Canonical transforms. III. Configuration and phase descriptions of quantum systems possessing an $sl(2,R)$ dynamical algebra

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The purpose of this article is to present a detailed analysis on the quantum mechnical level of the canonical transformation between coordinate-momentum and number-phase descriptions for systems possessing an $sl(2,R)$ dynamical algebra, specifically, the radial harmonic oscillator and pseudo-Coulomb systems. The former one includes the attractive and repulsive oscillators and the free particle, each with an additional "centrifugal" force, while the latter includes the bound, free and threshold states with an added "centrifugal" force. This is implemented as a unitary mapping—canonical transform—between the usual Hilbert space $\mathcal{L}^2$ of quantum mechanics and a new set of Hilbert spaces on the circle whose coordinate has the meaning of a phase variable. Moreover, the UIR's D_k^+ of the universal covering group of $SL(2,R)$ realized on the former space are mapped unitarily onto the latter.

1. INTRODUCTION

In this series of articles we have explored the question of canonical transformations in classical mechanics and their translation to quantum mechanics as unitary mappings between Hilbert spaces. These mappings have been given the general name of *canonical transforms*. In Ref. 1 we considered the set of (complex) linear transformations of phase space which preserved the Heisenberg algebra of coordinate and momentum variables (resp. operators) in classical (resp. quantum) mechanics, while in Ref. 2, upon examining the radial part of such an n-dimensional transformation, we found that the translation to quantum mechanics could be implemented asking for the preservation of a radial $sl(2,R)\approx su(1,1)\approx so(2,1)$ algebra built out of the n-dimensional underlying Heisenberg algebra. In this paper we will develop the unitary representation (canonical transform) of the transformation which can be formulated as follows.

Consider a classical system *possessing an $sl(2,R)$ dynamical algebra*. This means in our context that (i) there exist three quantities $\mathcal{J}_i(r,p_r)$, $i=1,2,3$ (where r and p_r are canonically conjugate variables: $\{r,p_r\}=1$) which under the Poisson bracket operation exhibit the $sl(2,R)$ Lie bracket relations

$$\{\mathcal{J}_1,\mathcal{J}_2\}=-\mathcal{J}_3,\quad \{\mathcal{J}_2,\mathcal{J}_3\}=\mathcal{J}_1,\quad \{\mathcal{J}_3,\mathcal{J}_1\}=\mathcal{J}_2, \tag{1.1}$$

and such that (ii) the Hamiltonian H of the system belongs to the algebra, i.e., it can be written as a linear combination of the $\mathcal{J}_i(r,p_r)$. Now, through $SL(2,R)$ group transformations, we can always redefine the basis of the algebra so that H coincides with *one* of the three *orbit representatives* given by $\mathcal{J}_3,\mathcal{J}_1$, or $\mathcal{J}_1+\mathcal{J}_3$ corresponding, respectively, to elliptic, hyperbolic, or parabolic orbits. In each one of these cases we can define as action and phase variables.

$$p_\phi=\mathcal{J}_3,\quad \phi=\arctan(\mathcal{J}_2/\mathcal{J}_1), \tag{1.2a}$$

$$p_\zeta=\mathcal{J}_1,\quad \zeta=\operatorname{arctanh}(\mathcal{J}_2/\mathcal{J}_3), \tag{1.2b}$$

$$p_\xi=\mathcal{J}_1+\mathcal{J}_3,\quad \xi=\mathcal{J}_2/(\mathcal{J}_1+\mathcal{J}_3), \tag{1.2c}$$

and in each of these cases one can verify that (1.1) implies that α and p_α ($\alpha=\phi,\zeta,\xi$) are canonically conjugate variables ($\{\alpha,p_\alpha\}=1$). The mapping $(r,p_r)\rightarrow(\alpha,p_\alpha)$ is a *canonical* transformation in the classical sense since the Heisenberg algebras are preserved, i.e., $\{r,p_r\}=1\quad\{\alpha,p_\alpha\}=1$, between the configuration and phase descriptions. The purpose of this article is to explore the quantum mechanical formulation of such canonical transformations. We shall see that the translation is possible when the Hamiltonian takes the standard from $\frac{1}{2}p_r^2+V(r)$ and the generators $\mathcal{J}_i(r,p_r)$ are *up-to-second order functions of p_r*. In this case (1.1) gives a set of coupled differential equations which severely restrict the types of potentials which can be considered, and in fact the possible realizations of the algebra (1.1) are essentially reduced to

$$\mathcal{J}_1=\tfrac{1}{4}(p_r^2-r^2+gr^{-2})=\tfrac{1}{4}(\mathbf{p}^2-\mathbf{r}^2+g|\mathbf{r}|^{-2}), \tag{1.3a}$$

$$\mathcal{J}_2=\tfrac{1}{2}rp_r=\tfrac{1}{2}\mathbf{r}\cdot\mathbf{p}, \tag{1.3b}$$

$$\mathcal{J}_3=\tfrac{1}{4}(p_r^2+r^2+gr^{-2})=\tfrac{1}{4}(\mathbf{p}^2+\mathbf{r}^2+g|\mathbf{r}|^{-2}), \tag{1.3c}$$

with arbitrary g, where $\mathbf{r}$ and $\mathbf{p}$ are n-dimensional vectors. The systems which can be described in this case are the attractive and repulsive harmonic oscillators and the free particle, all with an arbitrary additional "centrifugal" potential, corresponding to the elliptic, hyperbolic, and parabolic orbits mentioned above.

By *quantization* of (1.3) we mean the construction of self-adjoint operators on the usual Hilbert space of Lebesgue square-integrable functions $\mathcal{L}^2(R^n)$. This procedure is unique[3,4] for (1.3) and yields an $sl(2,R)$ algebra of operators $I_i(r,\partial_r)$ under the commutator bracket, self-adjoint in the "radial" space $\mathcal{L}^2(0,\infty)$. We will show in this article that we can perform a unitary mapping of $\mathcal{L}^2(0,\infty)$ onto Hilbert spaces $\mathcal{H}_k^+$ (to be described below) where the operators $\hat{p}_\alpha$ defined in (1.2) are realized as $-i\partial/\partial\alpha$. The difficulties of giving a meaning in quantum mechanics to (1.2) can be seen clearly for the harmonic oscillator case (1.2a) to stem from the following problems: (i) The operator $-i\partial/\partial\phi$ is required to have a *discrete* spectrum which is incompatible with the existence of a phase operator "$\hat{\phi}$" such that $[\hat{\phi},\hat{p}_\phi]=\mathbb{1}$. (ii) When the operator $\hat{p}_\phi$ is realized as $-i\partial/\partial\phi$ on $\mathcal{L}^2(-\pi,\pi)$, its spectrum turns out *not* to be positive-definite. The methods of treating these (and the related problem of angular momentum and angle observables) difficulties[5,6] have been through replacing the phase operator with some closely related ones, e.g., Toeplitz operators[7] such as $\sin\phi$ and $\cos\phi$, and/or constructing a representation of the Heisenberg algebra which cannot be integrated to the group.[6]

In our construction, Hilbert spaces are constructed so that $\hat{p}_\alpha$ is a self-adjoint operator represented by $-i\partial/\partial\alpha$ *with the appropriate spectrum*. The phase variable α retains the meaning of an underlying space. Its operator realization (multiplication by α) is not Hermitean. The $sl(2,R)$ algebra and group representations are preserved and take the place of the Heisenberg algebra and Weyl group respectively in the definition and determination of the quantum canonical transformation corresponding to (1.2), as a unitary mapping between Hilbert spaces. The integral transform realization of such a mapping is the associated *canonical transform*. Furthermore, the unitary mapping is implemented for the pseudo-Coulomb system with the classical generators[8–10]

$$\mathcal{K}_1 = \tfrac{1}{2}[r(\mathbf{p}^2 - 1) + g' r^{-1}], \qquad (1.4a)$$

$$\mathcal{K}_2 = \mathbf{r}\cdot\mathbf{p}, \qquad (1.4b)$$

$$\mathcal{K}_3 = \tfrac{1}{2}[r(\mathbf{p}^2 + 1) + g' r^{-1}], \qquad (1.4c)$$

by establishing the connection of this system with the harmonic oscillator. Although the complete dynamical groups for the two systems are different (the symplectic group $Sp(n,R)$ for the oscillator and $O(n,2)$ for the Coulomb system), the representations of the $\widetilde{SL}(2,R)$ subgroup are isomorphically related and appear to play a fundamental role in both systems.

The developments presented here have a group-theoretical significance of their own: On the algebra level, we connect the realization of the $sl(2,R)$ algebra generators on the line, as second-order differential operators, with their realization as first-order ones on the circle. On the group level, we relate the action of $\widetilde{SL}(2,R)$—the universal covering group of $SL(2,R)$—as conformal transformations of the circle with its nonlocal action on the line.

In Sec. 2 we construct the Hilbert spaces $\mathcal{H}_k^+$ where $\hat{p}_\alpha$ has the required properties and its unitary mapping to $\mathcal{L}^2(0,\infty)$. In Sec. 3 we relate the bound, free and threshold Coulomb systems with the three harmonic oscillator systems (1.2). In the Appendix we establish the connection between our spaces $\mathcal{H}_k^+$ and the spaces of analytic functions on the disk[11–13] and half-plane[14] used for the description of the $sl(2,R)$ D_k^+ unitary irreducible representations (UIR's).

2. THE HARMONIC OSCILLATOR SYSTEMS AND THE CIRCLE

A. Elliptic case

We begin with the quantum Hamiltonian for the n-dimensional harmonic oscillator with an extra "centrifugal" potential of strength g

$$H = \tfrac{1}{2}(-\nabla^2 + r^2 + g r^{-2}), \qquad (2.1)$$

where ∇^2 is the n-dimensional Laplacian and $0 \le r^2 = |\mathbf{r}^2| < \infty$. Since we are interested in the radial part of H only, we separate (2.1) and its eigenfunctions into their radial and angular variables and write in place of the angular part of (2.1) its well-known eigenvalues

$$\lambda = -L(L+n-2), \quad L = 0,1,2,\cdots, \qquad (2.2)$$

viz.

$$H = \tfrac{1}{2}\{-\partial_{rr} - [(n-1)/r]\,\partial_r + r^2 + (g-\lambda)/r^2\}. \qquad (2.3)$$

Now, the usual measure in n-dimensional radial configuration spaces is $r^{n-1}dr$; however, to facilitate our calculations, we can make the similarity transformation $H \to r^{(n-1)/2} H r^{-(n-1)/2}$, which brings the measure to simply dr with the corresponding formal differential operator

$$I_3 \equiv \tfrac{1}{2} r^{(n-1)/2} H r^{-(n-1)/2} = \tfrac{1}{4}\{-\partial_{rr} + r^2 + [(2k-1)^2 - \tfrac{1}{4}]/r^2\}, \qquad (2.4a)$$

where

$$2k = 1 \pm [(\tfrac{1}{2}n + L - 1)^2 + g]^{1/2}. \qquad (2.5)$$

Now, for $k \ge 1$, the spectral analysis of (2.4a) is well known[15] and there is a unique self-adjoint extension such that the normalized eigenvectors are

$$\psi_N^k(r) = [2N!/\Gamma(N+2k)]^{1/2} e^{-r^2/2} r^{2k-1/2} L_N^{(2k-1)}(r^2), \qquad (2.6a)$$

where

$$I_3 \psi_N^k(r) = (N+k)\psi_N^k(r), \quad N = 0,1,2,\cdots, \qquad (2.6b)$$

and where $L_N^{(\alpha)}(z)$ are the associated Laguerre polynomials.[16] In the case that $(2k-1)^2 < 1$, both solutions to the eigenvalue problem for I_3 are square-integrable in the neighborhood of $r = 0$, and we must implement an additional boundary condition there. In this article we are interested in exploring the eigenvalue problems for I_3 whose spectra are bounded from below corresponding to the discrete series of representations D_k^+ of $SL(2,R)$. This corresponds, for the spectral analysis of I_3 with $\tfrac{1}{2} \le k < 1$, to implementing two different boundary conditions which yield $\{\psi_N^k\}$ and $\{\psi_N^{-k+1}\}$ separately as complete sets of orthonormal eigenvectors. The second set can be described equivalently by extending the range of k to $0 < k < 1$. Indeed the richer structure displayed in this interval has been noticed by Sally[13] and Montgomery and O'Raifertaigh.[17] Other boundary conditions corresponding to different self-adjoint extensions of I_3 give rise to the supplementary series of $\widetilde{SL}(2,R)$.

We complete the Lie algebra of $\widetilde{SL}(2,R)$ by adding the generators[10,18–20]

$$I_1 = \tfrac{1}{4}\{-\partial_{rr} - r^2 + [(2k-1)^2 - \tfrac{1}{4}]/r^2\}, \qquad (2.4b)$$

$$I_2 = -\tfrac{1}{2} i (r\partial_r + \tfrac{1}{2}). \qquad (2.4c)$$

It is straightforward to verify that (2.4) satisfy the well-known commutation relations

$$[I_1, I_2] = -iI_3, \; [I_3, I_1] = iI_2, \quad [I_2, I_3] = iI_1 \qquad (2.7a)$$

and

$$I^2 = I_1^2 + I_2^2 - I_3^2 = k(1-k). \qquad (2.7b)$$

The common invariant domain where the operators (2.3) as well as the Lie products (2.7) are densely defined is taken as $\{f \in \mathcal{L}^2(0,\infty) : I_3^2 f \in \mathcal{L}^2(0,\infty)\}$. Furthermore, as discussed previously, the generators (2.4) can be integrated[20] to a unique unitary representation of $\widetilde{SL}(2,R)$. For the general element of $SL(2,R)$

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R), \quad ad-bc=1. \tag{2.8a}$$

$\widetilde{SL}(2,R)$ is defined from the universal covering group of the compact subgroup $SO(2)$. Explicitly, for the matrix

$$\begin{pmatrix} \cos\frac{1}{2}\omega & \sin\frac{1}{2}\omega \\ -\sin\frac{1}{2}\omega & \cos\frac{1}{2}\omega \end{pmatrix} \longleftrightarrow \exp(-i\omega I_3), \tag{2.8b}$$

we now allow $-\infty < \omega < \infty$. The other one-parameter subgroups are, with their corresponding representations,

$$\begin{pmatrix} \cosh\frac{1}{2}\alpha & \sinh\frac{1}{2}\alpha \\ \sinh\frac{1}{2}\alpha & \cosh\frac{1}{2}\alpha \end{pmatrix} \longleftrightarrow \exp(-i\alpha I_1), \tag{2.8c}$$

where $0 \leqslant \alpha < \infty$ and

$$\begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix} \longleftrightarrow \exp(-i\beta I_2). \tag{2.8d}$$

Associated with a general element of $SL(2,R)$ with $b \neq 0$ we have the group action[2,10,21]

$$(T_g f)(r) = |b|^{-1} \exp(\mp i\pi k \operatorname{sgn} b) \int_0^\infty dr'(rr')^{1/2} \times \exp\left(\frac{i}{2b}(ar'^2+dr^2)\right) J_{2k-1}\left(\frac{rr'}{|b|}\right) f(r'), \tag{2.9}$$

where $f \in \mathcal{L}^2(0,\infty)$ and $g \in SL(2,R)$. The integral is understood to be in the sense of limit in the mean. Equation (2.9) can be extended to the entire range of the parameter ω in (2.8b) and thus to the whole universal covering group $\widetilde{SL}(2,R)$ through $\exp(-2i\pi I_3) = \exp(-2i\pi k)$. When $b=0$, we have the *local* action

$$(T_g f)(r) = |a|^{-1/2} \exp[(ic/2|a|)r^2] f(|a|^{-1}). \tag{2.10}$$

We mention here that the ordinary $(g=0)$ n-dimensional oscillator of angular momentum L belongs to the UIR of $\widetilde{SL}(2,R)$ with $k=\frac{1}{2}L+\frac{1}{4}n$, i.e., $D^+_{L/2+n/4}$. For $n=3$, the oscillator states are spanned by the direct sum of UIR's $D^+_{3/4} \oplus D^+_{5/4} \oplus \cdots$. For the case $n=1$ $(g=0)$ the situation is somewhat different: The differential operator (2.4a) is no longer singular at the origin and the $O(n)$ rotational symmetry represented by the quantum number L is replaced by the two-element group C_2 of reflections, whose two representations are given by $L=0$ and 1 in (2.2). The corresponding $\widetilde{SL}(2,R)$ UIR's are $D^+_{1/4}$ and $D^+_{3/4}$ corresponding to even and odd functions respectively.

We shall now construct a unitary isomorphism of the Lie algebra $sl(2,R)$ and covering group $\widetilde{SL}(2,R)$ representations on $\mathcal{L}^2(0,\infty)$ onto the corresponding algebra and group representations on the circle S^1 with a suitably defined inner product. Our realization for the Lie algebra $sl(2,R)$ on S^1 is the algebra of formal differential operators[11,12]

$$I_3 = -i\partial_\phi, \quad I_\pm = e^{\pm i\phi}(-i\partial_\phi \pm k), \tag{2.11}$$

where $I_\pm \equiv I_1 \pm iI_2$ and I_1, I_2, I_3 satisfy (2.7). For the discrete series of UIR's D^+_k $(k \geqslant \frac{1}{2})$ of $\widetilde{SL}(2,R)$, the infinitesimal generators satisfy the well-known relations

$$I_3 g^k_m = m g^k_m, \tag{2.12a}$$

$$I_+ g^k_m = \vartheta_m[m(m+1)+k(1-k)]^{1/2} g^k_{m+1}, \tag{2.12b}$$

$$I_- g^k_m = \frac{1}{\vartheta_{m-1}}[m(m-1)+k(1-k)]^{1/2} g^k_{m-1}, \tag{2.12c}$$

where $|\vartheta_m|=1$, on a normalized set of basis vectors $\{g^k_m\}$ with the spectrum $m=k, k+1, \cdots$, thus $I_- g^k_k = 0$. Putting

$$g^k_m(\phi) = \gamma_m(k) e^{im\phi}, \tag{2.13a}$$

one can see after a straightforward calculation that

$$\gamma_m(k) = [\Gamma(m+k)/\Gamma(2k)\Gamma(m-k+1)]^{1/2} \tag{2.13b}$$

with $\vartheta_m = 1$.

We will now construct an inner product on S^1. This can be done by demanding that the $\{g^k_m\}$ form an orthonormal basis; however, we prefer to derive our inner product in the manner of Refs. 1 and 2, which elucidates the type of functions we are working with. We write down a general bilinear functional on a "nice" space of functions on S^1 and require the operators (2.11) to be Hermitean. It is easy to see that this inner product cannot be of the usual type for $\mathcal{L}^2(S^1)$, $(f,g) = \int_{-\pi}^{\pi} d\phi f(\phi)^* g(\phi)$ *unless* $k=\frac{1}{2}+i\rho$ with ρ real. This is the principal series of UIR's of $SL(2,R)$. Since we are treating the discrete series D^+_k this is not in general the case (except for $D^+_{1/2}$).

Now from the outset it is clear that we are dealing with multivalued representations, where the multivaluedness is determined by the real number $k \geqslant \frac{1}{2}$. We therefore consider the space $\mathcal{J}_k$ of infinitely differentiable functions on S^1 such that $f(\phi+2\pi) = \exp(2\pi ik) f(\phi)$. Furthermore, consider the space $\Phi(\mathcal{J}_k)$ of continuous linear functionals[22] on $\mathcal{J}_k$

$$\Omega(f) \equiv (\Omega, f) = \int_{S^1} d\phi' \Omega(\phi,\phi') f(\phi'). \tag{2.14}$$

We can define the inner product

$$(f_1,f_2)_k = (f_1, \Omega(f_2)) = \int\int_{S^1} d\phi\, d\phi' \Omega(\phi,\phi') f_1(\phi)^* f_2(\phi'). \tag{2.15}$$

The $\Omega(\phi,\phi')$ can be determined from the hermiticity conditions for the generators (2.11). First, demanding the hermiticity of I_3, i.e., $(I_3 f_1, f_2)_k = (f_1, I_3 f_2)_k$, we find the conditions

$$\Omega(\phi,\phi') = \Omega(\phi-\phi'), \quad \Omega(\phi+2\pi) = \exp(2\pi ik)\Omega(\phi). \tag{2.16}$$

Now any $f \in \mathcal{J}_k$ can be expanded uniformly in a Fourier series,

$$f(\phi) = \exp(ik\phi) \sum_{n=-\infty}^{\infty} a_n (2\pi)^{-1/2} \exp(in\phi), \tag{2.17}$$

and, by applying to it the lowering and raising operators $I_\pm$ it is clear that $\mathcal{J}_k$ is reducible since the subspace of functions $\mathcal{J}^+_k$ with Fourier coefficients $a_n=0$ for n negative is invariant under the action of (2.11). The space $\mathcal{J}_k$ is *not* completely reducible, however, but the restriction of the Lie algebra representation (2.11) to $\mathcal{J}^+_k$ is irreducible.

For any $f_1, f_2 \in \mathcal{J}^+_k$ consider the hermiticity conditions

$$(f_1, I_\pm f_2)_k = (I_\mp f_1, f_2)_k. \tag{2.18}$$

A straightforward calculation involving integrations by parts yields the condition

$$\int \int_{S^1} d\phi\, d\phi' f_1(\phi)^* e^{i\phi'} f_2(\phi')[i(e^{i\theta}-1)\Omega_\theta + (k-1)(e^{i\theta}+1)\Omega] = 0. \tag{2.19}$$

where $\theta \equiv \phi - \phi'$ and $\Omega_\theta = d\Omega/d\theta$. One is tempted to set the term in brackets in the integrand equal to zero and solve the resulting differential equation. Upon doing so, the solution is $\widetilde{\Omega}(\theta) = c(1-\cos\theta)^{k-1}$. We can verify that the Fourier expansion (2.17) of $\widetilde{\Omega}$ contains only *negative-n* partial wave coefficients and thus is a member of $\Phi(\widetilde{\mathcal{J}}_k^+)$, where $\widetilde{\mathcal{J}}_k^+$ is the complement of $\mathcal{J}_k^+$ in $\mathcal{J}_k$. Hence $(f_1, \widetilde{\Omega}(f_2)) = 0$ for *any* $f_1, f_2 \in \mathcal{J}_k^+$ and such a solution is worthless to us.

By inspecting (2.17) a bit closer it is seen that the vanishing of the terms in the square bracket of (2.19) is only a sufficient condition for the vanishing of the integral. Indeed, (2.17) is satisfied if the term in square brackets is orthogonal to $\exp(i\phi')f_2(\phi') \in \mathcal{J}_{k+1}^+$. So a *necessary and sufficient* condition for (2.17) to hold is

$$i[\exp(i\theta)-1]\Omega_\theta + (k-1)[\exp(i\theta)+1]\Omega = \widetilde{\omega}(\theta) + c\exp(ik\theta), \tag{2.20}$$

where $\widetilde{\omega}(\theta) \in \Phi(\widetilde{\mathcal{J}}_k^+)$ and c is a constant. Since any member of $\Phi(\widetilde{\mathcal{J}}_k^+)$ is useless to us as an inner product for $\mathcal{J}_k^+$ we discard $\widetilde{\omega}(\theta)$ and look for a solution $\Omega \in \Phi(\mathcal{J}_k^+)$ of

$$i[\exp(i\theta)-1]\Omega_\theta + (k-1)[\exp(i\theta)+1]\Omega = c\exp(ik\theta). \tag{2.21}$$

When we propose as a solution of (2.21) a series of the kind (2.17) with coefficients ω_n, we find this provides two independent solutions: One, for $n \geq 0$, yields the recursion $\omega_n = \omega_0 n!/(2k)_n$ in terms of the independent constant ω_0, while the second one, for $n < 0$, yields the recursion in terms of ω_{-1}. The latter series gives rise to $\widetilde{\Omega}$ and we thus discard it. The former series is thus our solution $\Omega \in \Phi(\mathcal{J}_k^+)$ and, choosing $\omega_0 = 1/4\pi^2$,

$$\begin{aligned}\Omega(\theta) &\equiv \sum_{m=k}^{\infty} \lambda_m(k)\exp(im\theta)\\ &= \frac{1}{4\pi^2}\sum_{N=0}^{\infty}\frac{N!}{(2k)_N}\exp[i(k+N)\theta]\\ &= \frac{1}{4\pi^2}\exp(ik\theta)F(1,1;2k;\exp(i\theta)).\end{aligned} \tag{2.22}$$

This series[16] converges absolutely for $k > 1$, conditionally for $\frac{1}{2} < k \leq 1$ (excluding $\theta = 0, 2\pi, \cdots$), and for $k = \frac{1}{2}$ it diverges on S^1. In the last case appropriate limiting arguments must be used in order to evaluate the double integral (2.15). For $0 < k < \frac{1}{2}$ the series (2.22) can still define a scalar product even though the series diverges.[13] Comparing the coefficients in (2.13) and (2.22), we find the important relation

$$\lambda_m(k) = [2\pi\gamma_m(k)]^{-2} \tag{2.23}$$

which quarantees that $\{g_m^k\}$ is an orthonormal set under the scalar product (2.15). Equation (2.23) would have defined $\lambda_m(k)$ in the series (2.22) had we decided to find Ω from the requirement that $\{g_m^k\}$ form under (2.15) an orthonormal basis.

Now consider the inner product (2.15). We have for any $f_1, f_2 \in \mathcal{J}_k^+$, after some integrations,

$$(f_1, f_2)_k = \sum_{m=k}^{\infty} a_m^* b_m \lambda_m(k), \tag{2.24}$$

where a_m and b_m are the ordinary Fourier coefficients for f_1 and f_2 respectively. We find from (2.22), for $k > \frac{1}{2}$ and $m = k+N$, N nonnegative integer, that $0 < 4\pi^2\lambda_m(k) < 1$ and $\lambda_m(k) \to 0$ as $m \to 0$ while $4\pi^2\lambda_m(\frac{1}{2}) = 1$. Thus the norm

$$0 < \|f\|_k^2 \equiv (f,f)_k = \sum_{m=k}^{\infty} |a_m|^2 \lambda_m(k) \leq \sum_{m=k}^{\infty} |a_m|^2 \leq \sum_{m=-\infty}^{\infty} |a_m|^2 < \infty \tag{2.25a}$$

is dominated by the Hardy–Lebesgue norm[22] H^2 as well as $\mathcal{L}^2(-\pi,\pi)$. The members of H^2 are the boundary values almost everywhere on S^1 of functions analytic in the unit disc $|z| < 1$ completed with respect to the norm

$$\|f\|_k^2 = \sup_{0<r<1} \int_{-\pi}^{\pi} d\phi\, |f(r\exp(i\phi))|^2 = \sum_{m=k}^{\infty} |a_m|^2. \tag{2.25b}$$

Thus, for $k = \frac{1}{2}$, closure gives the Hilbert space H^2. Notice also that when $0 < k < \frac{1}{2}$ the first inequality in (2.25a) is reversed; nevertheless the norm $\|f\|_k$ is defined by its series. Norms of this type were discussed by Sally[13] and are related to certain reproducing kernel spaces.

Using (2.23) and (2.24) we have

$$(f_1, f_2)_k = \sum_{m=k}^{\infty} (f_1, g_m^k)_k (g_m^k, f_2)_k \tag{2.26}$$

for $f_1, f_2 \in \mathcal{J}_k^+$. Indeed, from (2.25) we can extend (2.26) to all functions $f_1, f_2 \in H^2$. Now H^2 is not closed with respect to the norm $\|f\|_k$, but by adjoining the limit points we obtain a Hilbert space which we denote by $\mathcal{H}_k^+$. The connection between the Hilbert spaces $\mathcal{H}_k^+$ and those of analytic functions on the disc will be ellaborated upon in the Appendix.

Some further interesting properties of the linear functional $\Omega(f)$ defined by the kernel (2.22) can be seen by viewing Ω as a Hermitean operator on $\mathcal{L}^2(-\pi,\pi)$. It annihilates all $\widetilde{f} \in \widetilde{\mathcal{J}}_k^+$ and hence all members of $\mathcal{L}^2(-\pi,\pi)$ which are limits of such $\widetilde{f}$. For $k > \frac{1}{2}$ it is compact (completely continuous) and hence self-adjoint with eigenvalues $\lambda_m(k)$. Gel'fand and collaborators[14] have used such operators (for $k = 1, \frac{3}{2}, 2, \cdots$) to describe equivalences between representations labeled by k and $-k+1$.

Another linear functional in $\Phi(\mathcal{J}_k^+)$ which can be extended to all of $\mathcal{H}_k^+$ is the reproducing functional given by the formal series

$$K(\phi, \phi') = \sum_{m=k}^{\infty} g_m^k(\phi) g_m^k(\phi')^* = \exp[ik(\phi-\phi')](1-\exp[i(\phi-\phi')])^{-2k}. \tag{2.27}$$

Clearly this series diverges at $\phi = \phi'$, but nevertheless defines a continuous linear functional on $\mathcal{J}_k^+$, viz.,

$$f(\phi) = \int \int_{S^1} d\phi''\, d\phi'\, \Omega(\phi''-\phi') K(\phi, \phi'') f(\phi'). \tag{2.28}$$

We will now construct a unitary mapping which maps $\mathcal{L}^2(0,\infty)$ onto $\mathcal{H}_k^+$ and the infinitesmial generators (2.4) onto (2.11) and conversely. The statement that the Hilbert space $\mathcal{H}_k^+$ maps unitarily onto $\mathcal{L}^2(0,\infty)$ and conversely is almost trivial, since all separable Hilbert spaces are unitarily equivalent. We see easily that $\mathcal{L}^2(0,\infty)\approx l^2 \approx \mathcal{H}_k^+$, where l^2 denotes the space of generalized Fourier coefficients $\{c_N\}$, $N=0,1,2,\cdots$, such that $\sum_{N=0}^{\infty}|c_N|^2<\infty$. We have for any $\psi\in\mathcal{L}^2(0,\infty)$

$$\psi(r)=\sum_{N=0}^{\infty}c_N\psi_N^k(r), \qquad (2.29)$$

where $\{\psi_N^k\}$ are given by (2.6) and convergence is in the mean. Thus $(\psi,\psi)=\sum_{N=0}^{\infty}|c_N|^2<\infty$. But from (2.22) for any $\{c_N\}\in l^2$ we have an $f\in\mathcal{H}_k^+$ such that

$$f(\phi)=\sum_{N=0}^{\infty}c_N g_{k+N}^k(\phi) \qquad (2.30)$$

converges in the mean and hence $(f,f)_k=\sum_{N=0}^{\infty}|c_N|^2=(\psi,\psi)$.

It is clear that the above statements are if and only if statements with the only proviso that both $\psi(r)$ and $f(\phi)$ are defined up to sets of measure zero. It is now a simple task to construct this mapping explicitly as

$$(A\psi)(\phi)=\text{l.i.m.}\int_0^{\infty}dr\,A(\phi,r)\psi(r) \qquad (2.31)$$

for $\psi\in\mathcal{L}^2(0,\infty)$, where

$$A(\phi,r)=\sum_{N=0}^{\infty}g_N^k(\phi)\psi_N^k(r)^* =[2/\Gamma(2k)]^{1/2}r^{2k-1/2}\exp(ik\phi)[1-\exp(i\phi)]^{-2k} \times\exp[(r^2/2)(e^{i\phi}+1)/(e^{i\phi}-1)]. \qquad (2.32)$$

This kernel is singular at $\phi=0$, which in an intuitive sense is offset by the strong convergence in the $\mathcal{H}_k^+$ norm. The inverse mapping is given by

$$(A^{-1}f)(r)=\text{l.i.m.}\int\int_{S^1}d\phi\,d\phi'\,\Omega(\phi-\phi')A(\phi,r)^*f(\phi'), \qquad (2.33)$$

for any $f\in\mathcal{H}_k^+$. We stress that the unitary transformation kernel $A(\phi,r)$ is a unitary representation in quantum mechanics of the classical canonical transformation (1.2a). This is what we call a unitary canonical transform.

Now the important consequence of the unitary mappings (2.31) and (2.33) is that the group representations, or equivalently the Lie algebra representations (2.4) and (2.11) are unitarily equivalent. A straightforward computation shows that the operators $I_\pm=I_1\pm iI_2$ in the representation (2.4) satisfy the Lie algebra identities (2.12). Then using (2.31)—(2.33) and a simple integration by parts yields the desired results. The domain of the Lie algebra products is mapped onto each other and as a subspace of l^2 is given by all $\{c_N\}\in l^2$ such that $\sum_{m=k}^{\infty}m^4|c_N|^2<\infty$. Furthermore, the $\widetilde{SL}(2,R)$ group representation on $\mathcal{H}_k^+$ can be obtained from (2.9) and (2.31) by $U_g=AT_gA^{-1}$ yielding explicitly

$$(U_gf)(\phi)=|1+\gamma^*\exp(-i\phi)|^{-2k}(1-|\gamma|^2)^k \times f(\exp(i\omega)[\gamma+\exp(i\phi)]/[1+\gamma^*\exp(i\phi)]), \qquad (2.34)$$

for $f\in\mathcal{H}_k^+$. Here we have used the $SU(1,1)$ variables defined from (2.8a) as

$$\alpha=\tfrac{1}{2}[a+d+i(c-b)],\quad \beta=\tfrac{1}{2}[a-d-i(b+c)], \quad \gamma=\beta/\alpha,\quad \omega=2\arg\alpha. \qquad (2.35)$$

We mention that the representation (2.34) is equivalent to the representation $U^+(g,k)$ of Sally if we replace in (2.5.5) of Ref. 13 the complex variable z by its boundary $e^{i\phi}$ and perform the similarity transformation $\exp(ik\phi)U_g\exp(-ik\phi)$. For the connection between the representations described in this section and the usual treatment on the unit disc $\mathcal{M}$, the reader is referred to the Appendix.

We now pass to the description of a basis where a noncompact subgroup generator is diagonal.[23] As is well known, there are three orbits in the Lie algebra $sl(2,R)$ under the adjoint action of the group $SL(2,R)$. One of these orbits (the elliptic one) gives rise to the basis described previously (i.e., I_3 is diagonal). We proceed to give a brief description of the remaining two cases.

B. Parabolic case

In this case an orbit representative of the generators (2.4) is given by the radial free Hamiltonian

$$I_1+I_3=\tfrac{1}{2}\{-\partial_{rr}+[(2k-1)^2-\tfrac{1}{4}]/r^2\}. \qquad (2.36)$$

The eigenvalue problem thus gives rise to the generalized orthonormal eigenfunctions

$$\psi_s^k(r)=(rs)^{1/2}J_{2k-1}(rs) \qquad (2.37)$$

with eigenvalues $\frac{1}{2}s^2$. We also mention that an orbit representative which is simpler but with no physical meaning is $I_3-I_1=\frac{1}{2}r^2$. The relation between the two is given by $\exp(i\pi I_3)(I_3+I_1)\exp(-i\pi I_3)=I_3-I_1$. We emphasize that harmonic analysis[21,24] in terms of the latter is simpler than in terms of the former. Nevertheless, it is the former we are interested in, because of its physical meaning.

Our unitary mapping (2.31) can be extended in the usual way to operate on a suitable space of generalized functions[22] containing the eigenfunctions (2.37). This means that the generalized eigenfunctions have a meaning as the kernel of a particular transform (in this case the well-known Hankel transform) when applied to any $\psi\in\mathcal{L}^2(0,\infty)$. In this sense then the basis elements (2.37) are mapped unitarily onto generalized eigenfunctions $g_S^k(\phi)$ of the operator I_1+I_3 realized on the circle. In terms of the realization (2.11) we find

$$I_1+I_3=-i[(1+\cos\phi)\partial_\phi-k\sin\phi]. \qquad (2.38)$$

This operator becomes more transparent under the stereographic projection of the circle onto the real line given by

$$\xi=\tan\tfrac{1}{2}\phi,\quad -\pi\le\phi<\pi,\ -\infty<\xi<\infty. \qquad (2.39)$$

First we note that the space $\mathcal{J}_k^+$ on S^1 maps onto the space (called again $\mathcal{J}_k^+$) of infinitely differentiable functions which decrease at infinity as ξ^{-2k} (see the Appendix). The multivaluedness of functions on S^1 implies definite phase properties for the corresponding functions of ξ as $\xi\to\pm\infty$. This is specified by choosing the principal branch of $\ln z$ to correspond to the range $-\pi\le\phi<\pi$, so that

$$\exp(ik\phi)=\exp\left(k\ln\frac{i-\xi}{i+\xi}\right)\equiv\left(\frac{i-\xi}{i+\xi}\right)^k. \qquad (2.40)$$

Then (2.38) in the ξ-space realization becomes

$$I_1 + I_3 = -i\left(\partial_\xi - k\frac{2\xi}{\xi^2+1}\right). \quad (2.41)$$

The generalized eigenfunctions of (2.41) then have the form of a multiplier times the Fourier transform kernel with the phase inherited from the unitary mapping (2.31). Actually it is a simple calculation to obtain the eigenfunctions directly by applying (2.31) to the orthonormal basis functions (2.37), viz.

$$g_s^k(\phi) \equiv (A\psi_s^k)(\phi) = \sum_{N=0}^{\infty} g_{N+k}^k(\phi)(\psi_N^k, \psi_s^k), \quad (2.42a)$$

where (ψ_N^k, ψ_s^k) are the overlap functions between the canonical basis (2.6) and the parabolic basis (2.37). These overlap functions become trivial to calculate if we transform the ψ_s^k to a point on the orbit where $I_3 - I_1 = \frac{1}{2}r^2$ is diagonal with generalized eigenfunctions $\widetilde{\psi}_s^k(r) = \exp(i\pi k)\delta(r-s)$. We find

$$(\psi_N^k, \psi_s^k) = (\exp(i\pi I_3)\psi_N^k, \exp(i\pi I_3)\psi_s^k)$$
$$= (\exp(i\pi N + k)\psi_N^k, \widetilde{\psi}_s^k) = \exp(-i\pi N)\psi_N^k(s)^*. \quad (2.42b)$$

Hence, the properly normalized (including phase) generalized eigenfunctions on the circle are, using (2.32),

$$g_s^k(\phi) = \exp(i\pi k)A(\phi - \pi, s). \quad (2.42c)$$

This calculation shows the close connection between the unitary mapping of $\mathcal{L}^2(0,\infty)$ onto $\mathcal{H}_k^+$ and the parabolic basis. In terms of the ξ-space realization we find the form

$$g_s^k(\phi(\xi)) = [\Gamma(2k)]^{-1/2}(\tfrac{1}{2}s)^{2k-1/2}(1+\xi^2)^k \exp(\tfrac{1}{2}is^2\xi). \quad (2.42d)$$

It is readily checked that these functions are eigenfunctions of (2.41) with eigenvalues $\frac{1}{2}s^2$. Actually, since $s^2 \geq 0$, this is the half-space Fourier transform which is in complete accord with the fact, as discussed in the Appendix, that the members $f(\xi) \in \mathcal{H}_k^+$ in the ξ-space realization are the boundary values of functions $f(w)$ analytic in the upper half-plane $\mathrm{Im}w > 0$ with $\mathrm{Re}w = \xi$.

C. Hyperbolic case

In this case an orbit representative is given by the generator I_1 which is one-half the Hamiltonian for the repulsive harmonic oscillator. The eigenvalue problem is

$$I_1\psi_\nu^k = \tfrac{1}{2}\nu\psi_\nu^k. \quad (2.43a)$$

However, a much simpler orbit representative is given by the generator I_2 with the relation

$$\exp(\tfrac{1}{2}\pi i I_3)I_2\exp(-\tfrac{1}{2}\pi i I_3) = I_1.$$

The eigenvalue problem for I_2 is

$$I_2\widetilde{\psi}_\nu^k(r) = \tfrac{1}{2}\nu\widetilde{\psi}_\nu^k(r), \quad (2.43b)$$

with normalized generalized eigenfunctions given by the well-known Mellin transform kernel

$$\widetilde{\psi}_\nu^k(r) = (2\pi)^{-1/2}r^{i\nu-1/2}, \quad (2.44)$$

with $-\infty < \nu < \infty$. Using (2.9) to transform these functions to the corresponding basis functions for I_1, we find

$$\psi_\nu^k(r) = (2\pi r)^{-1/2}\exp(i\pi k)\exp(\tfrac{1}{4}\pi\nu)2^{i\nu/2}$$
$$\times[\Gamma(k+\tfrac{1}{2}i\nu)/\Gamma(2k)]M_{i\nu/2,k-1/2}(-ir^2), \quad (2.45)$$

where $M_{i\nu/2,k-1/2}(z)$ is a Whittaker function.[16] We wish to effect the mapping of the functions (2.45) to the generalized eigenfunctions on S^1. These will be eigenfunctions of the operator

$$I_1 = -i(\cos\phi\,\partial_\phi - k\sin\phi), \quad (2.46)$$

which satisfy (2.43a). Again, using the stereographic projection given by

$$\zeta = \tan[\tfrac{1}{2}(\phi + \tfrac{1}{2}\pi)], \quad -\tfrac{3}{2}\pi \leq \phi < \tfrac{1}{2}\pi, \quad (2.47)$$

we can write (2.46) as

$$I_1 = -i[\zeta\partial_\zeta - k(\zeta^2-1)/(\zeta^2+1)]. \quad (2.48)$$

Now the unnormalized generalized eigenfunctions of (2.48) which satisfy (2.43a) are $(\zeta^2+1)^k\zeta^{-k}\zeta_\pm^{i\nu/2}$, where $\zeta_+ = \zeta$ for $\zeta > 0$ and 0 for $\zeta < 0$, while $\zeta_- = -\zeta$ for $\zeta < 0$ and 0 for $\zeta > 0$. The correct normalization and phase for these eigenfunctions can be determined from the mapping (2.31). Alternatively, following the same procedure as in the parabolic case, we can write the eigenfunctions on the circle in terms of the Mellin transform of A, viz.

$$g_\nu^k(\phi) = (2\pi)^{-1/2}\int_0^\infty dr\,A(\phi + \tfrac{1}{2}\pi, r)r^{i\nu-1/2}. \quad (2.49)$$

Integrating this expression, we find explicitly

$$g_\nu^k(\phi) = \exp[\mp\tfrac{1}{4}i\pi\,(2k+i\nu)]\exp(i\pi k)2^{i\nu/2-k-1}[\pi\Gamma(2k)]^{-1/2}$$
$$\times\Gamma(k+\tfrac{1}{2}i\nu)\{\sin[\tfrac{1}{2}(\phi+\tfrac{1}{2}\pi)]\}^{-2k}\left|\tan[\tfrac{1}{2}(\phi+\tfrac{1}{2}\pi)]\right|^{k+i\nu/2}, \quad (2.50)$$

where $\mp$ is taken for $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$ and $-\frac{3}{2}\pi < \phi < -\frac{1}{2}\pi$ respectively. In terms of the variable ζ the eigenfunctions are

$$g_\nu^k(\phi(\zeta)) = \exp[\mp\tfrac{1}{4}i\pi\,(2k+i\nu)]\exp(i\pi k)2^{i\nu/2-k-1}[\pi\Gamma(2k)]^{-1/2}$$
$$\times\Gamma(k+\tfrac{1}{2}i\nu)(\zeta^2+1)^k|\zeta|^{i\nu/2-k}. \quad (2.51)$$

We remark that in the process of evaluating the integral (2.49) we have evaluated the more difficult integral of $A(\phi, r)$ in (2.32) with the Whittaker basis functions (2.45). This demonstrates the power of the group theoretical approach in obtaining special functions relations and is in the spirit of Refs. 21 and 24, where more difficult integrals are obtained. One further point is that the multiplicity of the hyperbolic decomposition for the representations D_k^+ is *one* in contradistinction to multiplicity *two* for the principal series[11,23] of $\widetilde{SL}(2,R)$. This is apparent in the $\mathcal{L}^2(0,\infty)$ realization, but in S^1 it is deeply hidden in the nonlocal measure. For example, from (2.51) one is led to think that the multiplicity is two—one Mellin transform for each half-axis. However, as discussed in the Appendix, the Hilbert space $\mathcal{H}_k^+$ in the ζ-space realization consists of boundary values of functions $f(w)$ analytic in the upper half-plane $\mathrm{Im}w > 0$ with $\mathrm{Re}w = \zeta$; hence, one can relate the two apparently independent Mellin transforms by using Cauchy's integral formula.

3. THE PSEUDO-COULOMB SYSTEM

The Hamiltonian for the n'-dimensional Coulomb system with an extra centrifugal force of strength g' is given by

$$H = \tfrac{1}{2}(\mathbf{p}^2 + 2qr^{-1} + g'r^{-2}) \quad (3.1)$$

where $r = |\mathbf{r}|$. It is to be noted that (3.1) is relevant in the relativistic Coulomb problem.[25] The standard trick[9] for introducing the $sl(2,R)$ Lie algebra is essentially to turn the standard eigenvalue problem for the energy (3.1) into an eigenvalue problem for the charge q by multiplying (3.1) by r, viz.,

$$(\tfrac{1}{2}r\mathbf{p}^2 - Er + \tfrac{1}{2}g'r^{-1} - q)\Phi(\mathbf{r}) = 0. \tag{3.2}$$

Then upon introducing the Lie algebra generators

$$K_1 = \tfrac{1}{2}[r(\mathbf{p}^2 - 1) + g'r^{-1}], \tag{3.3a}$$

$$K_2 = \mathbf{r}\cdot\mathbf{p} - i(n'-2) = rp_r - i(n'-2), \tag{3.3b}$$

$$K_3 = \tfrac{1}{2}[r(\mathbf{p}^2 + 1) + g'r^{-1}], \tag{3.3c}$$

Eq. (3.2) can be written as

$$[(\tfrac{1}{2} - E)K_3 + (\tfrac{1}{2} + E)K_1 - q]\Phi(\mathbf{r}) = 0. \tag{3.4}$$

A. Elliptic orbit (bound states)

There are three different solutions to (3.4) depending on which orbit the operator (3.4) lies. The case $E<0$ gives rise to the bound state solutions of the H atom, while for $E>0$ and $E=0$ one finds the scattering and threshold solutions. For $E<0$, the automorphism $\exp(i\theta I_2)$ called "tilting" by Barut and Kleinert,[9] where

$$\tanh\theta = \frac{E+\frac{1}{2}}{E-\frac{1}{2}} = \frac{|E|-\frac{1}{2}}{|E|+\frac{1}{2}}, \tag{3.5}$$

transforms (3.4) into

$$[(-2E)^{1/2}K_3 - q]\tilde{\Phi}(\mathbf{r}) = 0, \tag{3.6}$$

where $\tilde{\Phi}(\mathbf{r}) = \exp(i\theta I_2)\Phi(\mathbf{r})$.

Now we could insert (3.3c) into (3.6) and find the standard differential equation; however, we already know that for the UIR D_k^+ of $\widetilde{SL}(2,R)$ the spectrum of K_3 is simply $m=k+N$. Thus we have

$$q = (-2E)^{1/2}(k+N), \quad N = 0,1,2,\cdots, \tag{3.7a}$$

where

$$2k = 1 + [(n'+2L'-2)^2 + 4g']^{1/2}, \quad L' = 0,1,2,\cdots, \tag{3.7b}$$

[Note the *difference* between (3.7b) and (2.5)]. Turning Eq. (3.7a) around as an eigenvalue problem for E, we find the usual (at least for integer k, i.e., $g'=0$) result

$$E = -\tfrac{1}{2}q^2/(k+N)^2. \tag{3.7c}$$

It is this interpretation of (3.6) as an eigenvalue problem for E which suggests the name *pseudo*-Coulomb[10] for Eq. (3.4) and the Lie algebra (3.3). Indeed, the transition between the two problems is canonical *only for fixed* E, as can be seen from the transformation of the coordinate r under the "tilting" operation

$$\exp(i\theta K_2) r \exp(-i\theta K_2) = (-2E)^{1/2} r \equiv \rho, \tag{3.8a}$$

and thus it is seen that $\tilde{\Phi}(\mathbf{r}) = \Phi(\rho)$. Moreover, by using (3.3b) the canonical conjugate variable to r, p_r, transforms as

$$\exp(i\theta K_2) p_r \exp(-i\theta K_2) = (-2E)^{-1/2} p_r \equiv p_\rho. \tag{3.8b}$$

It is emphasized that what we have shown here is that the "tilting" operation of Barut and Kleinert is *equivalent* to the replacement of r and p_r in the generators (3.3) by the pair ρ, p_ρ. Furthermore, if r, p_r is a canonical pair then ρ, p_ρ is a canonical pair *only when E is constant*. Again, turning the problem around, we can start with ρ, p_ρ as a canonical pair obtaining r, p_r as one only for constant E. This is the pseudo-Coulomb problem, and it is this problem which can be mapped canonically by a simple point transformation onto the multidimensional harmonic oscillator[10,26] and hence onto the circle S^1 through the analysis of the preceeding section. Nevertheless, this group-theoretical treatment[9] of the hydrogen atom has had remarkable success in calculating transition amplitudes, form factors, etc.

Rewriting the operators (3.3) in terms of the variables ρ, p_ρ defined in (3.8), we see that (3.4) becomes the differential equation for the radial part of $\Phi(\boldsymbol{\rho})$ which we denote by $\varphi(\rho)$,

$$\tfrac{1}{2}[-\rho\partial_{\rho\rho} - (n'-1)\partial_\rho - (\lambda' - g')\rho^{-1} + \rho]\varphi(\rho) = m\varphi(\rho), \tag{3.9a}$$

where as before $m = k+N$ and is related to E through (3.7a), and, as in (2.2),

$$\lambda' = -L'(L'+n'-2), \quad L' = 0,1,2,\cdots. \tag{3.9b}$$

Now again the spectral analysis of (3.9) with the proper boundary condition on $\varphi(\rho)$ yields the allowed values of m as

$$m = N + k = N + \tfrac{1}{2} + [(\tfrac{1}{2}n' + L' - 1)^2 + g']^{1/2}, \tag{3.10}$$

where we have introduced k in (3.7b). Equation (3.9a) can now be turned into the analog of Eq. (2.6) with an operator Hermitean with respect to the measure $d\rho$ ($\rho \in [0,\infty)$) through a similarity transformation mapping functions as $\varphi(\rho) \to \psi'(\rho) = \rho^{n/2-1}\varphi(\rho)$ and operators as $K_i \to K_i' = \rho^{n/2-1}K_i\rho^{-n/2-1}$, viz.,

$$K_3'\psi_N'(\rho) = m\psi_N'(\rho), \tag{3.11a}$$

$$K_3' = \tfrac{1}{2}[-\rho\partial_{\rho\rho} - \partial_\rho + \rho + (k-\tfrac{1}{2})^2\rho^{-1}], \tag{3.11b}$$

$$\psi_N'(\rho) = [2N!/\Gamma(2k+N)]^{1/2}\rho^{k-1/2}e^{-\rho}L_N^{(2k-1)}(2\rho), \tag{3.11c}$$

and similarly for the operators (3.3a, b):

$$K_1' = \tfrac{1}{2}[-\rho\partial_{\rho\rho} - \partial_\rho - \rho + (k-\tfrac{1}{2})^2\rho^{-1}], \tag{3.11d}$$

$$K_2' = -i(\rho\partial_\rho + \tfrac{1}{2}). \tag{3.11e}$$

It is to be noted that the ordinary ($g'=0$) n'-dimensional pseudo-Coulomb problem with angular momentum L' has $k = L' + \frac{1}{2}(n'-1)$ and thus belongs to the UIR $D^+_{L'+(n'-1)/2}$ of $\widetilde{SL}(2,R)$. For $n'=3$, the bound states of the system belong to the direct sum $D_1^+ \oplus D_2^+ \oplus \cdots$.

We can now establish the link with the harmonic oscillator system. Indeed, if we take Eqs. (2.4a), (2.6a), and (2.6b) and effect the following:

(i) A change of variable $\rho = \frac{1}{2}r^2$ as suggested by the classical analogue (1.5); we obtain an operator (resp. eigenstates) Hermitean (resp. orthogonal) with respect to the measure $dr = (2\rho)^{-1/2}d\rho$ by simply following the chain rule for the derivatives.

(ii) A similarity transformation $\psi(\rho) \to \psi'(\rho) = (2\rho)^{-1/4}\psi(\rho)$ and $I_j \to K_j' = (2\rho)^{-1/4}I_j(2\rho)^{1/4}$ takes us to eigenstates (resp. operators) which are identical with (3.11c) [resp. (3.11b)] when

(iii) We identify

$$g=4g', \quad L=2L', \quad n=2n'-2. \tag{3.12}$$

Implementing this transformation, the spectrum-generating algebra of the pseudo-Coulomb system is obtained from the operators (2.4a, c) yielding precisely the operators (3.11d, e). We see that the ordinary ($g'=0$) n'-dimensional pseudo-Coulomb system of angular momentum L' belongs to the UIR $D^{+}_{L'+(n'-1)/2}=D^{+}_{L/2+n/4}$ of $\widetilde{SL}(2,R)$. Thus, for example, the states of the three-dimensional Hydrogen atom ($n'=3$, $L'=0,1,2,\cdots$) are mapped onto the even-angular momentum states of the *four*-dimensional harmonic oscillator[27] ($n=4$, $L=0,2,4,\cdots$) with the representation given by $D_1^{+}\oplus D_2^{+}\oplus\cdots$. We emphasize that the condition (3.12) and hence the mapping between the two systems is *not* a necessary one. Other possible mappings of the Hamiltonians were discussed in Ref. 26. Our choice (3.12) has the advantages of associating extra centrifugal potentials with each other as well as mapping states of zero angular momentum onto states of zero angular momentum. For $n'=2$, $n=2$, the mapping is the one described in Ref. 10.

A similar analysis can be effected for the two noncompact orbits.

B. Parabolic orbit (threshold states)

As the energy here is constant (zero), this is the only truly canonical mapping between the real Coulomb system and the system (2.4). In this case (3.4) becomes simply

$$[\tfrac{1}{2}(K_3+K_1)-q]\Phi(\mathbf{r})=0, \tag{3.13}$$

where, from (3.3),

$$K_3+K_1=r\mathbf{p}^2+g'r^{-1}. \tag{3.14}$$

Implementing the necessary similarity transformations which led to Eqs. (3.11) and replacing the variables r by ρ and p_r by p_ρ, the corresponding generator becomes

$$K_3'+K_1'=-\rho\partial_{\rho\rho}-\partial_\rho+(k-\tfrac{1}{2})^2\rho^{-1}. \tag{3.15}$$

Making again the simple change of variables as well as the similarity transformation (ii) and the identification (3.2), we find precisely the operator for the radial free particle (3.26) with the generalized eigenfunctions

$$\psi'^{k}_{s}(\rho)=s^{1/2}J_{2k-1}(s(2\rho)^{1/2}). \tag{3.16}$$

We mention here that in complete analogy with the parabolic orbit in Sec. 2 the harmonic analysis in terms of the operator $K_3'-K_1'=\rho$ is much simpler.

C. Hyperbolic orbit (scattering states)

The case $E>0$ gives rise to the Coulomb scattering states.[9] Now Eq. (3.4) can be brought to the eigenvalue problem for K_1 by the "tilting" operator $\exp(i\theta K_2)$, where now

$$\tanh\theta=\frac{E+\frac{1}{2}}{E-\frac{1}{2}}=\frac{|E|+\frac{1}{2}}{|E|-\frac{1}{2}}, \tag{3.17}$$

and we arrive at

$$[(2E)^{1/2}K_1-q]\widetilde{\Phi}=0, \tag{3.18}$$

which again is equivalent to the replacement of r and p_r^2 by ρ and p_ρ^2 respectively. Under the transformation (3.16) we have

$$\rho=(2E)^{1/2}r, \quad p_\rho=(2E)^{-1/2}p_r, \tag{3.19}$$

in lieu of (3.8). Again it is emphasized that for calculation purposes it is much easier to deal with $K_2=\exp(-\frac{1}{2}\pi iK_3)K_1\exp(\frac{1}{2}\pi iK_3)$ and the corresponding Mellin transform. Here we simply write down the eigenfunctions of the operator K_1' [(3.11d)] obtained from (2.45) by the point and similarity transformations (i) and (ii) described above with the identification (3.12), viz.,

$$\widetilde{\psi}'^{k}_{\nu}(\rho)=(2\pi)^{-1/2}\exp(i\pi k)\exp(\tfrac{1}{4}\pi\nu)2^{i\nu/2}\frac{\Gamma(k+\frac{1}{2}i\nu)}{\Gamma(2k)}(2\rho)^{-1/2}\times M_{i\nu/2,k-1/2}(-2i\rho). \tag{3.20}$$

We have shown that the spectrum-generating algebra $so(2,1)\approx su(1,1)\approx sl(2,R)$ [as well as its universal covering group $\widetilde{SL}(2,R)$] for the pseudo-Coulomb problem maps unitarily onto the radial harmonic oscillator system and thus through the composition maps onto the circle S^1. It is emphasized that this rotor has a nonlocal scalar product in order to preserve the positive definiteness of the bound-state spectrum. A similar situation can be found in the original work of Barut and Kleinert,[9] and Fronsdal,[8] where a nonlocal scalar product appears on the Fock sphere to insure a unitary representation of the $SO(4,2)$ group, or equivalently the $SO(2,1)$ subgroup. Since we have singled out the latter by studying the radial problem, the symmetry group $SO(4)$ does not appear here. It should be mentioned that the stereographic projection of the circle S^1 can be related to the radial pseudo-Coulomb problem through a transform with a Fourier type kernel. The connection of this with the momentum space and the embedding in the Fock sphere will be studied elsewhere.

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APPENDIX

We shall relate here the representation theory of $SL(2,R)$ on the circle S^1 as presented in Sec. 2 to the better known representation of the discrete series on the unit disc as described by Bargmann[11] for single-valued UIR's of $SL(2,R)$, by Sally[13] for the multivalued UIR's and Gel'fand[14] for single-valued representations on the complex upper half-plane.

Let $f\in\mathcal{J}_k^{+}$; then we can expand f together with all its derivatives in a Fourier series with positive partial waves as

$$f^{(n)}(\phi)=\exp(ik\phi)\sum_{N=0}^{\infty}a_N^n\exp(iN\phi). \tag{A1a}$$

Moreover, for $z\equiv r\exp(i\phi)$ with $r\leqslant 1$,

$$f^{(n)}(\phi)\geqslant\exp(ik\phi)\sum_{N=0}^{\infty}a_N^n r^N\exp(iN\phi)$$
$$=\exp(ik\phi)\sum_{N=0}^{\infty}a_N^n z^N, \tag{A1b}$$

and hence the series

$$g(z) \equiv \sum_{N=0}^{\infty} a_n z^n \tag{A2}$$

defines an analytic function whose radius of convergence is greater than 1. Thus for every $f \in \mathcal{J}_k^+$ we can associate a function g analytic in a region R_g containing the closed unit disc $\overline{\mathcal{M}} = \{z \in c : |z| \leq 1\}$ such that $\exp(-ik\phi)f(\phi)$ is the boundary value of $g(z)$ as $|z| \to 1$, and conversely, for every analytic function g in $R_g \supset \overline{\mathcal{M}}$ we can construct the uniformly converging series (A1). Following Bargmann,[11] we equip the space of analytic function on the open disc with the inner product

$$(g_1, g_2)_k = (2k-1)\pi^{-1} \int \int_{\mathcal{M}} r\,dr\,d\phi (1-r^2)^{2k-2} g_1(z)^* g_2(z) \tag{A3}$$

and the norm $\|g\|_k = (g,g)_k^{1/2} < \infty$. The r integral is understood to be in the sense of limit in the mean. Now if g_1, g_2 are analytic on all of $\overline{\mathcal{M}}$, we can write a Cauchy integral representation

$$g(z) = \frac{1}{2\pi i} \oint_{|z'|=1} \frac{g(z')\,dz'}{z-z'} . \tag{A4}$$

Substituting (A4) into (A3) and performing the r and ϕ integrals, we find

$$(g_1, g_2)_k = \int \int_{S^1} d\phi\, d\phi'\, \Omega(\phi - \phi') f_1(\phi)^* f_2(\phi') \tag{A5}$$

with $f_i(\phi) = \exp(ik\phi) \lim_{|z| \to 1} g_i(z)$ and $\Omega(\phi - \phi')$ given precisely by (2.22). However, since the norms (A3) and (2.15) are equivalent on $\mathcal{J}_k^+$, mean convergence in one is the same as mean convergence in the other, and so the space of functions analytic in $\mathcal{M}$ with finite norm (A3) is a realization of the Hilbert space $\mathcal{H}_k^+$. The members of $\mathcal{H}_k^+$ on S^1 are the boundary values almost everywhere of analytic functions in $\mathcal{M}$ with finite norm (A3). Moreover, as demonstrated by Bargmann[11] and Sally,[13] mean convergence in $\mathcal{H}_k^+$ implies pointwise convergence of analytic functions in $\mathcal{M}$.

We can easily express the Lie algebra generators (2.8) and group representation (2.34) on $\mathcal{M}$ by replacing $\exp(i\phi)$ by z. Then the mapping (2.31) is a mapping from $\mathcal{L}^2(0,\infty)$ to the $\mathcal{H}_k^+$ realization on the disc. This mapping was mentioned previously by Bargmann[28] and studied in detail by Sally.[13]

The well-known conformal mapping of the unit disc $\mathcal{M}$ onto the upper half-plane $C_+ = \{w \in C : \mathrm{Im} w > 0\}$ is the analog of the mapping (2.39). Explicitly, for $z \in C$ we write

$$z = \frac{i-w}{i+w}, \quad w = i\frac{1-z}{1+z}. \tag{A6}$$

Then it is easy to see that $|z| < 1$ implies $\mathrm{Im} w > 0$; moreover, the boundary $|z| = 1$ of $\mathcal{M}$ maps onto the real line $\mathrm{Im} w = 0$ including the point at infinity. Thus (A6) defines a homeomorphism of the closed unit disc $\overline{\mathcal{M}}$ onto the one-point compactification of the upper half-plane[14] $\overline{C}_+ = \{w \in C : \mathrm{Im} w \geq 0\} \cup \{\infty\}$. Under this mapping the scalar product (A3) becomes

$$(f_1, f_2)_k = (2k-1)\pi^{-1} \int_{C^+} dw (\mathrm{Im} w)^{2k-2} f_1(w)^* f_2(w), \tag{A7a}$$

where

$$f_i(w) = 2^{2k-2}(i+w)^{-2k} g_i(z(w)). \tag{A7b}$$

As a result f is analytic in $C_+(\overline{C}_+)$ when g is analytic in $\mathcal{M}(\overline{\mathcal{M}})$. Moreover, analyticity of $g \in \overline{\mathcal{M}}$ and therefore of $f \in \overline{C}_+$ implies the condition at infinity

$$f(w) \underset{|w| \to \infty}{\sim} |w|^{-2k}. \tag{A8}$$

The realization of $\mathcal{J}_k^+$ on $\overline{C}_+$ is the space of all functions analytic in $\overline{C}_+$ satisfying the condition (A8). The realization of $\mathcal{H}_k^+$ on C_+ is the space of all functions analytic in C_+ with finite norm $\|f\|_k = (f,f)_k^{1/2}$ given from (A7a), and $\mathcal{H}_k^+$ is the completion of $\mathcal{J}_k^+$ with respect to this norm.

[1]K. B. Wolf, J. Math. Phys. 15, 1295 (1974).
[2]K. B. Wolf, J. Math. Phys. 15, 2102 (1974).
[3]R. Hermann, *Lie Groups for Physicists* (Benjamin, New York, 1966), Chap. 16.
[4]K. B. Wolf, "The Heisenberg-Weyl Ring in Quantum Mechanics" in *Group Theory and Applications. III*, edited by E. Loebl, (Academic, New York, to appear).
[5]D. Judge and J. T. Lewis, Phys. Lett. 5, 190 (1963); L. Susskind and J. Glogower, Physics 1, 49 (1964); P. Carruthers and M. M. Nieto, Rev. Mod. Phys. 40, 411 (1968); R. Jackiw, J. Math. Phys. 12, 1021 (1971); Ref. 4, Sec. VI.
[6]J. C. Garrison and J. Wong, J. Math. Phys. 11, 2242 (1970); here the number operator $-i\partial_\phi$ is not essentially self-adjoint on the commutator domain; moreover, the spectrum is empty there.
[7]M. Rosenblum, Proc. Am. Math. Soc. 13, 590 (1962).
[8]C. Fronsdal, Phys. Rev. 156, 1665 (1967); K. B. Wolf, Suppl. Nuovo Cimento 5, 1041 (1967).
[9]A. O. Barut and H. Kleinert, Phys. Rev. 156, 1541 (1967); 157, 1180 (1967); 160, 1149 (1967); H. Kleinert, "Group Dynamics of the Hydrogen Atom," Lectures in Theoretical Physics, Boulder, Col. (1967); A. O. Barut, SIAM J. Appl. Math. 25, 247 (1973).
[10]M. Moshinsky, T. H. Seligman, and K. B. Wolf, J. Math. Phys. 13, 901 (1972).
[11]V. Bargmann, Ann. Math. 48, 568 (1947).
[12]L. Pukánsky, Math. Ann. 156, 96 (1964); A. O. Barut and C. Fronsdal, Proc. Roy. Soc. London 287, 532 (1965).
[13]P. J. Sally Jr., Mem. Amer. Math. Soc. 69 (1967).
[14]I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions, Vol.* 5 (Academic, New York, 1966), Chap. VII.
[15]T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966), Chap. VI, Sec. 4; E. C. Titchmarsh, *Eigenfunction Expansions, Vol.* 1 (Oxford U. P., Oxford, 1962).
[16]A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vols. 1, 2.
[17]W. Montgomery and L. O'Raifertaigh, J. Math. Phys. 15. 380 (1974).
[18]H. Bacry and J. L. Richards, J. Math. Phys. 8, 2230 (1967); N. Mukunda and B. Radhakrishnan, J. Math. Phys. 14, 254 (1973).
[19]U. Niederer, Helv. Phys. Acta 46, 191 (1973); U. Niederer,

to appear in Helv. Phys. Acta.
[20]C. P. Boyer, Comunicaciones Técnicas CIMAS 5, No. 68 (1974), to appear in Helv. Phys. Acta; note that the group SL $(2, R) \otimes O(n)$ appears as the symmetry group of the time-dependent Schrödinger equation with Hamiltonian (2.3).
[21]E. G. Kalnins and W. Miller Jr., J. Math. Phys. 15, 1728 (1974).
[22]K. Yosida, *Functional Analysis* (Springer-Verlag, Berlin, 1965).
[23]N. Mukunda, J. Math. Phys. 8, 2210 (1967); A. O. Barut and E. C. Phillips, Commun. Math. Phys. 8, 52 (1968); G. Lindblad and B. Nagel, Ann. Inst. H. Poincaré A 13, 27 (1970); C. Itzykson, J. Math. Phys. 10, 1109 (1969).
[24]C. P. Boyer, E. G. Kalnins, and W. Miller Jr., J. Math. Phys. 16, 499 (1975).
[25]A. O. Barut and G. L. Bornzin, J. Math. Phys. 12, 841 (1971).
[26]D. Bergman and Y. Frishman, J. Math. Phys. 6, 1855 (1965).
[27]M. Boiteux, Physica 65, 381 (1973); for an alternative approach, see D. G. Ravenhall, R. T. Sharp, and W. J. Pardee, Phys. Rev. 164, 1950 (1967).
[28]V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961), p. 203.