization has provided a degree of uniformity in the treatment of the discrete series[19] of UIRs on the one hand and the continuous series[21] of UIRs on the other. In this article it has enabled us to evaluate, in a straightforward and unified way, the UIR matrix elements and integral kernels of finite SL(2, R) elements. In contrast with some of the previous investigations, this approach deals with the general SL(2, R) group element, rather than with specific one-parameter subgroups. Although Bargmann's results on UIRs of SL(2, R) in the compact subgroup basis[30] are well known, it is also true that other continuous noncompact and mixed-basis reductions have so far not received uniform consideration[2,9,10,12,31–33] and are scattered in the literature. The discrete series of UIRs in all subgroup reductions was undertaken by Boyer and Wolf[34] using canonical transforms. We repeat their results here since the journal is not generally available and the article contains some errata. The mixed-basis matrix elements of the continuous series were treated by Kalnins,[31] who gave expressions for one-parameter subgroups in terms of Whittaker and Laguerre functions of the second kind.[35] All our expressions are given in terms of confluent and Gauss hypergeometric functions, and have uniformity of notation, normalization, and phase conventions. The purpose of this paper is to give a comprehensive derivation and listing of all subgroup reductions.

The plan of the article is as follows. In Sec. 2 we display the needed formulas from the theory of canonical transforms for the general method of construction and, since we want to describe all UIR matrix elements and integral kernels, we organize the notation properly in due accordance with Bargmann's conventions. In Sec. 3 and 4 we give the results for the discrete and continuous (nonexceptional and exceptional) representation series. The first subsection of each lists the subgroup-adapted basis functions, the second treats the mixed-basis expressions, while the third subsection treats the subgroup reductions, i.e., the cases when the row and column variables refer to the same subgroup. These are ex-

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pressed as Gauss or confluent hypergeometric functions and, alternatively, as cylinder and Whittaker functions[36,37] of the three independent SL(2, R) parameters. Certain cases of interest are pointed out in a further subsection. Comparison with alternative derivations available in the literature is pointed out whenever we are aware of such results.

The representation matrix elements for the compact subgroup chain were obtained by Bargmann as solutions to differential equations[38] with boundary conditions imposed by the group identity. We come to the evaluation of an integral as the last step to the same end. We make use of a method by Majumdar and Basu[32] on hypergeometric series Mellin expansions to solve three of the six chains in each series. In the special case of the continuous series in the compact subgroup reduction, such an integral (a Gaussian of imaginary width times two Whittaker functions, one with a rescaled argument) is not available in the literature. Through Bargmann's result this is evaluated.

In Sec. 5 we point out that the six different mixed-basis and subgroup-reduced representation matrix elements constitute six families of SL(2, R) integral and discrete transforms, as well as series expansions, of which the set of canonical transforms is but one. The Appendix summarizes some information about the groups SU(1, 1), SL(2, R), and their UIRs as classified by Bargmann. Throughout this article Z and R stand for the set of integers and real numbers. Boldfaced symbols indicate vectors or matrices. For brevity, we shall speak of UIR *matrix elements* encompassing both the ordinary and generalized (i.e., integral transform kernel) cases.

As a general observation, we should remark that the canonical transform realization of SL(2, R) can be regarded as a complementary alternative to Bargmann's treatment of the same group. The latter is simpler in certain respects, particularly when dealing with the compact subgroup chain, while the former seems to be most appropriate for noncompact subgroup chains.

2. CANONICAL TRANSFORMS

A. The construction of SL(2, R) representations

The determination of representation matrices (or integral kernels) for group elements $g \in G$ may proceed as follows: Provided (i) one has a Hilbert space $\mathscr{H}$ of functions $f(r)$, r in some carrier space X, endowed with a sesquilinear positive definite inner product $(\cdot,\cdot)$, where the action of G is well defined and onto,

$$f(r) \xrightarrow{g} f_g(r) = [\mathbb{C}_g f](r), \quad f, f_g \in \mathscr{H} \tag{2.1}$$

(ii) one has a complete orthonormal, or generalized Dirac-orthonormal basis for $\mathscr{H}$, $\{\psi_\lambda(r)\}_{\lambda\in\Lambda}$ (Λ being the range of the label specifying the basis vectors uniquely), one can build a representation $\mathbf{D}$: $G \rightarrow \mathrm{Hom}\Lambda$ as

$$\mathbf{D}(g) = \|D_{\lambda,\lambda'}(g)\|, \tag{2.2a}$$

$$D_{\lambda,\lambda'}(g) = (\psi_\lambda, \mathbb{C}_g \psi_{\lambda'}). \tag{2.2b}$$

The completeness of the (possibly generalized) basis function set will then guarantee the representation property

$$\underset{\lambda'\in\Lambda}{\mathrm{S}} D_{\lambda,\lambda'}(g_1) D_{\lambda',\lambda''}(g_2) = D_{\lambda,\lambda''}(g_1 g_2), \tag{2.2c}$$

where the symbol $\mathrm{S}_{\lambda'\in\Lambda}$ stands for summation in the case of proper, and integration in the case of generalized, bases. The unitarity and irreducibility properties of $\mathbf{D}$ follow from similar requirements for the action (2.1) on $\mathscr{H}$.

The reasons for which this straightforward program often fails to provide a definite result have to do more with knowing the "best" choice of basis functions $\{\psi_\lambda(r)\}_{\lambda\in\Lambda}$ and the problem of explicit computation of the integral in (2.2b), than with matters of principle. The bases are usually chosen as the eigenvectors of one or more operators in the Lie algebra—so that subgroup reductions result—while the space $\mathscr{H}$ is an $\mathscr{L}^2(X)$ space on a coset manifold $X = G/H$ (or $H\setminus G$) with some convenient subgroup $H \subset G$. A closely related approach to part (ii) of evaluation of (2.2b) calls for (ii′) finding these functions for various one-parameter subgroups of G as solutions of differential equations obtained from the subgroup generators, subject to the boundary conditions $\mathbf{D}(e) = \mathbf{1}$ at the group identity $e \in G$.

The group G which we consider here is SL(2, R):

$$\left\{ \mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in R, \quad \det \mathbf{g} = 1 \right\}. \tag{2.3}$$

Starting with Bargmann[1] a number of authors have implemented the program (i)–(ii) or (i)–(ii′), using for the supporting space X the coset space provided by the Iwasawa decomposition $NA\setminus NAK = S_1$ (i.e., the circle) and Bargmann's multiplier action.[29] This is unitary in $\mathscr{L}^2(S_1)$ for the continuous nonexceptional representation series[29]; for the continuous exceptional and discrete series it is $\mathscr{L}^2_{\Omega^C}(S_1)$ and $\mathscr{L}^2_{\Omega^D}(S_1)$ with nonlocal measures[39,40] Ω^C and Ω^D. The latter is equivalent[20] to a space of analytic functions on the unit disk[29] or on the complex half-plane.[16] These realizations are very appropriate for finding the SL(2, R) representation matrices reduced with respect to the compact SO(2) subgroup, since, the ensuing analysis makes use of Fourier series on $\mathscr{L}^2(S_1)$ for UIRs belonging to the continuous class, or Hardy spaces for those belonging to the discrete series.[39] When one makes use of the same action and spaces for the reduction under a noncompact subgroup, calculations become awkward.

The Hilbert spaces and SL(2, R) action we use in this article have been developed in Refs. 9, 15, 19, 21, and 22 for Sp(2, R)$\simeq$SL(2, R), as well as the oscillator representation[14,18] of Sp(2N, R) on an N-dimensional carrier space R^N. As we shall see in implementing part (ii) of the program outlined above, these techniques are best suited for noncompact subgroup reduction.

B. The discrete series $D_k^\pm$

The oscillator representation of the subgroup SO(2)×SL(2, R) of Sp(4, R), restricted to a given one-dimensional UIR M of SO(2), $M \in Z$, generates the conjugate SL(2, R) representation[15,19,22,27] belonging to the discrete series D_k^+ with $k = (1 + |M|)/2$. When the two-dimensional carrier space R^2 is parametrized in polar coordinates, this representation is realized as an integral transform group on the radial variable $r\in R^+$ and defines the *k-radial canonical transform* on the Hilbert space $\mathscr{L}^2(R^+)$. The inner product is thus the standard one,

$$(f,h)=\int_0^\infty dr\, f(r)^* h(r), \tag{2.4}$$

and the action of the group element **g** is given by

$$[\mathbb{C}_{\mathbf{g}}^k f](r)=\int_0^\infty dr' C_{\mathbf{g}}^k(r,r')f(r'),\quad \mathbf{g}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \tag{2.5a}$$

where the integral kernel $C_{\mathbf{g}}^k(r,r')$ is given by an imaginary Gaussian times a Bessel function:

$$C_{\mathbf{g}}^k(r,r')$$
$$=e^{-i\pi k}b^{-1}(rr')^{1/2}\exp(i[dr^2+ar'^2]/2b)J_{2k-1}(rr'/b), \tag{2.5b}$$

$$2k-1=0,1,2,\ldots,\quad \text{i.e., } k=\tfrac{1}{2},1,\tfrac{3}{2},2,\ldots. \tag{2.5c}$$

When **g** is a lower-triangular matrix ($b=0$) one finds from the asymptotic properties of the Bessel function[41] that Eq. (2.5a) becomes the multiplier action

$$\left[\mathbb{C}^k\begin{pmatrix} a & 0\\ c & a^{-1}\end{pmatrix}f\right](r)=(\operatorname{sgn} a)^{2k}|a|^{-1/2}\exp(icr^2/2a)f(r/|a|). \tag{2.5d}$$

We shall write $\mathbb{C}^k(\mathbf{g})$ for $\mathbb{C}_{\mathbf{g}}^k$ whenever **g** is displayed as a matrix. The k-canonical transform (2.5) is unitary under the inner product (2.4) and a Parseval relation (f,h) $=(\mathbb{C}_{\mathbf{g}}^k f,\mathbb{C}_{\mathbf{g}}^k h)$ holds.

The Lie generators of $\mathbb{C}_{\mathbf{g}}^k$ are *second*-order differential operators[42] given by

$$J_1^\gamma=\frac{1}{4}\left(-\frac{d^2}{dr^2}+\frac{\gamma}{r^2}-r^2\right), \tag{2.6a}$$

$$J_2^\gamma=-\frac{i}{2}\left(r\frac{d}{dr}+\tfrac{1}{2}\right), \tag{2.6b}$$

$$J_0^\gamma=\frac{1}{4}\left(-\frac{d^2}{dr^2}+\frac{\gamma}{r^2}+r^2\right), \tag{2.6c}$$

on a space dense in $\mathscr{L}^2(R^+)$, and γ is related to k through

$$\gamma=(2k-1)^2-\tfrac{1}{4}, \tag{2.7}$$

so that $\gamma=-\tfrac{1}{4},\tfrac{3}{4},\tfrac{15}{4},\ldots$. These generators close into a Lie algebra sl(2, R) under commutation. We shall also come to use

$$J_+^\gamma=J_0^\gamma+J_1^\gamma=\frac{1}{2}\left(-\frac{d^2}{dr^2}+\frac{\gamma}{r^2}\right), \tag{2.8a}$$

$$J_-^\gamma=J_0^\gamma-J_1^\gamma=\tfrac{1}{2}r^2. \tag{2.8b}$$

The Casimir invariant of sl(2, R) is a multiple of the identity:

$$Q=(J_1^\gamma)^2+(J_2^\gamma)^2-(J_0^\gamma)^2=q\mathbf{1}, \tag{2.9a}$$

$$q=-\tfrac{1}{4}\gamma+\tfrac{3}{16}=k(1-k), \tag{2.9b}$$

i.e., $q=\tfrac{1}{4},0,-\tfrac{3}{4},-2,\ldots$.

The association of (2.6)–(2.8) with the one-parameter subgroups of SL(2, R) is as follows

$$\exp(i\alpha J_1)\mapsto\mathbf{M}_1(\alpha)$$
$$=\begin{pmatrix}\cosh\alpha/2 & -\sinh\alpha/2\\ -\sinh\alpha/2 & \cosh\alpha/2\end{pmatrix}\in \mathrm{SO}(1,1)_1, \tag{2.10a}$$

$$\exp(i\beta J_2)\mapsto\mathbf{M}_2(\beta)=\begin{pmatrix}\exp(-\beta/2) & 0\\ 0 & \exp(\beta/2)\end{pmatrix}\in\mathrm{SO}(1,1)_2, \tag{2.10b}$$

$$\exp(i\gamma J_0)\mapsto\mathbf{M}_0(\gamma)=\begin{pmatrix}\cos(\gamma/2) & -\sin(\gamma/2)\\ \sin(\gamma/2) & \cos(\gamma/2)\end{pmatrix}\in\mathrm{SO}(2)_0, \tag{2.10c}$$

$$\exp(ibJ_+)\mapsto\mathbf{M}_+(b)=\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}\in\mathrm{E}(1)_+, \tag{2.10d}$$

$$\exp(icJ_-)\mapsto\mathbf{M}_-(c)=\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\in\mathrm{E}(1)_-. \tag{2.10e}$$

All nonequivalent one-parameter subgroups of SL(2, R) are present in (2.10): the compact rotation elliptic subgroup SO(2), the noncompact Euclidean parabolic subgroup E(1), and the boost hyperbolic subgroup SO(1, 1). For the latter two we have the following equivalence relations between the equivalent pairs (2.10a)–(2.10b) and (2.10d)–(2.10e):

$$\mathbf{SM}_2(\zeta)\mathbf{S}^{-1}=\mathbf{M}_1(\zeta),\quad \mathbf{S}=2^{-1/2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \tag{2.11a}$$

$$\mathbf{FM}_-(z)\mathbf{F}^{-1}=\mathbf{M}_+(z),\quad \mathbf{F}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\mathbf{S}^{-2}. \tag{2.11b}$$

The spectrum of J_0^γ in (2.6c) for $\gamma\geqslant\tfrac{3}{4}$ in $\mathscr{L}^2(R^+)$ has a lower bound given by its corresponding $k\geqslant 1$. (For $k=\tfrac{1}{2}$ or $\gamma=-\tfrac{1}{4}$ this is also the case for the self-adjoint extension specified in Sec. 3) The k-radial canonical transforms (2.5) thus belong to the lower-bound UIRs D_k^+ of SL(2, R).

The UIRs D_k^- are obtained from the D_k^+ ones through the sl(2, R) outer automorphism[43]

$$J_0^\gamma\leftrightarrow -J_0^\gamma,\quad J_1^\gamma\leftrightarrow -J_1^\gamma,\quad J_2^\gamma\leftrightarrow J_2^\gamma,\quad J_\pm^\gamma\leftrightarrow -J_\pm^\gamma. \tag{2.12a}$$

This exchanges the raising and lowering operators with a change of sign:

$$J_\uparrow^\gamma\leftrightarrow -J_\downarrow^\gamma,\quad J_{\uparrow\downarrow}^\gamma=J_1^\gamma\pm iJ_2^\gamma. \tag{2.12b}$$

The automorphism acts on the SL(2, R) group elements[44] as

$$\mathbf{g}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\leftrightarrow\begin{pmatrix}a & -b\\ -c & d\end{pmatrix}=\mathbf{g}^A. \tag{2.12c}$$

The D_k^- matrix elements can be thus expressed in terms of the corresponding D_k^+ ones, as will be detailed for the various subgroup reductions, at the end of the next section.

C. The continuous nonexceptional series C_q^ϵ

The oscillator representation of Sp(4, R) can also be reduced with respect to an O(1, 1)$\times$SL(2, R) subgroup[11,21,22] by making use of hyperbolic coordinates on the plane. The resulting reduction, on being restricted to a definite UIR $(p,2s)$ of O(1, 1), $p=\pm 1$, $s\in R$, yields a conjugate reduction of SL(2, R) to one of the continuous series of UIRs C_q^ϵ. The case of vector ($\epsilon=0$) and spinor ($\epsilon=\tfrac{1}{2}$) representations correspond to even ($p=+1$) and odd($p=-1$) parity representations of O(1, 1) with $q=\tfrac{1}{4}+s^2\geqslant\tfrac{1}{4}$. Since hyperbolic coordinates require two coordinate patches to cover the plane, the "hyperbolic radial" carrier space will be $X=R^++R^+$ and the Hilbert space correspondingly a two-component $\mathscr{L}^2$ space of functions

$$\mathbf{f}(r)=\begin{pmatrix} f_1(r) \\ f_{-1}(r)\end{pmatrix}=\|f_j(r)\|,\quad j=1,\ -1,\quad f_j(r)\in\mathscr{L}^2(R^+). \tag{2.13}$$

The inner product in this Hilbert space $\mathscr{L}^2_{\text{II}}(R^+) = \mathscr{L}^2(R^+) \dot{+} \mathscr{L}^2(R^+)$ will be

$$(\mathbf{f},\mathbf{h})=\sum_{j=\pm 1}\int_0^\infty dr\, f_j(r)^* h_j(r). \tag{2.14}$$

Calling $k=\frac{1}{2}+is$, this reduction leads to the (ϵ, k)-hyperbolic canonical transform

$$[\mathbb{C}^{\epsilon,k}_{\mathbf{g}}\mathbf{f}]_j(r)=\sum_{j'=\pm 1}\int_0^\infty dr'\,[\mathbf{C}^{\epsilon,k}_{\mathbf{g}}]_{j,j'}(r,r')f_{j'}(r'). \tag{2.15a}$$

The 2×2 matrix integral kernel $\mathbf{C}^{\epsilon,k}_{\mathbf{g}}(r,r')$ is given by a Gaussian times Hankel and Macdonald functions of imaginary index. For $2k-1=2is$, $s\in R$, $p_0=1$, $p_{1/2}=-1$, we can write[45]

$$[\mathbf{C}^{\epsilon,k}_{\mathbf{g}}]_{jj'}(r,r')=G_{\mathbf{g},jj'}(r,r')H^{\epsilon,k}_{jj'}(-rr'/b), \tag{2.15b}$$

$$G_{\mathbf{g},jj'}(r,r')=(2\pi|b|)^{-1}(rr')^{1/2}\exp(i[djr^2+aj'r'^2]/2b), \tag{2.15c}$$

$$\begin{aligned} H^{\epsilon,k}_{1,1}(\zeta) &= p_\epsilon H^{\epsilon,k}_{-1,-1}(\zeta)=p_\epsilon H^{\epsilon,k}_{1,1}(-\zeta)=H^{\epsilon,1-k}_{1,1}(\zeta)\\ &= i\pi[e^{-\pi s}H^{(1)}_{2is}(\zeta+i0^+)-p_\epsilon e^{\pi s}H^{(2)}_{2is}(\zeta-i0^+)]\\ &= 2i\pi(-\operatorname{sgn}\zeta)^{2\epsilon}[-g_{1/2-\epsilon}(k)J_{2is}(|\zeta|)\\ &\quad + i\,g_\epsilon(k)Y_{2is}(|\zeta|)], \end{aligned} \tag{2.15d}$$

$$\begin{aligned} H^{\epsilon,k}_{1,-1}(\zeta) &= p_\epsilon H^{\epsilon,k}_{-1,1}(\zeta)=p_\epsilon H^{\epsilon,k}_{1,-1}(-\zeta)=p_\epsilon H^{\epsilon,1-k}_{1,-1}(\zeta)\\ &= 4(-\operatorname{sgn}\zeta)^{2\epsilon}g_\epsilon(k)K_{2is}(|\zeta|), \end{aligned} \tag{2.15e}$$

$$\epsilon=0:\ \begin{cases} k-\frac{1}{2}=is, & s\geqslant 0\\ k-\frac{1}{2}=\sigma, & 0<\sigma<\frac{1}{2}\end{cases},\quad g_0(k)=\sin\pi k=\begin{cases}\cosh\pi s\\ \cos\pi\sigma\end{cases}, \tag{2.15f}$$

$$\epsilon=\tfrac{1}{2}:\ k-\tfrac{1}{2}=is,\quad s>0,\quad g_{1/2}(k)=i\cos\pi k=\sinh\pi s. \tag{2.15g}$$

In the last two equations we are defining the function $g_\epsilon(k)$ for values of k which will make it applicable to the exceptional continuous series discussed in the next subsection. Note that for $\zeta<0$, $\arg(\zeta\pm i0^+)=\pm\pi$, so (2.15d) valuates $H^{(1)}_{2is}$ *above* the branch cut of the function (placed along the negative real half-axis), and $H^{(2)}_{2is}$ is valuated *below* the cut.

When $\mathbf{g}$ in Eq. (2.3) is lower-triangular ($b=0$), as for the oscillator radial case (2.5), one finds from the asymptotic properties of the cylinder functions that Eq. (2.15a) becomes the multiplier action

$$\left[\mathbb{C}^{\epsilon k}\begin{pmatrix} a & 0\\ c & a^{-1}\end{pmatrix}\mathbf{f}\right]_j(r)=(\operatorname{sgn} a)^{2\epsilon}|a|^{-1/2}\exp(ijcr^2/2a)f_j(r/|a|). \tag{2.15h}$$

The (ϵ, k)-hyperbolic canonical transform is unitary under (2.14), and a corresponding Parseval relation holds.

Here too, the Lie generators of the integral transform action are second-order differential operators, but arranged in 2×2 matrix form. In terms of the formal operators (2.6) they are[11,21]

$$\mathbf{J}^\gamma_1=\begin{pmatrix} J^\gamma_1 & 0\\ 0 & -J^\gamma_1\end{pmatrix}=\|j\delta_{jj'}J^\gamma_1\|, \tag{2.16a}$$

$$\mathbf{J}^\gamma_2=\begin{pmatrix} J^\gamma_2 & 0\\ 0 & J^\gamma_2\end{pmatrix}=\|\delta_{jj'}J^\gamma_2\|, \tag{2.16b}$$

$$\mathbf{J}^\gamma_0=\begin{pmatrix} J^\gamma_0 & 0\\ 0 & -J^\gamma_0\end{pmatrix}=\|j\delta_{jj'}J^\gamma_0\|, \tag{2.16c}$$

$$\mathbf{J}^\gamma_\pm=\begin{pmatrix} J^\gamma_\pm & 0\\ 0 & -J^\gamma_\pm\end{pmatrix}=\|j\delta_{jj'}J^\gamma_\pm\|. \tag{2.16d}$$

Again γ is related to k through (2.7), but now as k is in the range (2.15f) and (2.15g) [instead of (2.5c)], we have $\gamma\leqslant -\frac{1}{4}$. As the subgroup assignments (2.10) are representation-independent statements, they continue to hold here as well. The Casimir invariant of SL(2, R) is now $q\geqslant\frac{1}{4}$, corresponding to the continuous nonexceptional series of UIRs. The one point we must clarify in this regard (See the Appendix) is that for spinor representations ($\epsilon=\frac{1}{2}$) the hyperbolic canonical transforms (2.15) do not include the point $k=\frac{1}{2}$ (i.e., $s=0$ or $q=\frac{1}{4}$). Indeed, from (2.15e) we can verify that for $k=\frac{1}{2}+is$, $s\to 0^+$ the off-diagonal kernel elements ($j\neq j'$) vanish and hence the two j-component spaces uncouple. The diagonal elements are now $\sim J_0(\zeta)$, that is, they are the $D^+_{1/2}$ ($k=\frac{1}{2}$)-radial canonical transform kernel for the upper component, and the $D^-_{1/2}$ one for the lower component, as is clearly suggested by (2.12a)–(2.16).

D. The continuous exceptional series C^0_q

The oscillator representation of Sp(4, R) does not contain the exceptional continuous representation series of any of its SL(2, R) subgroups. However, there exist unique self-adjoint extensions[46] of the generators (2.16) in $\mathscr{L}^2_{\text{II}}(R^+)$, which enable us to reach this series by analytic continuation in the variable k in (2.15f) to values off $k=\frac{1}{2}$, in the range

$$\tfrac{1}{2}<k<1\ \text{(i.e., } 0<2k-1=2\sigma<1\text{), for } \epsilon=0\ \ (p_\epsilon=1). \tag{2.17}$$

For these UIRs $-\frac{1}{4}<\gamma<\frac{3}{4}$, i.e., $0<q<\frac{1}{4}$.

The features one must check are that the integral kernels corresponding to these values of k continue to map $\mathscr{L}^2_{\text{II}}(R^+)$ functions into functions in the same space, and that the representation property (2.2c) holds. That this is the case follows from the integrability properties of cylinder functions in the range $(-1, 1)$ of the index, in particular their behavior at zero and infinity, and from the completeness relations for the similarly extended basis functions, to be seen in Sec. 4.

Again, as for the $\epsilon=\frac{1}{2}$, $k=\frac{1}{2}+is$, $s\to 0^+$ case seen above, when $\epsilon=0$ and $k\to 1^-$ the integral kernel matrix (2.13) becomes diagonal and the two j components uncouple. In the limit, the upper and lower-diagonal components become proportional to $J_1(\zeta)$, and belong to the D^+_1 and D^-_1 representations.

We have assembled in the last subsections the tools for the calculation of the matrix elements of SL(2, R) in point (i) of our program. In the next two sections we shall implement point (ii) for the discrete and continuous UIRs.

E. Notation

A word about notation: we shall use the eigenbases of J^γ_α, $\alpha=0, 1, 2, +, -$, generating the discrete UIRs $D^{\pm}_k$. We denote their eigenfunctions by ${}^\alpha\Phi^k_\lambda(r)$, λ being a function of the eigenvalue. When J^γ_α is in the elliptic orbit ($\alpha=0$) the eigenvalue set of J_0^γ is discrete and we shall denote its eigenvalues λ by m. The range will be understood by the context. When J_α^γ is in the hyperbolic orbit ($\alpha = 1, 2$) or in the parabolic orbit ($\alpha = +, -$), its eigenvalue set is continuous. In the first case λ will be denoted by $\mu \in R$, the eigenvalue under $J_{1,2}^\gamma$ being μ. In the second case λ will be called $\rho \in R^+$, the eigenvalues of $J_\pm^\gamma$ being $\rho^2/2$. Eigenbases for the D_k^- UIRs will not be needed separately. In the continuous series C_q^ϵ the eigenbases of J_α^γ will be similarly denoted by ${}^\alpha\Psi_\lambda^{\epsilon,k}(r)$, these are two-component functions with elements ${}^\alpha\Psi_{\lambda,j}^{\epsilon,k}$, $j = 1, -1$. We use m again for λ, the eigenvalue under J_0^γ. The multiplicity of the eigenvalues of the generators in the hyperbolic and parabolic orbits is now doubled, however. For the former we use for λ the pair (κ, μ), $\kappa = \pm 1$, $\mu \in R$, and for the latter $(\mathrm{sgn}\,\rho, |\rho|) = \rho$, $\rho \in R$, the eigenvalues being again μ and $\rho^2/2$ under the respective J^γ's.

The representations $\mathbf{D}(\mathbf{g})$ constructed in (2.2) have their matrix elements

$$ {}^{\alpha,\beta}D_{\lambda,\lambda'}^k(\mathbf{g}) = ({}^\alpha\Phi_\lambda^k, \mathbb{C}_\mathbf{g}^k\, {}^\beta\Phi_{\lambda'}^k) = [{}^{\beta,\alpha}D_{\lambda',\lambda}^k(\mathbf{g}^{-1})]^*, \tag{2.18a} $$

$$ {}^{\alpha,\beta}D_{\lambda,\lambda'}^{\epsilon,k}(\mathbf{g}) = ({}^\alpha\Psi_\lambda^{\epsilon,k}, \mathbb{C}_\mathbf{g}^{\epsilon,k}\, {}^\beta\Psi_{\lambda'}^{\epsilon,k}) = [{}^{\beta,\alpha}D_{\lambda',\lambda}^{\epsilon,k}(\mathbf{g}^{-1})]^*, \tag{2.18b} $$

in the appropriate inner product. When $\alpha = \beta$ we write ${}^\alpha D_{::}$ for ${}^{\alpha,\alpha}D_{::}$. The cases $\alpha \neq \beta$ and $\alpha = \beta$ in (2.18) will be called *mixed-basis* and *subgroup-reduced* UIR matrix elements. We shall work mostly with the D_k^+ UIRs and use (2.18a). In Sec. 3D, when we express the D_k^- UIRs in terms of the D_k^+ ones, we shall write ${}^{\cdot}D_{::}^{k(-)}$ and ${}^{\cdot}D_{::}^{k(+)}$ to distinguish between them.

3. THE DISCRETE SERIES $D_k^\pm$

In this section we present the evaluation of the matrix elements (or integral kernels) of finite SL(2, R) transformations for the UIRs belonging to the discrete series $D_k^\pm$. The first subsection gives the E(1), SO(1, 1), and SO(2) subgroup-adapted eigenfunctions, while the second and third subsections provide the explicit evaluation of D_k^+ mixed-basis and subgroup-reduced cases respectively. The last subsection relates these results to those of the D_k^- representations.

A. The subgroup-adapted eigenfunctions

i. E(1)$\subset$*SL*(2, *R*). The two operators generating conjugate E(1) subgroups [c.f. Eqs. (2.10d) and (2.10e)] are, as given by (2.8a), and (2.8b), J_+^γ and J_-^γ. They are unitarily equivalent through the Hankel transform (2.11b).

The eigenfunctions of J_+^γ in $\mathscr{L}^2(R^+)$ are, for $\gamma = (2k-1)^2 - \frac{1}{4}$,

$$ {}^+\Phi_\rho^k(r) = e^{i\pi k}(\rho r)^{1/2} J_{2k-1}(\rho r), \quad \rho \in R^+, \quad k = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \tag{3.1} $$

with eigenvalue $\rho^2/2 \in R^+$. The phase has been chosen so that the phase of the ${}^-\Phi_\rho^k$ functions, below, be as simple as possible.

A more convenient operator in the E(1) orbit is J_-^γ, as its eigenfunctions are simply

$$ {}^-\Phi_\rho(r) = \delta(\rho - r) = [\mathbb{C}_\mathrm{F}^k\, {}^+\Phi_\rho^k](r), \quad r \in R^+, \tag{3.2a} $$

with eigenvalue $\rho^2/2$. These are Dirac-orthonormal and complete:

$$ ({}^-\Phi_\rho, {}^-\Phi_{\rho'}) = \delta(\rho - \rho'), $$

$$ \int_0^\infty d\rho\, {}^-\Phi_\rho(r)^*\, {}^-\Phi_\rho(r') = \delta(r - r'), \tag{3.2b} $$

and independent of k.

ii. SO(1, 1)$\subset$*SL*(2, *R*). Here again we have two operators generating conjugate SO(1, 1) subgroups [c.f. Eqs. (2.10a) and (2.10b) and (2.11a)]: J_1^γ and J_2^γ, as given by (2.5a) and (2.5b). The latter is the simpler one, and its eigenfunctions are

$$ {}^2\Phi_\mu(r) = \pi^{-1/2} r^{-1/2 + 2i\mu}, \quad \mu \in R, \tag{3.3a} $$

with eigenvalue μ. They are Dirac-orthonormal and complete:

$$ ({}^2\Phi_\mu, {}^2\Phi_{\mu'}) = \delta(\mu - \mu'), \quad \int_{-\infty}^\infty d\mu\, {}^2\Phi_\mu(r)^*\, {}^2\Phi_\mu(r') = \delta(r - r'), \tag{3.3b} $$

and independent of k. The expansion in terms of them is—up to a factor—the positive Mellin transformation,[47] so an appropriate phase choice has been made.

The J_1^γ Dirac-normalized eigenfunctions may be found from (3.3a) and (2.11a) to be

$$ \begin{aligned} {}^1\Phi_\mu^k(r) &= [\mathbb{C}_\mathrm{S}^k\, {}^2\Phi_\mu](r) \\ &= C_\mu^k e^{i\pi k/2} r^{-1/2} M_{i\mu,k-1/2}(-ir^2) \\ &= C_\mu^k r^{2k-1/2} e^{ir^2/2}\, {}_1F_1\left[\begin{matrix} k - i\mu \\ 2k \end{matrix}; -ir^2\right], \end{aligned} \tag{3.4a} $$

$$ C_\mu^k = e^{i\pi k/2} 2^{i\mu} \pi^{-1/2} e^{\pi\mu/2} \Gamma(k + i\mu)/\Gamma(2k). \tag{3.4b} $$

and where $M_{..}(\cdot)$ is one of the Whittaker functions.[48] They correspond to eigenvalue μ under J_1^γ, and are Dirac-orthonormal and complete as in (3.3b).

iii. SO(2)$\subset$*SL*(2, *R*). The compact SO(2) subgroup is generated by J_0^γ as given in Eq. (2.6c). Its normalized eigenfunctions are given by

$$ \begin{aligned} {}^0\Phi_m^k(r) &= [2n!/(2k+n-1)!]^{1/2} r^{2k-1/2} e^{-r^2/2} L_n^{(2k-1)}(r^2) \\ &= [2(2k+n-1)!/n!(2k-1)!]^{1/2} r^{-1/2} M_{m,k-1/2}(r^2) \\ &= [2(2k+n-1)!/n!]^{1/2} [(2k-1)!]^{-1} r^{2k-1/2} e^{-r^2/2} \\ &\quad \times {}_1F_1\left[\begin{matrix} -n \\ 2k \end{matrix}; r^2\right], \\ & m = k + n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.5a} $$

with eigenvalue $m = k, k+1, \cdots$. The phase of these functions has been chosen following Bargmann's convention,[49] namely, such that the raising and lowering operators $J_1^\gamma \pm iJ_2^\gamma$ have real, positive, matrix elements. They are orthonormal and complete (dense) in $\mathscr{L}^2(R^+)$:

$$ ({}^0\Phi_m^k, {}^0\Phi_{m'}^k) = \delta_{m,m'}, \quad \sum_{m=k}^\infty {}^0\Phi_m^k(r)^*\, {}^0\Phi_m^k(r') = \delta(r - r'). \tag{3.5b} $$

B. The mixed-basis matrix elements

i. E(1)$\subset$*SL*(2, *R*)$\supset$*SO*(2). For all $\mathbf{g} \in$ SL(2, R) we may perform the Iwasawa decomposition

$$\mathbf{g}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\bar a & 0\\ \bar c & \bar a^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta/2 & -\sin\theta/2\\ \sin\theta/2 & \cos\theta/2\end{pmatrix}, \tag{3.6a}$$

where

$$e^{i\theta}=(a-ib)/(a+ib),\quad \bar a=(a^2+b^2)^{1/2},\quad \bar a\,\bar c=ac+bd. \tag{3.6b}$$

Application of $\mathbb{C}^k_{\mathbf{g}}$ decomposed as above, multiplies the J^γ_0 eigenfunction by $e^{im\theta}$, followed subsequently by a multiplier Lie transformation, Eq. (2.5d). Thus

$$\begin{aligned}[\mathbb{C}^k_{\mathbf{g}}\,{}^0\Phi^k_m](r)&=[(a-ib)/(a+ib)]^m(a^2+b^2)^{-1/4}\\ &\quad\times\exp(ir^2[ac+bd]/2[a^2+b^2])\\ &\quad\times{}^0\Phi^k_m(r/[a^2+b^2]^{1/2}).\end{aligned} \tag{3.7}$$

Since the J^γ_- eigenfunctions are simple Dirac deltas, we immediately obtain

$$\begin{aligned}{}^{-,0}D^k_{\rho m}\begin{pmatrix}a & b\\ c & d\end{pmatrix}&=\left({}^-\Phi_\rho,\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}{}^0\Phi^k_m\right)\\ &=\left[\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}{}^0\Phi^k_m\right](\rho)={}^{+,0}D^k_{\rho m}\begin{pmatrix}c & d\\ -a & -b\end{pmatrix}\\ &=\left[{}^{0,-}D^k_{m\rho}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\right]^*\\ &=\left(\frac{a-ib}{a+ib}\right)^m\left[\frac{2\Gamma(k+m)}{(m-k)!}\right]^{1/2}\frac{(a^2+b^2)^{-k}}{\Gamma(2k)}\\ &\quad\times\rho^{2k-1/2}\exp\left(-\frac{\rho^2}{2}\frac{d-ic}{a+ib}\right)\\ &\quad\times{}_1F_1\left[\begin{matrix}-(m-k)\\ 2k\end{matrix};\frac{\rho^2}{a^2+b^2}\right].\end{aligned} \tag{3.8}$$

The overlap coefficient between the $\mathrm{E}(1)_-$ and $\mathrm{SO}(2)_0$ subgroup chains is obtained by setting $\mathbf{g}=\mathbf{1}$, i.e., $a=1=d$, $b=0=c$ in Eq. (3.8). This is ${}^0\Phi^k_m(\rho)$, i.e., this change of basis is basically the Laguerre series expansion of functions of $\rho\in R^+$.

ii. $SO(1,1)\subset SL(2,R)\supset SO(2)$. This mixed basis element is essentially the Mellin transform of Eq. (3.8), and is given by[50]

$$\begin{aligned}&{}^{2,0}D^k_{\mu m}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\\ &=\left({}^2\Phi_\mu,\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}{}^0\Phi^k_m\right)\\ &={}^{1,0}D^k_{\mu m}\begin{pmatrix}2^{-1/2}(a-c) & 2^{-1/2}(b-d)\\ 2^{-1/2}(a+c) & 2^{-1/2}(b+d)\end{pmatrix}\\ &=2^{k-i\mu}\left[\frac{\Gamma(k+m)}{2\pi(m-k)!}\right]^{1/2}\frac{\Gamma(k-i\mu)}{\Gamma(2k)}\\ &\quad\times(a+ib)^{-m}(a-ib)^{m-k+i\mu}(d-ic)^{-k+i\mu}\\ &\quad\times{}_2F_1\left[\begin{matrix}-(m-k),k-i\mu\\ 2k\end{matrix};\frac{2}{(a-ib)(d-ic)}\right]\\ &=(-1)^{m-k}2^{m-i\mu}[2\pi(m-k)!\Gamma(k+m)]^{-1/2}\Gamma(m-i\mu)\\ &\quad\times(a+ib)^{-m}(a-ib)^{i\mu}(d-ic)^{-m+i\mu}\\ &\quad\times{}_2F_1\left[\begin{matrix}-(m-k),1-k-m\\ 1-m+i\mu\end{matrix};\tfrac12(a-ib)(d-ic)\right].\end{aligned} \tag{3.9}$$

In all power-function factors, the principal branch of this function is to be taken in an obvious way. The hypergeometric function is a polynomial of degree $m-k=n$ so no multivaluation problems occur on its account.

The overlap coefficient between these two chains in the discrete series is obtained by setting $\mathbf{g}=\mathbf{1}$. Using an identity for the hypergeometric function[51] we find

$$\begin{aligned}({}^2\Phi_\mu,{}^0\Phi^k_m)&={}^{2,0}D^k_{\mu m}(\mathbf{1})\\ &=(-1)^{m-k}2^{k-i\mu}\frac{\Gamma(m-i\mu)}{[2\pi(m-k)!\Gamma(k+m)]^{1/2}}\\ &\quad\times{}_2F_1\left[\begin{matrix}-(m-k),k+i\mu\\ 1-m+i\mu\end{matrix};-1\right].\end{aligned} \tag{3.10a}$$

Correspondingly

$$\begin{aligned}&({}^1\Phi^k_\mu,{}^0\Phi^k_m)\\ &={}^{2,0}D^k_{\mu m}\left(2^{-1/2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right)=e^{-im\pi/2}({}^2\Phi_\mu,{}^0\Phi^k_m),\end{aligned} \tag{3.10b}$$

which may be compared with prior results.[52]

iii. $E(1)\subset SL(2,R)\supset SO(1,1)$. The application of $\mathbb{C}^k_{\mathbf{g}}$ to ${}^2\Phi_\mu$ in Eq. (3.3a) is up to a factor the Mellin transform of the k-canonical transform kernel (2.5b) with respect to the second argument r'. Although integrals of this type appear in the standard tables,[53] if we want to have expressions valid for all group parameters, positive as well as negative, care must be taken to choose the appropriate parameter products and ratios so that the ensuing complex power function be evaluated in a definite way: We choose here the principal sheet (with the branch cut along the negative real axis). Following the general method of finding the Mellin transforms of hypergeometric functions due to Majumdar and Basu,[39] which will be explained in some detail in the next section, we find the value of the integral to be

$$\begin{aligned}{}^{-,2}D^k_{\rho\mu}\begin{pmatrix}a & b\\ c & d\end{pmatrix}&=\left({}^-\Phi_\rho,\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}{}^2\Phi_\mu\right)=\left[\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}{}^2\Phi_\mu\right](\rho)\\ &=e^{-i\pi k}2^{-k+i\mu}\pi^{-1/2}\frac{\Gamma(k+i\mu)}{\Gamma(2k)}\\ &\quad\times b^{-2k}(-ia/b)^{-k-i\mu}\\ &\quad\times\rho^{2k-1/2}\exp(id\rho^2/2b)\\ &\quad\times{}_1F_1\left[\begin{matrix}k+i\mu\\ 2k\end{matrix};\frac{-i\rho^2}{2ab}\right].\end{aligned} \tag{3.11}$$

The complex-power function argument $-ia/b$ lies, for all signs of a and b on the imaginary axis.[54] Valuation on the principal sheet means that the phase of $-ia/b$ is $-\pi/2$ for $\mathrm{sgn}\,ab=1$ and $\pi/2$ for $\mathrm{sgn}\,ab=-1$.

The overlap coefficient between these two chains may be obtained as the limit $\mathbf{g}\to\mathbf{1}$ in Eq. (3.11), or directly, as

$$({}^-\Phi_\rho,{}^2\Phi_\mu)={}^{-,2}D^k_{\rho\mu}(\mathbf{1})=\pi^{-1/2}\rho^{-1/2+2i\mu}, \tag{3.12}$$

which is[47] $2^{1/2}$ times the positive Mellin transform kernel, of argument 2μ, between a function of $\rho\in R^+$ and its transform function of $\mu\in R$.

C. The matrix elements in the subgroup bases

i. $E(1)\subset SL(2,R)$. In this generalized basis the integral kernel is the simplest to obtain, as no integrations need be performed:

$$ {}^{-}D^{k}_{\rho\rho'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left({}^{-}\Phi_{\rho}, \mathbb{C}^{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^{-}\Phi_{\rho'} \right) = C^{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\rho,\rho') $$
$$ = e^{-i\pi k} b^{-1}(\rho\rho')^{1/2}\exp(i[d\rho^{2}+a\rho'^{2}]/2b)J_{2k-1}(\rho\rho'/b) $$
$$ = 2(2ib)^{-2k}[\Gamma(2k)]^{-1}(\rho\rho')^{2k-1/2} $$
$$ \times\exp(i[d\rho^{2}-2\rho\rho'+a\rho'^{2}]/2b)\ {}_{1}F_{1}\begin{bmatrix} 2k-\tfrac{1}{2} \\ 4k-1 \end{bmatrix}; \frac{2i\rho\rho'}{b} \Bigg]. \tag{3.13} $$

For $g\in E(1)$, the subgroup generated by J^{γ}_{-} [c.f. Eq. (2.10e)], the kernel becomes diagonal. In fact, it is diagonal for the two-parameter subgroup generated by the first-order differential operators, for which (2.13) converges weakly to

$$ {}^{-}D^{k}_{\rho\rho'}\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = (\operatorname{sgn} a)^{2k}|a|^{-1/2} $$
$$ \times\exp(ic\rho^{2}/2a)\delta(\rho'-\rho/|a|). \tag{3.14} $$

From this form it is manifest that ${}^{-}D^{k}_{\rho\rho'}(\mathbf{1}) = \delta(\rho-\rho')$, the unit operator in $\mathscr{L}^{2}(R^{+})$, while ${}^{-}D^{k}_{\rho\rho'}(-\mathbf{1}) = (-1)^{2k}\delta(\rho-\rho')$. The composition property is satisfied, i.e., Eq. (2.2c) under $\int_{R^{+}} d\rho$.., as under this measure the eigenbasis is Dirac-orthonormal and complete.

The matrix elements between the J^{γ}_{+} eigenfunctions can now be immediately computed:

$$ {}^{+}D^{k}_{\rho\rho'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left({}^{+}\Phi^{k}_{\rho}, \mathbb{C}^{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^{+}\Phi^{k}_{\rho'} \right) $$
$$ = {}^{-}D^{k}_{\rho\rho'}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \tag{3.15} $$

The matrix elements (3.14) and (3.15) are manifestly unitary. This is a direct consequence of the unitarity of the canonical transforms.

The E(1) reduction shows in particular that the Bessel functions in ${}^{+}\Phi^{k}_{\rho}(r)$ are self-reciprocating[55] under the k-radial canonical transforms, i.e., the $\mathbb{C}^{k}_{\mathbf{g}}$-transform of ${}^{+}\Phi^{k}_{\rho}$ may be written as a multiplier function times a function of the transformed argument:

$$ \left[\mathbb{C}^{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^{+}\Phi^{k}_{\rho}\right](r) $$
$$ = \left[\mathbb{C}^{k}\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \exp(-iba^{-1}J^{\gamma}_{+})\ {}^{+}\Phi^{k}_{\rho}\right](r) $$
$$ = |a|^{-1/2}\exp(-ib\rho^{2}/2a)\exp(icr^{2}/2a)\ {}^{+}\Phi^{k}_{\rho}(r/|a|). \tag{3.16} $$

Here we have made use of the decomposition of $\mathbf{g}$ as a lower-triangular matrix times $\mathbf{M}_{+}(b/a)$ [c f. Eqs. (2.10d) and (2.10e)]; the latter factor gives rise to the phase $\exp(-ib\rho^{2}/2a)$ while the former is the point transformation as given by Eq. (2.5c). Similar self-reciprocation formulas hold for other subgroup-reduced matrix elements throughout this article.

ii. $SO(1,1)\subset SL(2,R)$. This matrix element[56] is essentially the Mellin transform of Eq. (3.11) with respect to the argument ρ. Again, as the general method for evaluating Mellin transforms of hypergeometric functions[39] is presented in the next section, we simply quote here the result:

$$ {}^{2}D^{k}_{\mu\mu'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left({}^{2}\Phi_{\mu}, \mathbb{C}^{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^{2}\Phi_{\mu'} \right) $$
$$ = {}^{1}D^{k}_{\mu\mu'}\begin{pmatrix} (a-b-c+d)/2 & (a+b-c-d)/2 \\ (a-b+c-d)/2 & (a+b+c+d)/2 \end{pmatrix} $$
$$ = e^{-i\pi k}2^{i(\mu'-\mu)}\frac{\Gamma(k-i\mu)\Gamma(k+i\mu')}{2\pi\Gamma(2k)} $$
$$ \times b^{-2k}\left(\frac{-ia}{b}\right)^{-k-i\mu'}\left(\frac{-id}{b}\right)^{-k+i\mu} $$
$$ \times\ {}_{2}F_{1}\begin{bmatrix} k-i\mu, k+i\mu' \\ 2k \end{bmatrix}; \frac{1}{ad}\Bigg]. \tag{3.17} $$

As in (3.11), we give this expression in terms of complex power functions, taking care that these variables be evaluated for points along the imaginary axis, in the principal sheet of the power functions, where the cut is chosen along the negative real half-axis.[57] An alternative expression in terms of the absolute values of a, b, and d may be written through

$$ b^{-2k}(-ia/b)^{-k-i\mu'}(-id/b)^{-k+i\mu'} $$
$$ = (\operatorname{sgn} b)^{2k}\exp(i\tfrac{1}{2}\pi[k+i\mu']\operatorname{sgn} ab) $$
$$ \times\exp(i\tfrac{1}{2}\pi[k-i\mu]\operatorname{sgn} bd)|a|^{-k-i\mu'} $$
$$ \times|b|^{i(\mu'-\mu)}|d|^{-k+i\mu}. \tag{3.18} $$

One can obtain from these expressions the diagonal and antidiagonal cases

$$ {}^{2}D^{k}_{\mu\mu'}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = (\operatorname{sgn} a)^{2k}|a|^{-2i\mu}\delta(\mu-\mu'), \tag{3.19} $$

$$ {}^{2}D^{k}_{\mu\mu'}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} $$
$$ = e^{-i\pi k}2^{-2i\mu}[\Gamma(k-i\mu)/\Gamma(k+i\mu)]\delta(\mu+\mu') $$
$$ = \exp i(-\pi k - 2\mu\ln 2 + 2\arg[k-i\mu])\delta(\mu+\mu'). \tag{3.20} $$

From (3.19) we verify that ${}^{2}\mathbf{D}^{k}(\pm\mathbf{1}) = (\pm 1)^{2k}\mathbf{1}$, while (3.20) is the Fourier–Hankel transform in the Mellin basis. The representations are unitary in all cases. The direct evaluation of (3.20) allows us to give alternative forms for (3.17) through

$$ {}^{2}D^{k}_{\mu,\mu'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{-\infty}^{\infty} d\mu''\, {}^{2}D^{k}_{\mu,\mu''}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^{2}D^{k}_{\mu'',\mu'}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} $$
$$ = e^{-i\pi k}2^{-2i\mu}[\Gamma(k-i\mu)/\Gamma(k+i\mu)]\ {}^{2}D^{k}_{-\mu,\mu'}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} $$
$$ = e^{-i\pi k}2^{2i\mu'}[\Gamma(k+i\mu')/\Gamma(k-i\mu')]\ {}^{2}D^{k}_{\mu,-\mu'}\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}. \tag{3.21} $$

iii. $SO(2)\subset SL(2,R)$. This matrix element is the inner product of Eq. (3.7) with ${}^{0}\Phi^{k}_{m}$. The resulting integral in available from the tables.[58] It is

$$
\begin{aligned}
{}^0D^k_{mm'}\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \left({}^0\Phi^k_m, \mathbb{C}^k\begin{pmatrix} a & b \\ c & d\end{pmatrix}{}^0\Phi^k_{m'}\right)\\
&= 2^{2k}\Gamma(m+m')[\Gamma(k+m)\Gamma(1-k+m)\Gamma(k+m')\Gamma(1-k+m')]^{-1/2}\\
&\quad\times[(d-a)-i(b+c)]^{m-k}[(a-d)-i(b+c)]^{m'-k}[(a+d)+i(b-c)]^{-m-m'}\\
&\quad\times {}_2F_1\left[\begin{matrix}-(m-k),\ -(m'-k)\\ 1-m-m'\end{matrix};\frac{a^2+b^2+c^2+d^2+2}{a^2+b^2+c^2+d^2-2}\right]\\
&= (-1)^{m-k}\Gamma(m+m')[\Gamma(k+m)\Gamma(1-k+m)\Gamma(k+m')\Gamma(1-k+m')]^{1/2}\\
&\quad\times\alpha^{*-m-m'}\beta^{m-k}\beta^{*m'-k}\,{}_2F_1\left[\begin{matrix}-(m-k),\ -(m'-k)\\ 1-m-m'\end{matrix};\frac{|\alpha|^2}{|\beta|^2}\right].
\end{aligned}
\tag{3.22}
$$

In the last expression we have given the SL(2, R) representation matrix elements in terms of the complex SU(1, 1) parameters of Bargmann through (A3). The hypergeometric function appearing above is actually a polynomial of degree min($m-k$, $m'-k$). One also checks easily that ${}^0\mathbf{D}^k(\pm\mathbf{1}) = (\pm 1)^{2k}\mathbf{1}$ and that the representation matrix is unitary.

The expression (3.22) for the UIR matrix elements gives the value of the group unit at the point at infinity of the hypergeometric function. We can bring[59] (3.22) to coincide with the form given by Bargmann,[60] which values the group unit at the zero of the hypergeometric function, taking care to distinguish the cases $m\geqslant m'$ from $m\leqslant m'$.

D. The D_k^- representations

The discrete representation series D_k^- is obtained from the D_k^+ series through the group automorphism (2.12c), i.e., $\mathbf{D}^{k(-)}(\mathbf{g}) = \mathbf{D}^{k(+)}(\mathbf{g}^A)$. The basis functions ${}^\alpha\Phi^k_\lambda(r)$ are now to be taken as eigenfunctions of the algebra generators $\tilde\sigma_\alpha J^\gamma_\alpha$, where $\tilde\sigma_\alpha = -1$ for $\alpha = 0, 1, +, -$ and $\tilde\sigma_\alpha = 1$ for $\alpha = 2$, with eigenvalue $\tilde\sigma_\alpha$ times the eigenvalue of the J^γ_α representation generator. In addition, for the SO(2) subgroup chain, if we are to follow Bargmann's phase convention[49] of having the raising and lowering operators represented by matrices with positive elements, (2.12b) implies that the phase of the basis functions ${}^0\Phi^k_m(r)$ must be multiplied by a sign factor $\tau^m_0 = (-1)^{m-k}$ [recall (3.5b)]. For convenience we set $\tau^\lambda_\alpha = 1$ for all other $\alpha\neq 0$. We can then write all D_k^- mixed-basis and subgroup-reduced matrix elements in terms of the D_k^+ expressions given above in this section as

$$
{}^{\alpha,\beta}D^{k(-)}_{\lambda,\lambda'}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \tau^\lambda_\alpha\tau^{\lambda'}_\beta\,{}^{\alpha,\beta}D^{k(+)}_{\sigma_\alpha\lambda,\sigma_\beta\lambda'}\begin{pmatrix} a & -b \\ -c & d\end{pmatrix},
\tag{3.23a}
$$

$$
\sigma_\alpha = \begin{cases} 1, & \alpha = 2, +, - \\ -1, & \alpha = 0, 1\end{cases},
$$

$$
\tau^\lambda_\alpha = \begin{cases} 1, & \alpha = 1, 2, +, - \\ (-1)^{m-k}, & \alpha = 0\end{cases}.
\tag{3.23b}
$$

4. THE CONTINUOUS SERIES C_q^ϵ

In this section we follow the same general strategy in finding the unitary irreducible matrix elements (or integral kernels) corresponding to the continuous series C^ϵ_q. The difference is that here we use the hyperbolic canonical transforms of Sec. 2C, rather than the radial ones employed above. The function space has now two components, the inner product is given by Eq. (2.14), the group action by (2.15), and the subgroup generators by Eqs. (2.16). The noncompact subgroup generators $\mathbf{J}_-$ and $\mathbf{J}_2$ of E(1)$_-$ and SO(1, 1)$_2$ are just as simple as those in the last section—although their spectra are doubly degenerate. The eigenfunctions of $\mathbf{J}_0$ and $\mathbf{J}_1$ are in general less simple: linear combinations of the first and second solutions of the confluent hypergeometric differential equation. Although the $\mathbf{J}_0$ eigenfunctions sum up to a Whittaker function,[61] the $\mathbf{J}_1$ eigenfunctions do not.

A. The subgroup-adapted eigenfunctions

i. $E(1)\subset SL(2, R)$. The simplest operator in the parabolic orbit, as for its discrete counterpart, is $\mathbf{J}_-$, given by (2.16c). Its generalized eigenfunctions are

$$
{}^-\boldsymbol{\Psi}_\rho(r) = \begin{cases} \begin{pmatrix}\delta(\rho - r)\\ 0\end{pmatrix}, & \rho\geqslant 0\\ \begin{pmatrix} 0 \\ \delta(|\rho| - r)\end{pmatrix} = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}{}^-\boldsymbol{\Psi}_{|\rho|}(r), & \rho < 0,\end{cases}
\tag{4.1a}
$$

with eigenvalue (sgn ρ) $\rho^2/2$. The spectrum of $\mathbf{J}_-$ in the continuous series UIRs thus ranges over R, rather than over R^+ as in the discrete ones. In (4.1a) a definite choice of phase has been made. The set of functions (4.1a) is Dirac-orthonormal and complete in $\mathscr{L}^2_{\mathrm{II}}(R^+)$:

$$
({}^-\boldsymbol{\Psi}_\rho, {}^-\boldsymbol{\Psi}_{\rho'}) = \delta(\rho-\rho'), \quad \int_{-\infty}^{\infty} d\rho\, \Psi_{\rho,j}(r)^*\Psi_{\rho,j'}(r') = \delta_{j,j'}\delta(r-r').
\tag{4.1b}
$$

From Eqs. (4.1a) and the hyperbolic inverse Fourier canonical transform [Eqs. (2.15) for $\mathbf{F}^{-1}$ as given in (2.11b)] we find the $\mathbf{J}^\gamma_+$ generalized eigenfunctions to be

$$
{}^+\boldsymbol{\Psi}^{\epsilon k}_\rho(r) = \frac{(\rho r)^{1/2}}{2\pi}\begin{pmatrix} H^{\epsilon k}_{1,1}(-\rho r)\\ H^{\epsilon k}_{-1,1}(-\rho r)\end{pmatrix} = \begin{pmatrix}(2\pi)^{-1/2}[e^{-i\pi/4}W_{0,2k-1}(2i\rho r) + p_\epsilon e^{i\pi/4}W_{0,2k-1}(-2i\rho r)]\\ (2/\pi)^{1/2}p_\epsilon g_\epsilon(k)W_{0,2k-1}(2\rho r)\end{pmatrix}, \quad \rho\geqslant 0,
\tag{4.2a}
$$

$$ {}^{+}\Psi_{\rho}^{\epsilon k}(r) = \frac{(|\rho| r)^{1/2}}{2\pi}\begin{pmatrix} H^{\epsilon k}_{1,-1}(\rho r) \\ H^{\epsilon k}_{-1,-1}(\rho r)\end{pmatrix} = p_{\epsilon}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} {}^{+}\Psi^{\epsilon k}_{|\rho|}(r), \quad \rho \leqslant 0, \tag{4.2b} $$

where the $H^{\epsilon k}_{jj'}(\zeta)$ are given in (2.15d)–(2.15e). We have expressed the Hankel and Macdonald functions in terms of Whittaker functions[61] of argument phase 0 and $\pm\pi/2$. As in (4.1a), (4.2) correspond to the eigenvalue $(\operatorname{sgn}\rho)\rho^2/2\in R$. Recall that for the continuous nonexceptional series $2k-1=2is$, $s\geqslant 0$ for $\epsilon=0$ and $s>0$ for $\epsilon=\frac{1}{2}$, while for the exceptional interval $\epsilon=0$, $2k-1=2\sigma$, $0<\sigma<\frac{1}{2}$.

ii. $SO(1,1)\subset SL(2,R)$. The simplest operator in the hyperbolic orbit is $\mathbf{J}_2$, as given by (2.16b). Notice that the signs of the entries are the same. The spectrum of J_2 covers R once in $\mathscr{L}^2(R^+)$, while that of $\mathbf{J}_2$ does so twice in $\mathscr{L}^2_{\rm II}(R^+)$. The normalized eigenfunctions ${}^2\Psi_{\kappa,\mu}(r)$ thus require an extra dichotomic index $\kappa=\pm 1$, and are

$$ {}^2\Psi_{\kappa,\mu}(r) = (2\pi)^{-1/2}\begin{pmatrix}1\\ \kappa\end{pmatrix} r^{-1/2+2i\mu}, \quad \kappa=\pm 1, \mu\in R, \tag{4.3a} $$

belonging to the eigenvalue μ under $\mathbf{J}_2$. The dichotomic index κ has been introduced by Mukunda and Radhakrishnan[11]; it can be seen as the eigenvalue of ${}^2\Psi_{\kappa,\mu}(r)$ under a transformation in $\mathscr{L}^2_{\rm II}(R^+)$ given by $A: f_j(r)\to f_{-j}(r)$, which may be represented[62] as

$$ \mathbf{A} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. $$

The statement of Dirac orthonormality and completeness is

$$ ({}^2\Psi_{\kappa,\mu}, {}^2\Psi_{\kappa',\mu'}) = \delta_{\kappa,\kappa'}\delta(\mu-\mu'), $$
$$ \sum_{\kappa=\pm 1}\int_{-\infty}^{\infty} d\mu\, {}^2\Psi_{\kappa,\mu,j}(r)^{*}\, {}^2\Psi_{\kappa,\mu,j'}(r') = \delta_{jj'}\delta(r-r'). \tag{4.3b} $$

The eigenfunctions ${}^1\Psi^{\epsilon,k}_{\kappa,\mu}(r)$ of $\mathbf{J}^{\gamma}_1$ [Eq. (2.16a)], on the other hand, using (2.11a) are given by[63]

$$ \begin{aligned} {}^1\Psi^{\epsilon,k}_{\kappa,\mu,j}(r) &= [\mathbb{C}^{\epsilon,k}_{\mathbf{S}}\, {}^2\Psi_{\kappa,\mu}]_j(r) = (-1)^{2\epsilon}(2\pi)^{-3/2}2^{i\mu+1}g_{\epsilon}(k) \\ &\times [e^{-ij\pi(k+i\mu)/2}\{p_{\epsilon}G^{k}_{\mu,j}(r) + G^{1-k}_{\mu,j}(r)\} \\ &+ \kappa e^{ij\pi(k+i\mu)/2}\{G^{k}_{\mu,j}(r) + G^{1-k}_{\mu,j}(r)\}], \end{aligned} \tag{4.4a} $$

$$ G^{k}_{\mu,j}(r) = \Gamma(1-2k)\Gamma(k+i\mu) r^{2k-1} e^{ijr^2/2} \times {}_1F_1(k-i\mu;\, 2k;\, -ijr^2). \tag{4.4b} $$

They are obtained from Eqs. (4.17)–(4.18), below.

iii. $SO(2)\subset SL(2,R)$. For the continuous series C^{ϵ}_{q} of UIRs belonging to the nonexceptional or exceptional series, the eigenfunctions of the compact generator $\mathbf{J}^{\gamma}_0$ are given by

$$ {}^0\Psi^{\epsilon,k}_{m}(r) = \frac{g_{\epsilon}(k)}{\pi r^{1/2}} \times\begin{pmatrix} (-1)^{m-\epsilon}[2\Gamma(k-m)\Gamma(1-k-m)]^{1/2}W_{m,k-1/2}(r^2) \\ [2\Gamma(k+m)\Gamma(1-k+m)]^{1/2}W_{-m,k-1/2}(r^2)\end{pmatrix}. \tag{4.5a} $$

These eigenfunctions belong to the eigenvalue m under $\mathbf{J}^{\gamma}_0$. We have chosen the phase in accordance with Bargmann's convention,[64] i.e., such that the raising and lowering operators have positive matrix elements. They are orthonormal and complete in $\mathscr{L}^2_{\rm II}(R^+)$:

$$ ({}^0\Psi^{\epsilon,k}_{m}, {}^0\Psi^{\epsilon,k}_{m'}) = \delta_{m,m'}, $$
$$ \sum_{m\in Z} {}^0\Psi^{\epsilon,k}_{m,j}(r)^{*}\, {}^0\Psi^{\epsilon,k}_{m,j'}(r') = \delta_{j,j'}\delta(r-r'). \tag{4.5b} $$

B. The mixed-basis matrix elements

i. $E(1)\subset SL(2,R)\supset SO(2)$. Application of $\mathbb{C}^{\epsilon k}_{\mathbf{g}}$ decomposed as in (3.6) gives

$$ \begin{aligned} [\mathbb{C}^{\epsilon,k}_{\mathbf{g}}\, {}^0\Psi^{\epsilon,k}_{m}]_j(r) &= \left(\frac{a-ib}{a+ib}\right)^{m}(a^2+b^2)^{-1/4} \\ &\times \exp\left(\frac{ijr^2[ac+bd]}{2[a^2+b^2]}\right) \\ &\times {}^0\Psi^{\epsilon,k}_{m,j}(r/[a^2+b^2]^{1/2}). \end{aligned} \tag{4.6} $$

This formula displays the Whittaker functions (4.5a) as self-reciprocating under the corresponding hyperbolic canonical transforms.[65] Since the $\mathbf{J}_{-}$ eigenfunctions are simple Dirac deltas, we obtain[66]

$$ \begin{aligned} {}^{-,0}D^{\epsilon,k}_{\rho,m}\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= \left({}^{-}\Psi_{\rho}, \mathbb{C}^{\epsilon,k}\begin{pmatrix}a & b\\ c & d\end{pmatrix} {}^0\Psi^{\epsilon,k}_{m}\right) = \left[\mathbb{C}^{\epsilon,k}\begin{pmatrix}a & b\\ c & d\end{pmatrix} {}^0\Psi^{\epsilon,k}_{m}\right]_{\operatorname{sgn}\rho}(|\rho|) = {}^{+,0}D^{\epsilon,k}_{\rho,m}\begin{pmatrix}c & d\\ -a & -b\end{pmatrix} \\ &= (-\operatorname{sgn}\rho)^{m-\epsilon}\left(\frac{a-ib}{a+ib}\right)^{m}\frac{g_{\epsilon}(k)}{\pi|\rho|^{1/2}}\exp\left(\frac{\rho^2[a-ib\operatorname{sgn}\rho][d-ic\operatorname{sgn}\rho]}{2(a^2+b^2)}\right) \\ &\times\left[\left\{\Gamma(1-2k)\left[\frac{2\Gamma(k-m_\rho)}{\Gamma(1-k-m_\rho)}\right]^{1/2}\left[\frac{\rho^2}{a^2+b^2}\right]^{k} \right.\right. \\ &\left.\left.\times\, {}_1F_1\left[\begin{matrix}k-m_\rho\\ 2k\end{matrix};\frac{\rho^2}{a^2+b^2}\right]\right\} + \{k\leftrightarrow 1-k\}\right], \\ & m_\rho = m\operatorname{sgn}\rho. \end{aligned} \tag{4.7} $$

The overlap coefficient between the E(1)$_-$ and SO(2)$_0$ subgroup chains is easily found from (4.7) for $\mathbf{g} = \mathbf{1}$ and is ${}^0\Psi^{\epsilon,k}_{m,\mathrm{sgn}\rho}(|\rho|)$. This change of basis thus represents basically the Whittaker series expansion ($m\in Z$) of a function of $\rho\in R$.

ii. SO(1,1)$\subset$*SL*(2,*R*)$\supset$*SO*(2). The evaluation of this mixed-basis matrix element will be given in some detail because the method presented here has been used to obtain all the matrix elements carrying SO(1,1) reductions, both in the continuous and in the discrete series in the last section, where its discussion was postponed. The method[32] essentially consists of a Taylor expansion of $[\mathbb{C}^k_{\mathbf{g}}\, {}^0\Psi^{\epsilon,k}_m](r)$ followed by a Mellin–Barnes transformation.

The Taylor expansion of the Gaussian and ${}_1F_1$ functions appearing in (4.7) [for $|\rho|\mapsto r$ and $\mathrm{sgn}\,\rho\mapsto j$] yields, after an exchange of summations which allows us to recognize one of them as a ${}_2F_1$ series,

$$[\mathbb{C}^k_{\mathbf{g}}\, {}^0\Psi^{\epsilon,k}_m]_j(r) = (-j)^{m-\epsilon}\left(\frac{a-ib}{a+ib}\right)^m \frac{g_\epsilon(k)}{\pi}\,[2\Gamma(k-jm)\Gamma(1-k-jm)]^{1/2} \times [X^j_k + X^j_{1-k}], \qquad (4.8)$$

where

$$X^j_k = \left(-\frac{q_j}{t}\right)^{1/2-jm}\left(\frac{r}{|\bar a|}\right)^{1/2} \times \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1-2k-n)(-q_jr^2)^{k-1/2+n}}{n!\,\Gamma(1-k-jm-n)} \times {}_2F_1\left[\begin{matrix}k-jm,\ 1-k-jm\\ 1-k-jm-n\end{matrix};\ 1+\frac{q_j}{t}\right], \qquad (4.9a)$$

and where we are using the abbreviations from (3.6b) for $\bar a$ and $\bar c$, and

$$q_j = -(1-ij\bar a\,\bar c)/2\bar a^2,\quad t-1/\bar a^2,\quad j=\pm 1. \qquad (4.9b)$$

The terms in the sum over n are now recognized as the residues, at $z=z_n = -k-n,\ -1+k-n,\ (n=0,1,2,\ldots)$ of the following meromorphic function:

$$\chi^j(z) = \left(-\frac{q_j}{t}\right)^{1/2-jm}\left(\frac{r}{|\bar a|}\right)^{1/2}(-q_jr^2)^{-1/2-z} \times \frac{\Gamma(k+z)\Gamma(1-k+z)}{\Gamma(1+z-jm)} \times {}_2F_1\left[\begin{matrix}k-jm,\ 1-k-jm\\ 1+z-jm\end{matrix};\ 1+\frac{q_j}{t}\right]. \qquad (4.10)$$

Since for fixed ζ, $\Gamma(c)^{-1}\,{}_2F_1(a,b;c;\zeta)$ is an entire function of the parameters, $\chi^j(z)$ is a meromorphic function falling to zero rapidly as $|z|\to\infty$ in the region $\mathrm{Re}\,z<0$. The singularities of $\chi^j(z)$ are simple poles arising from the Gamma functions in the factor $\Gamma(k+z)\Gamma(1-k+z)$ and are located at the points $z=z_n$.

For the nonexceptional UIRs, $k-\frac12$ is pure imaginary and the poles lie symmetrically with respect to the real axis. For the exceptional UIRs k is real, but no two pole points z_n are coincident.

If we now choose a closed contour $\mathscr{C}$ consisting of the infinite semicircle $\mathscr{S}$ on the left, and the imaginary axis, we obtain

$$\frac{1}{2\pi i}\oint_{\mathscr{C}} dz\,\chi(z) = \sum_{n=0}^{\infty}\mathrm{Res}[\chi(z)]_{z=-k-n} + \sum_{n=0}^{\infty}\mathrm{Res}[\chi(z)]_{z=-1+k-n}. \qquad (4.11)$$

The first and second terms on the right-hand side, by our previous analysis, are respectively equal to X^j_k and X^j_{1-k} and hence the integral in (4.11) vanishes on $\mathscr{S}$, as can be easily verified. We obtain

$$X^j_k + X^j_{1-k} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,\chi^j(-i\lambda). \qquad (4.12)$$

This expression, replaced in (4.8), represents the solution of the problem of finding the integral of ${}^2\Psi_{\kappa,\mu}(r)$ with it, since the latter integral is essentially the Mellin transform of (4.8), integrated over r for the value $-\mu$; we note that (4.12) is expressed as an inverse Mellin transform of the coefficient (function of λ) of the $r^{-1/2+2i\lambda}$ factor in (4.10). The value of this coefficient for $z=-\mu$ and summed over the two j components will be the inner product of ${}^2\Psi_{\kappa,\mu}$ with (4.8). We thus obtain[67]

$$\begin{aligned}{}^{2,0}D^{\epsilon,k}_{\kappa,\mu;m}\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= \left({}^2\Psi_{\kappa,\mu},\ \mathbb{C}^{\epsilon,k}\begin{pmatrix}a & b\\ c & d\end{pmatrix}{}^0\Psi^{\epsilon,k}_m\right)\\ &= {}^{1,0}D^{\epsilon,k}_{\kappa,\mu;m}\begin{pmatrix}2^{-1/2}(a-c) & 2^{-1/2}(b-d)\\ 2^{-1/2}(a+c) & 2^{-1/2}(b+d)\end{pmatrix}\\ &= g_\epsilon(k)\pi^{-3/2}\Gamma(k-i\mu)\Gamma(1-k-i\mu)(a+ib)^{-m-i\mu}(a-ib)^{m-i\mu}\\ &\quad\times\sum_{j=\pm1}(-j)^{m-\epsilon}\kappa^{(1-j)/2}\frac{[\Gamma(k-jm)\Gamma(1-k-jm)]^{1/2}}{\Gamma(1-i\mu-jm)}\\ &\quad\times{}_2F_1\left[\begin{matrix}k-i\mu,\ 1-k-i\mu\\ 1-i\mu-jm\end{matrix};\ \tfrac12(a-ijb)(d-ijc)\right].\end{aligned} \qquad (4.13)$$

The overlap coefficient between these two chains[68] in the continuous series is obtained by setting $\mathbf{g}=\mathbf{1}$:

$$({}^2\Psi_{\kappa,\mu},\ {}^0\Psi^{\epsilon,k}_m) = g_\epsilon(k)\pi^{-3/2}\Gamma(k-i\mu)\Gamma(1-k-i\mu)\sum_{j=\pm1}(-j)^{m-\epsilon}\kappa^{(1-j)/2}\frac{[\Gamma(k-jm)\Gamma(1-k-jm)]^{1/2}}{\Gamma(1-i\mu-jm)} \times {}_2F_1\left[\begin{matrix}k-i\mu,\ 1-k-i\mu\\ 1-i\mu-jm\end{matrix};\ \tfrac12\right]. \qquad (4.14)$$

iii. $E(1)\subset SL(2,R)\supset SO(1,1)$. As in all cases involving E(1), the calculation here consists in applying $\mathbf{C}_{\mathbf{g}}^{\epsilon,k}$ on ${}^2\Psi_{\kappa,\mu}$, that is, performing the integral in

$$[\mathbf{C}_{\mathbf{g}}^{\epsilon,k}\,{}^2\Psi_{\kappa,\mu}]_j(r)=\sum_{j'=\pm 1}\int_0^\infty dr'\,[\mathbf{C}_{\mathbf{g}}^{\epsilon,k}]_{jj'}(r,r')\,{}^2\Psi_{\kappa,\mu,j'}(r'), \tag{4.15}$$

of the kernel $[\mathbf{C}_{\mathbf{g}}^{\epsilon,k}]_{jj'}(r,r')$ with the Mellin basis function. We resort to the expansion of the hyperbolic canonical transform kernel in Taylor series and to the Mellin–Barnes contour deformation presented above. We obtain

$$[\mathbf{C}_{\mathbf{g}}^{\epsilon,k}\,{}^2\Psi_{\kappa,\mu}]_j(r)=A_{\mathbf{g};\kappa,\mu,j}^{\epsilon,k}(r)+\kappa B_{\mathbf{g};\kappa,\mu,j}^{\epsilon,k}(r), \tag{4.16}$$

where

$$A\begin{pmatrix}a&b\\c&d\end{pmatrix}_{\kappa,\mu,1}^{\epsilon,k}(r)=\kappa p_\epsilon A\begin{pmatrix}-a&b\\c&-d\end{pmatrix}_{\kappa,\mu,-1}^{\epsilon,k}(r)=\frac{(\mathrm{sgn}b)^{2\epsilon}g_\epsilon(k)}{(2\pi)^{3/2}|b|}\left(\frac{-ia}{2b}\right)^{-1/2-i\mu}r^{1/2}\exp\left(\frac{idr^2}{2b}\right)$$
$$\times\left[p_\epsilon\left\{\Gamma(1-2k)\Gamma(k+i\mu)\left(\frac{ir^2}{2ab}\right)^{k-1/2}{}_1F_1\left[\begin{matrix}k+i\mu\\2k\end{matrix};\frac{-ir^2}{2ab}\right]\right\}+\{k\leftrightarrow 1-k\}\right], \tag{4.17}$$

$$B\begin{pmatrix}a&b\\c&d\end{pmatrix}_{\kappa,\mu,1}^{\epsilon,k}(r)=\kappa p_\epsilon B\begin{pmatrix}-a&b\\c&-d\end{pmatrix}_{\kappa,\mu,-1}(r)=\frac{(\mathrm{sgn}\,b)^{2\epsilon}g_\epsilon(k)}{(2\pi)^{3/2}|b|}\left(\frac{ia}{2b}\right)^{-1/2-i\mu}r^{1/2}\exp\left(\frac{idr^2}{2b}\right)$$
$$\times\left[\left\{\Gamma(1-2k)\Gamma(k+i\mu)\left(\frac{-ir^2}{2ab}\right)^{k-1/2}{}_1F_1\left[\begin{matrix}k+i\mu\\2k\end{matrix};\frac{-ir^2}{2ab}\right]\right\}+\{k\leftrightarrow 1-k\}\right]$$
$$=\frac{(\mathrm{sgn}b)^{2\epsilon+1}g_\epsilon(k)}{(2\pi)^{3/2}}\left(\frac{ia}{2b}\right)^{-i\mu}r^{-1/2}\exp\left(\frac{ir^2}{4ab}[ad+bc]\right)\Gamma(k+i\mu)\Gamma(1-k+i\mu)W_{-i\mu,k-1/2}(-ir^2/2ab), \tag{4.18}$$

which come, respectively, from the Mellin transforms of the on- and off-diagonal integral kernel elements. We remind the reader again that the complex power functions are to be evaluated inthe principal sheet.

Since the $E(1)_-$ basis has simple Dirac deltas, we immediately obtain[69]

$$^{-,2}D_{\rho;\kappa,\mu}^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\left({}^-\Psi_\rho,\,\mathbb{C}^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}{}^2\Psi_{\kappa,\mu}\right)$$
$$=A\begin{pmatrix}a&b\\c&d\end{pmatrix}_{\kappa,\mu,\mathrm{sgn}\rho}^{\epsilon,k}(|\rho|)+B\begin{pmatrix}a&b\\c&d\end{pmatrix}_{\kappa,\mu,\mathrm{sgn}\rho}^{\epsilon,k}(|\rho|). \tag{4.19}$$

The overlap coefficient between these two chains may be obtained upon letting $\mathbf{g}\to\mathbf{1}$, or directly as

$$({}^-\Psi_\rho,{}^2\Psi_{\kappa,\mu})={}^2\Psi_{\kappa,\mu,\mathrm{sgn}\rho}(|\rho|). \tag{4.20}$$

C. The matrix elements in the subgroup bases

i. $E(1)\subset SL(2,R)$. The integral kernel representations of SL(2, R) in this chain are given by the hyperbolic canonical transform integral kernel, which we may rewrite in terms of the confluent hypergeometric function as follows:

$$^-D_{\rho,\rho'}^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\left({}^-\Psi_\rho,\,\mathbb{C}^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}{}^-\Psi_{\rho'}\right)$$
$$=C^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}_{\mathrm{sgn}\rho,\mathrm{sgn}\rho'}(|\rho|,|\rho'|)$$
$$=(\mathrm{sgn}b)^{2\epsilon}p_\epsilon^{(1+\mathrm{sgn}\rho')/2}(\pi|b|)^{-1}g_\epsilon(k)$$
$$\times|\rho\rho'|^{1/2}\exp(i[dj\rho^2-2\eta\rho\rho'+aj'\rho'^2]/2b)$$
$$\times\left[\left\{\Gamma(1-2k)\left|\frac{\rho\rho'}{2b}\right|^{2k-1}\right.\right.$$
$$\times\left.{}_1F_1\left[\begin{matrix}2k-1/2\\4k-1\end{matrix};\frac{2i\rho\rho'}{\eta b}\right]\right\}$$
$$\left.+\{k\leftrightarrow 1-k\}\right], \tag{4.21}$$

where $\eta=1$ for $\mathrm{sgn}\rho=\mathrm{sgn}\rho'$, and $\eta=-i$ for $\mathrm{sgn}\rho\neq\mathrm{sgn}\rho'$. In particular, for the $b=0$ subgroup we have, as from Eqs. (2.15h),

$$^-D_{\rho,\rho'}^{\epsilon,k}\begin{pmatrix}a&0\\c&a^{-1}\end{pmatrix}$$
$$=(\mathrm{sgn}a)^{2k}|a|^{-1/2}\exp(i(\mathrm{sgn}\rho)c\rho^2/2a)\delta(\rho'-\rho/|a|). \tag{4.22}$$

In the $E(2)_+$ reduction, as in (3.17),

$$^+D_{\rho,\rho'}^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\left({}^+\Psi_\rho^{\epsilon,k},\,\mathbb{C}^{\epsilon,k}\begin{pmatrix}a&b\\c&d\end{pmatrix}{}^+\Psi_{\rho'}^{\epsilon,k}\right)$$
$$={}^-D_{\rho,\rho'}^{\epsilon,k}\begin{pmatrix}d&-c\\-b&a\end{pmatrix}. \tag{4.23}$$

ii. $SO(1,1)\subset SL(2,R)$. These matrix elements are essentially the Mellin transforms of (4.16)–(4.18), and can be obtained by the same technique[32] of Taylor expansion and Mellin–Barnes contour deformation. The Taylor expansion of, for example, the function (4.17) yields

$$A\begin{pmatrix}a&b\\c&d\end{pmatrix}_{\kappa,\mu,1}^{\epsilon,k}(r)$$
$$=\frac{(-\mathrm{sgn}b)^{2\epsilon}g_\epsilon(k)}{(2\pi)^{3/2}|b|}\left(\frac{-ir^2}{2ab}\right)^{-1/2-i\mu}r^{1/2}[Y_k+p_\epsilon Y_{1-k}], \tag{4.24}$$

with

$$Y_k=\exp(i\pi[2k-1][\alpha+\beta]/4)$$
$$\times\Gamma(1-2k)\Gamma(k+i\mu)|ad|^{1/2-k}$$
$$\times\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{-idr^2}{2b}\right)^{-1/2+k+n}{}_2F_1\left[\begin{matrix}-n,k+i\mu\\2k\end{matrix};\frac{1}{ad}\right], \tag{4.25}$$

where we denote for brevity $\alpha=\mathrm{sgn}(ab)$, $\beta=\mathrm{sgn}(bd)$. The terms in this series can be identified as the residues of the meromorphic function

$$\nu_k(z) = \Gamma(k+z)\left(\frac{-idr^2}{2b}\right)^{-1/2-z} {}_2F_1\left[\begin{matrix} k+z,\ k+i\mu \\ 2k \end{matrix};\frac{1}{ad}\right] \quad (4.26)$$

at the simple poles at $z = z_n = -k-n$. Through the same argument as in (4.9)–(4.12), we may express

$$Y_k = \exp(i\pi[2k-1][\alpha+\beta]/4)\Gamma(1-2k) \times\Gamma(k+i\mu)|ad|^{1/2-k}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\ \nu_k(-i\lambda). \quad (4.27)$$

As before, the function $\nu_k(z)$ on the integration contour in (4.27) contains the kernel $r^{-1/2+2i\lambda}$, so (4.27) is the inverse Mellin transform of the coefficient of that term in (4.26). The corresponding Mellin transform of B term (4.18) follows (4.24)–(4.27) with the same meromorphic function (4.26), but with different linear combination coefficients which originate from the corresponding coefficients in the two summands of (4.17) vs (4.18). We consequently find[70]

$$\begin{aligned} {}^2D^{\epsilon,k}_{\kappa,\mu,\kappa',\mu'}\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \left({}^2\Psi_{\kappa,\mu}, \mathbb{C}^{\epsilon k}\begin{pmatrix} a & b \\ c & d\end{pmatrix}{}^2\Psi_{\kappa',\mu'}\right) \\ &= (-\operatorname{sgn} b)^{2\epsilon}(2\pi)^{-2}g_\epsilon(k) \\ &\quad\times[(\tau_k+\kappa\kappa' p_\epsilon\tau_k^{-1}+\kappa'\theta_\kappa+\kappa p_\epsilon\theta_k^{-1})T_k \\ &\quad+(p_\epsilon\tau_{1-k}+\kappa\kappa'\tau_{1-k}^{-1}+\kappa'\theta_{1-k}+\kappa p_\epsilon\theta_{1-k}^{-1})T_{1-k}], \end{aligned} \quad (4.28a)$$

$$T_k = \Gamma(1-2k)\Gamma(k-i\mu)\Gamma(k+i\mu')|a|^{-k-i\mu'}|2b|^{i(\mu'-\mu)} |d|^{-k+i\mu}{}_2F_1\left[\begin{matrix} k-i\mu,\ k+i\mu' \\ 2k\end{matrix};\frac{1}{ad}\right], \quad (4.28b)$$

$$\tau_k = \exp(i\tfrac{1}{2}\pi[\{k+i\mu\}\operatorname{sgn} ab + \{k-i\mu\}\operatorname{sgn} bd]), \quad (4.28c)$$

$$\theta_k = \exp(i\tfrac{1}{2}\pi[-\{k+i\mu'\}\operatorname{sgn} ab + \{k-i\mu\}\operatorname{sgn} bd]). \quad (4.28d)$$

Whereas in the discrete series we are able to express the ${}^2D$ function as a meromorphic function in b, $-ia/b$, and $-id/b$ [c f. Eq. (3.19)] the corresponding continuous series functions do not have this property, and must be written in terms of powers of $|a|$, $|b|$, and $|d|$, with phase factors (4.28c) and (4.28d). This stems from the corresponding lack of meromorphicity of the hyperbolic canonical transform kernel (2.15d) and (2.15e), where the two Hankel functions are to be evaluated in the upper and lower half-planes, vis-á-vis the radial canonical transform kernel (2.5b), which is meromorphic in the group parameters. It has been pointed out before[21] that the continuous series UIRs cannot be subject to analytic continuation to a unitarizable representation of a subsemigroup of SL(2, C), such as may be done for the discrete series.[19]

Finally, it is easy to verify that our result is consistent with the expected behavior near the identity, namely

$${}^2D^{\epsilon,k}_{\kappa,\mu;\kappa',\mu'}\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = (\operatorname{sgn} a)^{2\epsilon}|a|^{-2i\mu}\delta_{\kappa,\kappa'}\delta(\mu-\mu'), \quad (4.29)$$

which acts as a reproducing kernel when we sum over κ and integrate over μ as in (4.3b). The Fourier transform case is

$${}^2D^{\epsilon k}_{\kappa,\mu;\kappa',\mu'}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = p_\epsilon\frac{g_\epsilon(i\mu)+\kappa g_\epsilon(k)}{\sin(\pi[k+i\mu])}2^{-2i\mu}\frac{\Gamma(k-i\mu)}{\Gamma(k+i\mu)}\delta_{\kappa,p,\kappa'}\delta(\mu+\mu'), \quad (4.30)$$

Remarks similar to those made on Eq. (4.28) apply here.

iii. SO(2)$\subset$*SL*(2,*R*). This matrix element should be obtained in the same way as the discrete series case given in Eq. (3.25a), with the basis functions which are now ${}^0\Psi^{\epsilon,k}_m(r)$ as given in (4.5a) [instead of the simpler ones ${}^0\Phi^k_m(r)$ in (3.5a)], and the inner product which is now the $\mathscr{L}^2_{11}(R^+)$ given in (2.14) [in place of the $\mathscr{L}^2(R^+)$ inner product (2.4)]. The application of the hyperbolic canonical transform $\mathbf{C}^{\epsilon,k}_g$ to ${}^0\Psi^{\epsilon,k}_m(r)$ is the exact analog of (3.6)–(3.7), namely, these functions are self-reciprocating[65] under $\mathbf{C}^{\epsilon,k}_g$. We can thus write

$$\begin{aligned} {}^0D^{\epsilon,k}_{m,m'}\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \left({}^0\Psi^{\epsilon,k}_m, \mathbb{C}^{\epsilon,k}\begin{pmatrix} a & b \\ c & d\end{pmatrix}{}^0\Psi^{\epsilon,k}_{m'}\right) \\ &= [(a-ib)/(a+ib)]^{m'}(a^2+b^2)^{-1/4}\sum_{j=\pm1}\int_0^\infty dr\, {}^0\Psi^{\epsilon,k}_{mj}(r)^* \\ &\quad\times\exp(ir^2[ac+bd]/2[a^2+b^2])\,{}^0\Psi^{\epsilon,k}_{m'j}(r/[a^2+b^2]^{1/2}) \\ &\left\{\begin{aligned} &= 2^{2m'}(m'!)^{-1}[\Gamma(k+m)\Gamma(1-k+m)/\Gamma(k+m')\Gamma(1-k+m')]^{1/2} \\ &\quad\times[(a+d)+i(b-c)]^{-m-m'}[(a-d)+i(b+c)]^{m-m'} \\ &\quad\times{}_2F_1(k-m', 1-k-m'; 1+m-m'; -\tfrac{1}{4}[a^2+b^2+c^2+d^2-2]),\quad m\geqslant m' \\ &= (-1)^{m'-m}2^{2m}(m!)^{-1}[\Gamma(k+m')\Gamma(1-k+m')/\Gamma(k+m)\Gamma(1-k+m)]^{1/2} \\ &\quad\times[(a+d)+i(b-c)]^{-m-m'}[(a-d)-i(b+c)]^{m'-m} \\ &\quad\times{}_2F_1(k-m, 1-k-m; 1+m'-m; -\tfrac{1}{4}[a^2+b^2+c^2+d^2-2]),\quad m\leqslant m' \end{aligned}\right\}. \end{aligned} \quad (4.31)$$

The right-hand term has been taken from Bargmann's work,[71] rewriting his phases and normalization constants, and using (A3) for the parameters. We have not been able to solve the integral in (4.31) directly: When we replace ${}^0\Psi^{\epsilon,k}_m(r)$ from (4.5a), we are confronted with a solution of a sum of two integrals whose integrands are each a product of two Whittaker functions, one of them with a rescaled argument, times an oscillating Gaussian function. This type of integral does not appear in the standard tables nor, apparently, does it yield easily to reduction to simpler forms. Bargmann's method of evaluation[38] of (4.31) does not make use of any explicit form of the basis functions $\Psi_m^{\epsilon,k}$. Instead, the function ${}^0D^{\epsilon,k}_{m,m'}(\mathbf{g})$ is shown to factorize into two exponentials of the first and third Euler angles, and the Bargmann-d function of the second Euler angle. The latter is subject to the differential relation stemming from (2.9) with $\mathbf{J}_\alpha$ expressed as operators on the group manifold. The condition ${}^0D^{\epsilon,k}_{m,m'}(\mathbf{1}) = \delta_{m,m'}$ provides the normalization and boundary conditions. This line of reasoning applies to any operator realization of the group belonging to that representation and subgroup reduction. The result provided by Bargmann[71] thus evaluates (4.31) and gives the solution for the integral. We can set $b = 0$, $a > 0$, and $r^2 = x$ in thus writing[72]

$$\sum_{j=\pm 1} |\Gamma(\tfrac{1}{2}+is-jm)\Gamma(\tfrac{1}{2}+is-jn)| \int_0^\infty dx\, x^{-1}\exp(icx/2a) W_{jm,is}(x) W_{jn,is}(x/a^2)$$
$$= \begin{Bmatrix} = \left(\frac{\pi}{g_\epsilon(k)}\right)^2 \frac{2^{2n}}{n!} \left|\frac{\Gamma(\frac{1}{2}+is+m)}{\Gamma(\frac{1}{2}+is+n)}\right| (a+a^{-1}-ic)^{-m-n}(a-a^{-1}+ic)^{m-n} \\ \times {}_2F_1(\tfrac{1}{2}+is-n, \tfrac{1}{2}-is-n; 1+m-n; -\tfrac{1}{4}[a^2+a^{-2}+c^2-2]), \quad m \geqslant n \\ = \left(\frac{\pi}{g_\epsilon(k)}\right)^2 \frac{2^{2m}}{m!} \left|\frac{\Gamma(\frac{1}{2}+is+n)}{\Gamma(\frac{1}{2}+is+m)}\right| (a+a^{-1}-ic)^{-m-n}(a-a^{-1}+ic)^{-m+n} \\ \times {}_2F_1(\tfrac{1}{2}+is-m, \tfrac{1}{2}-is-m; 1-m+n; -\tfrac{1}{4}[a^2+a^{-2}+c^2-2]), \quad m \leqslant n \end{Bmatrix} \tag{4.32}$$

where $\epsilon = 0(\frac{1}{2})$ for m, n integer (odd-half-integer), $g_\epsilon(k\,(s))$ is given by (2.15f) and (2.15g) and the range of s is, as above, $s > 0$ and $s = -i\sigma$, $0 \leqslant \sigma < \frac{1}{2}$ for $\epsilon = 0$.

D. The limits of continuous to discrete representations

i. $C_q^{1/2} \underset{q\to\frac{1}{4}}{\longrightarrow} D^+_{1/2} \dotplus D^-_{1/2}$. At the end of Sec. 2C we noted that the continuous series integral kernel $[\mathbf{C}_\mathbf{g}^{1/2,k}]_{j,j'}(r, r')$, for $k = \frac{1}{2} + is$, $s \to 0^+$, uncoupled in the sense of having its off-diagonal ($j \neq j'$) terms vanish. The hyperbolic canonical transform kernel becomes the direct sum of the $D^+_{1/2}$ radial canonical transform for the $j = 1$ component, and the $D^-_{1/2}$ one for the $j = -1$ component. In terms of the E(1) representation integral kernels,

$$ {}^-D^{1/2,\,1/2+is}_{\rho,\rho'}(\mathbf{g}) \underset{s\to 0^+}{\longrightarrow} \delta_{\mathrm{sgn}\rho,\mathrm{sgn}\rho'}\, {}^-D^{1/2(\mathrm{sgn}\,\rho)}_{|\rho|,|\rho'|}(\mathbf{g}), \tag{4.33}$$

as can be verified using (4.21) for the $C_q^{1/2}$ representation, (2.5b) for the $D^+_{1/2}$, and (3.23) for the $D^-_{1/2}$ representations. The SO(2) ⊂ SL(2, R) UIR matrices found by Bargmann follow (4.33) (replacing ρ, ρ' by m, m', and $-$ by 0). Indeed, after (4.7) we remarked that the E(1) ⊂ SL(2, R) ⊃ SO(2) overlap coefficient in the continuous series is ${}^0\Psi^{\epsilon,k}_{m,\mathrm{sgn}\rho}(|\rho|)$. From its functional form (4.5a) we can see that

$$ {}^0\Psi^{1/2,\,1/2+is}_{jm,j}(r) \underset{s\to 0^+}{\longrightarrow} j^{m-1/2}\, {}^0\Phi^{1/2}_m(r), \tag{4.34a}$$

$$ {}^0\Psi^{1/2,\,1/2+is}_{jm,-j}(r) \underset{s\to 0^+}{\longrightarrow} 0, \quad m = \tfrac{1}{2} + n, \quad n = 0, 1, 2, \ldots \tag{4.34b}$$

The continuous series UIR in the SO(2) basis thus also separates in block-diagonal form into the $D^+_{1/2}$ and $D^-_{1/2}$ representations:

$$ {}^0D^{1/2,\,1/2+is}_{m,m'}(\mathbf{g}) \underset{s\to 0^+}{\longrightarrow} \delta_{\mathrm{sgn}m,\mathrm{sgn}m'}\, {}^0D^{1/2(\mathrm{sgn}m)}_{|m|,|m'|}(\mathbf{g}). \tag{4.35}$$

The SO(1,1) subgroup-reduced integral kernels do separate, although not in block-diagonal form as in the former cases. The E(1) ⊂ SL(2, R) ⊃ SO(1,1) overlap coefficient in the continuous series (4.20) for $\mathbf{g} = \mathbf{1}$ are, in terms of those of the discrete series (3.14),

$$({}^-\Psi_\rho, {}^2\Psi_{\kappa,\mu}) = {}^2\Psi_{k,\mu,\mathrm{sgn}\rho}(|\rho|) = \begin{cases} 2^{-1/2}({}^-\Phi_{|\rho|}, {}^2\Phi_\mu), & \rho \geqslant 0 \\ \kappa 2^{-1/2}({}^-\Phi_{|\rho|}, {}^2\Phi_\mu), & \rho < 0, \end{cases} \tag{4.36}$$

and hence we obtain a sum of the $D^+_{1/2}$ and $D^-_{1/2}$ representations:

$$ {}^2D^{1/2,\,1/2+is}_{\kappa,\mu;\kappa',\mu'}(\mathbf{g}) \underset{s\to 0^+}{\longrightarrow} \frac{1}{2} \sum_{\tau=\pm 1} (\kappa\kappa')^{(1-\tau)/2}\, {}^2D^{1/2(\tau)}_{\mu,\mu'}(\mathbf{g}). \tag{4.37}$$

From this and the remark following (4.18) on the bilateral Mellin transform, it may appear more convenient to use $\mathbf{J}_2$ eigenfunctions whose dichotomic index label functions with upper or lower components only, instead of those used in (4.3a). This may be a useful alternative in some contexts, such as matching the two components of the bilateral Mellin transform kernel.[47] In some other cases, as in the study of an (uncoupled) hyperbolic Fourier transform class,[73] still another linear combination of the two ${}^-\Psi_\rho$ rows proves to be useful, as it diagonalizes the 2×2 kernel matrix.

ii. $C_q^0 \underset{q\to 0}{\longrightarrow} D_1^+ \dotplus D_1^-$. We also remarked at the end of Sec. 2D that the exceptional continuous series integral kernel $[\mathbf{C}_\mathbf{g}^{0k}]_{j,j'}(r, r')$ for $k = \frac{1}{2} + \sigma$, $\sigma \to (\frac{1}{2})^-$ also uncoupled into the D_1^+ and D_1^- radial canonical transform kernels:

$$ {}^-D^{0,\,1/2+\sigma}_{\rho,\rho'}(\mathbf{g}) \underset{\sigma\to(1/2)^-}{\longrightarrow} \delta_{\mathrm{sgn}\,\rho,\mathrm{sgn}\,\rho'}\, {}^-D^{1(\mathrm{sgn}\,\rho)}_{|\rho|,|\rho'|}(\mathbf{g}). \tag{4.38}$$

The significance of this limit is the same as for (4.33), and equations parallel to (4.34)–(4.37) follow for all other overlap coefficients and subgroup reductions. In particular, ${}^0\Psi_0^{0,1/2+\sigma}(r)$ vanishes as $\sigma \to (\frac{1}{2})^-$.

5. SL (2, R) TRANSFORMS AND SERIES

In Sec. 2 we introduced the SL(2, R) group of unitary k-canonical integral transforms for all UIR series of this group. The ensuing developments in Secs. 3 and 4 have detailed three families of bases for these spaces, associated with the E(1), SO(1, 1), and SO(2) families of subgroup reductions, and have given their overlap coefficients. These define as many families of integral transforms and series expansions.

A. The discrete series

i. $E(1)\subset SL(2,R)\supset E(1)$. For the discrete series, we can write in terms of the $\mathscr{L}^2(R^+)$ inner product and E(1) basis functions (3.2)

$$({}^-\Phi_r, f) = f(r), \quad r\in R^+. \tag{5.1a}$$

The k-radial canonical transform may be thus implemented as a change of coordinates

$$f(r) \xrightarrow{g} f_{\mathbf{g}}(r) = [\mathbb{C}^k_{\mathbf{g}} f](r) = ({}^-\Phi_r, \mathbb{C}^k_{\mathbf{g}} f)$$

$$= (\mathbb{C}^k_{\mathbf{g}^{-1}} {}^-\Phi_r, f) = \int_0^\infty dr' \, {}^-D^k_{r,r'}(\mathbf{g}) f(r'), \tag{5.1b}$$

from the Dirac-orthonormal E(1) eigenbasis $\{{}^-\Phi_r\}_{r\in R^+}$ to a similar family of bases $\{\mathbb{C}^k_{\mathbf{g}^{-1}} {}^-\Phi_r\}_{r\in R^+}$ of generalized eigenfunctions of $\mathbb{C}^k_{\mathbf{g}^{-1}} J_- \mathbb{C}^k_{\mathbf{g}}$, for every fixed $\mathbf{g}\in SL(2,R)$. The UIR matrix elements are the radial canonical transform kernels, as has been noted before. The transform inverse to (5.1b) has a kernel ${}^-D^k_{r,r'}(\mathbf{g}^{-1}) = [D^k_{r',r}(\mathbf{g})]^*$. The unitarity of the transform implies the Parseval identity (f, h) $= (f_{\mathbf{g}}, h_{\mathbf{g}})$. In particular, it contains the Hankel transform of $\mathbf{g} = \mathbf{F}$ [Eq. (2.11b)].

ii. $E(1)\subset SL(2,R)\supset SO(1,1)$. In the point of view we are developing in this section, the coordinates of f in the $SO(1,1)_2$ eigenbasis $\{{}^2\Phi_\mu\}_{\mu\in R}$ are

$$\begin{aligned}\hat{f}(\mu) &= ({}^2\Phi_\mu, f)\\ &= \int_0^\infty dr ({}^2\Phi_\mu, {}^-\Phi_r)({}^-\Phi_r, f)\\ &= \int_0^\infty dr\, \pi^{-1/2} r^{-1/2-2i\mu} f(r)\\ &= 2^{1/2} f^M_+(2\mu),\end{aligned} \tag{5.2a}$$

where f^M_+ is the positive Mellin transform[47] of f. The family of SL(2, R)-similar Dirac bases $\{\mathbb{C}^k_{\mathbf{g}^{-1}} {}^2\Phi_\mu\}_{\mu\in R}$ defines a corresponding SL(2, R)-parametrized family of integral transforms between $\mathscr{L}^2(R^+)$ and $\mathscr{L}^2(R)$,

$$f(r) \xrightarrow{(M)\mathbf{g}} \hat{f}^k_{\mathbf{g}}(\mu) = ({}^2\Phi_\mu, \mathbb{C}^k_{\mathbf{g}} f)$$

$$= (\mathbb{C}^k_{\mathbf{g}^{-1}} {}^2\Phi_\mu, f) = \int_0^\infty dr\, {}^{2,-}D^k_{\mu,r}(\mathbf{g}) f(r), \tag{5.2b}$$

whose kernel (3.11) contains in general a confluent hypergeometric function, with μ in one index and r in the argument. In particular, it contains the positive Mellin transform (5.2a) for $\mathbf{g} = \mathbf{1}$. The transform inverse to (5.2b) has a kernel ${}^{-,2}D^k_{r,\mu}(\mathbf{g}^{-1}) = [{}^{2,-}D^k_{\mu,r}(\mathbf{g})]^*$ and the integration is performed over $\mu\in R$. An obvious Parseval identity holds between (f, h) and $\hat{f}^k_{\mathbf{g}}(\mu)^* \hat{h}^k_{\mathbf{g}}(\mu)$ integrated over μ.

iii. $E(1)\subset SL(2,R)\supset SO(2)$. The coordinates of f in the SO(2)$\subset$SL(2, R)-similar eigenbases $\{\mathbb{C}^k_{\mathbf{g}^{-1}} {}^0\Phi^k_m\}_{m=k}^\infty$ define a mapping between $\mathscr{L}^2(R^+)$ and l^2_+ (lower-bound square-summable sequences):

$$f(r) \xrightarrow{(L)\mathbf{g}} f^k_{\mathbf{g},m} = ({}^0\Phi^k_m, \mathbb{C}^k_{\mathbf{g}} f)$$

$$= (\mathbb{C}^k_{\mathbf{g}^{-1}} {}^0\Phi^k_m, f) = \int_0^\infty dr\, {}^{0,-}D^k_{m,r}(\mathbf{g}) f(r), \tag{5.3}$$

which contains, essentially, the normalized Laguerre series analysis [in $L^{(2k-1)}_{m-k}(r^2)$] of $f(r)$ for $\mathbf{g} = \mathbf{1}$. The series synthesis is provided by the functions ${}^{-,0}D^k_{r,m}(\mathbf{g}^{-1}) = [{}^{0,-}D^k_{m,r}(\mathbf{g})]^*$ and a corresponding Parseval identity holds.

iv. $SO(1,1)\subset SL(2,R)\supset SO(2)$. We may also use the overlap coefficients between the SO(1, 1) and SO(2) bases to define the expansion of an $\mathscr{L}^2(R)$ function $\hat{f}(\mu)$ in a series of hypergeometric functions of argument $\frac{1}{2}$, as given by (3.10a), or its generalization for any fixed argument as given by (3.9), through the analysis

$$\hat{f}(\mu) \xrightarrow{(H)\mathbf{g}} \hat{f}^k_{\mathbf{g},m} = \int_{-\infty}^\infty d\mu\, {}^{0,2}D^k_{m,\mu}(\mathbf{g}) \hat{f}(\mu) \tag{5.4}$$

and the corresponding synthesis with $[{}^{0,2}D^k_{m,\mu}(\mathbf{g})]^*$, with an appropriate Parseval identity.

v. $SO(1,1)\subset SL(2,R)\supset SO(1,1)$. The SO(1, 1) subgroup decomposition of the discrete UIR series provides an SL(2, R)-parametrized family of unitary integral transforms between $\mathscr{L}^2(R)$ and itself,

$$\hat{f}(\mu) \xrightarrow{(F),\mathbf{g}} \hat{f}^k_{\mathbf{g}}(\mu) = \int_{-\infty}^\infty d\mu'\, {}^2D^k_{\mu,\mu'}(\mathbf{g}) \hat{f}(\mu'), \tag{5.5}$$

with a kernel involving hypergeometric functions of fixed argument, as given by (3.17). This is basically the Mellin transform of the k-radial canonical transform family (5.1).

vi. $SO(2)\subset SL(2,R)\supset SO(2)$. The SO(2) subgroup decomposition, finally, provides an SL(2, R)-parametrized family of mappings of discrete unitary transforms between l^2_+ and l^2_+ which repesents the well-known action of the group—for a fixed element $\mathbf{g}$ and k—on the space of sequences $\{f_m\}_{m=k}^\infty$.

The SL(2, R) D^+_k UIR matrix elements of the discrete series thus provide six different SL(2, R)-parametrized families of integral or discrete transforms, or series expansions between $\mathscr{L}^2(R^+)$, $\mathscr{L}^2(R)$, and l^2_+, of which the k-canonical radial transforms given in Sec. 2 are but one family.

B. The continuous series

The same pattern of six families of transforms hold for the continuous series of SL(2, R) UIRs, between spaces $\mathscr{L}^2_{\mathrm{II}}(R^+)$ [extendable to $\mathscr{L}^2(R)$ through $f(\rho) = f_{\mathrm{sgn}\rho}(|\rho|)$], $\mathscr{L}^2_{\mathrm{II}}(R)$ and l^2. These families include the k-hyperbolic canonical transforms given in Sec. 2, bilateral Mellin transforms, Whittaker and hypergeometric series and transforms.

C. Further extensions

Since these six families of transforms have a group-theoretical origin and parametrization, pairs of transforms belonging to one or two families (with the same k) may be applied in succession, respecting the mixed-basis transitivity properties, to give another transform of the same or of a different family. These are transforms which are all associated with the SL(2, R) group and its representations, so we would like to close our account of these with some comments on further extensions to this set, which have been published in the literature, and to other sets as yet not fully explored.

The first extension pertains consideration of the coverging group $\overline{\mathrm{SL}(2,R)}$. Indeed, the oscillator (metaplectic) representation is the two-fold covering of SL(2, R) [four-fold covering of SO(2, 1)] provided by $D^+_{1/4} \dot{+} D^+_{3/4}$. The case D^+_k, for real $k > 0$, has been described in Refs. 19, 20, and 34, but as yet it has not been as thoroughly analyzed as would be desirable. The continuous series of SL(2, R) have not been treated, although partial results exist. The subject of complex extensions of SL(2, R) to a semigroup of integral transforms,[17,19,28] possible for the discrete series—which includes the bilateral Laplace, Gauss–Weierstrass (heat diffusion), Bargmann[74] and Barut-Girardello[75] transforms—and the extension of SL(2, R) to W $\wedge$ SL(2, R) (W being the Heisenberg–Weyl group), has not been touched upon in this work, as it falls outside the scope of the title. Parts of it have appeared in various articles by one of the authors,[76] but the description of this last extension in various subgroup—and mixed bases is still wanting. Finally, the subject of nonsubgroup decompositions[77] in this context is still open.

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APPENDIX: THE UNITARY IRREDUCIBLE REPRESENTATIONS OF SL(2, R)

Bargmann[1] classified all UIRs of SU(1, 1) $\approx$ SL(2, R) $\approx$ Sp(2, R) $\overset{1:2}{\approx}$ SO(2, 1). We give here a summary of the results, nomenclature, and notation followed in this article.

We denote by SL(2, R) the special linear group in two dimensions over the real field, i.e., the group of 2×2 matrices

$$\mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a, b, c, d \in R, \quad \det\mathbf{g} = ad - bc = 1. \tag{A1}$$

Due to the unimodularity condition, (A1) also satisfy $\mathbf{g}\boldsymbol{\sigma}_p\mathbf{g}^T = \boldsymbol{\sigma}_p$, $\mathbf{g}^T$ being the transpose of $\mathbf{g}$, with the symplectic metric matrix

$$\boldsymbol{\sigma}_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The elements of the real symplectic group Sp(2, R) are thus also given by $\mathbf{g}$ as in (A1). The "1 + 1" unimodular pseudounitary group SU(1, 1), on the other hand, is the set of unimodular 2×2 complex matrices $\mathbf{u}$ satisfying $\mathbf{u}\boldsymbol{\sigma}_3\mathbf{u}^\dagger = \boldsymbol{\sigma}_3$, $\mathbf{u}^\dagger$ being the adjoint (transpose, complex conjugate) of $\mathbf{u}$, with the metric matrix

$$\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to show that the most general form of $\mathbf{u}$ is

$$\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \; \alpha, \beta \in C, \; \det\mathbf{u} = |\alpha|^2 - |\beta|^2 = 1. \tag{A2}$$

The link between SL(2, R) and SU(1, 1) matrices which relates the results of this article with those of Bargmann is given by the similarity transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{W}\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}\mathbf{W}^{-1} = \begin{pmatrix} \mathrm{Re}\alpha + \mathrm{Re}\beta & -\mathrm{Im}\alpha + \mathrm{Im}\beta \\ \mathrm{Im}\alpha + \mathrm{Im}\beta & \mathrm{Re}\alpha - \mathrm{Re}\beta \end{pmatrix}, \tag{A3a}$$

$$\mathbf{W} = 2^{-1/2}\begin{pmatrix} \omega^{-1} & \omega^{-1} \\ -\omega & \omega \end{pmatrix}, \quad \omega = e^{i\pi/4}. \tag{A3b}$$

Other isomorphisms found in the literature are determined by W's such as

$$2^{-1/2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \; 2^{-1/2}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix},$$

and

$$2^{-1/2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The latter yields the complex conjugate of (A3a). The 2:1 homomorphism between SU(1, 1) and the Lorentz group SO(2, 1) is often exploited through parametrizing the former in terms of Euler angles,

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} e^{-i\mu} & 0 \\ 0 & e^{i\mu} \end{pmatrix}\begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix}\begin{pmatrix} e^{-i\nu} & 0 \\ 0 & e^{i\nu} \end{pmatrix}. \tag{A4}$$

Our favored set of parameters are those in (A1), and in terms of those we express the UIR matrix elements. Of particular interest to many authors are the representations of the hyperbolic rotation (boost) subgroup in the middle factor of (A4). This is given by $\mathbf{M}_2(-2\zeta)$ in (2.10b).

Out of the matrix realization (A1)–(A2) Bargmann[1] finds the sl(2, R) Lie algebra. Without having to realize the algebra elements through differential operators, but only under the assumption of the existence of a Hilbert space endowed with a sesquilinear positive-definite inner product, one can find the self-adjoint irreducible representations of the algebra classified through the eigenvalues q of the Casimir operator (2.9), and through the usual raising- and lowering-operator techniques, the SO(2) representations m contained in any one SL(2, R) UIR are found.

The following are all nonequivalent single-valued representations of SL(2, R).

Discrete series $q = k(1-k)$ for $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ containing:

D^+_k positive discrete UIRs, $m = k, k+1, k+2, \ldots$
D^-_k negative discrete UIRs, $m = -k, -k-1, -k-2, \ldots$.

Continuous series

C^0_q the vector nonexceptional continuous UIRs
$q = k(1-k) \geqslant \frac{1}{4}$; $k = \frac{1}{2} + is$, $s \geqslant 0$,
C^0_q the (vector) exceptional continuous UIRs
$0 < q = k(1-k) < \frac{1}{4}$; $k = \frac{1}{2} + \sigma$, $0 < \sigma < \frac{1}{2}$,
$C^{1/2}_q$ the spinor (nonexceptional) continuous UIRs
$q = k(1-k) > \frac{1}{4}$; $k = \frac{1}{2} + is$, $s > 0$.

Values of k other than these give rise to nonunitary and/or multivalued representations of SL(2, R).

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[42]For example, see Ref. 28, Eqs. (9.75)–(9.77).
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[44]On the SU(1, 1) matrices of Bargmann, (A2), it acts through complex conjugation: See Ref. 1, Eqs. (9.1) and (9.20).
[45]Expressions (2.15) differ for $b<0$ from those presented in Ref. 11, Eqs. (2.15) and (2.16) and those in Ref. 21, Eq. (3.11) due to the fact that the integrals leading to the Hankel functions must take account of the appropriate Sommerfeld contour deformation: $0<\arg z<\pi$ for $H_\nu^{(1)}(z)$ and $-\pi<\arg z<0$ for $H_\nu^{(2)}(z)$. This implies that one should approach the real axis from above and below, respectively. This point was overlooked in the quoted references, and is quite crucial for subsequent calculations with these kernels.
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[61]Ref. 37, Eq. 9.220.4.
[62]Its meaning in the Sp(4, R) parent group, for the continuous UIR series, can be seen in Ref. 21, Eq. (2.9).
[63]Two of the four ${}_1F_1$ functions sum to a Whittaker W function [cf, Eq. (4.18b)]; the other two do not due to a phase difference. In Ref. 21, Eq. (4.19) the claim to reduce ${}^1\psi^{\epsilon,k}_{\kappa,\mu,j}(r)$ to a single W function is thus incorrect. This is one consequence of the imprecision in the phases of the hyperbolic canonical transform kernel of Eq. (3.11a) in Ref. 21 versus Eq. (2.15d) here.
[64]Ref. 1, Eqs. (6.22)–(6.26), (7.10)–(7.11), and (8.10)–(8.15).
[65]These results extend those presented in Ref. 55.
[66]Compare with Eq. (2.5) of Ref. 31 for the $SO(1,1)_2$ subgroup.
[67]Compare with Eq. (2.16) of Ref. 31 for the $SO(1,1)_2$ subgroup.
[68]This is the method and result of Ref. 32, Eq. (2.24). One can compare this result with Ref. 12, Eqs. (3.10), (3.24), and (3.25) after a ${}_2F_1$ transformation is used. The cases $\epsilon=0$ and $\epsilon=\frac{1}{2}$ are not distinguished there.
[69]Compare with Eqs. (2.26)–(2.27) of Ref. 31 for the $SO(1,1)_2$ subgroup. It should be noted that the symmetry $k\leftrightarrow 1-k$ (i.e., $\rho\leftrightarrow -\rho$ there) is not apparent. Caution should be excised as the dichotomic index in Ref. 31, and our κ are not the same.
[70]Compare with Ref. 12, Eqs. (3.18′)–(3.25). There is no indication of whether the $\epsilon=0$ or $\epsilon=\frac{1}{2}$ continuous series is being discussed.
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