Canonical transforms. I. Complex linear transforms

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Recent work by Moshinsky *et al.* on the role and applications of canonical transformations in quantum mechanics has focused attention on some complex extensions of linear transformations mapping the position and momentum operators $\hat{x}$ and $\hat{p}$ to a pair $\hat{\eta}$ and $\hat{\xi}$ of canonically conjugate, but not necessarily Hermitian, operators. In this paper we show that for a continuum of complex linear canonical transformations, a related Hilbert space of entire analytic functions exists with a scalar product over the complex plane such that the pair $\hat{\eta}, \hat{\xi}$ can be realized in the Schrödinger representation η and $-id/d\eta$. We provide a unitary mapping onto the ordinary Hilbert space of square-integrable functions over the real line through an integral transform. The transform kernels provide a representation of a subsemigroup of $SL(2,\mathbb{C})$. The well-known Bargmann transform is the special case when $\hat{\eta}$ and $i\hat{\xi}$ are the harmonic oscillator raising and lowering operators. The Moshinsky–Quesne transform is regained in the limit when the canonical transformation becomes real, a case which contains the ordinary Fourier transform. We present a realization of these transforms through hyperdifferential operators.

I. INTRODUCTION

The purpose of this work is to explore some of the consequences of the use of general canonical transformations in quantum mechanics. We shall concentrate here in studying complex linear transformations between the quantum mechanical operators of position and momentum $\hat{x}$ and $\hat{p}$, and a new pair of quantities given by

$$\hat{\eta} = a\hat{x} + b\hat{p},$$

$$\hat{\xi} = c\hat{x} + d\hat{p}, \quad a, b, c, d \in \mathbb{C} \text{ complex field,} \tag{1.1a}$$

with the unimodularity condition

$$ad - bc = 1 \tag{1.1b}$$

which ensures that, if $\hat{x}$ and $\hat{p}$ are canonically conjugate operators, then $\hat{\eta}$ and $\hat{\xi}$ will also be canonically conjugate, namely

$$[\hat{x}, \hat{p}] = i\mathbb{1} \Leftrightarrow [\hat{\eta}, \hat{\xi}] = i\mathbb{1} \tag{1.2}$$

in units where $\hbar \equiv 1$. In the usual Hilbert space $\mathcal{H}$ of quantum mechanical states,[1] we have the space of square integrable functions over the real line $\mathbb{R}$ with the scalar product

$$(f, g)_0 = \int_{\mathbb{R}} dx\, f(x)^* g(x) \tag{1.3}$$

for $f, g \in \mathcal{H}$. (The star denotes complex conjugation.) The Stone–von Neumann theorem states, moreover, that we can always (through a unitary transformation if necessary) use the Schrödinger realization of the realization of the Heisenberg algebra (1.2), i.e., represent $\hat{x}$ and $\hat{p}$ by x and $-id/dx$ over a set dense in $\mathcal{H}$.

When the transformation (1.1) is real, a scalar product where $\hat{\eta}$ and $\hat{\xi}$ are Hermitian and realized by the Schrödinger representation as η and $-i\,d/d\eta$ on functions of η in $\mathcal{H}' \approx \mathcal{H}$, with a scalar product analogous to (1.3) leads to the Moshinsky–Quesne transform[2] between $\mathcal{H}$ and $\mathcal{H}'$. The ordinary Fourier transform is a special case of this for $a = 0 = d$, $b = 1 = -c$.

The use of a complex linear transformation (1.1) with

$$a = 2^{-1/2} = d, \quad b = -i2^{-1/2} = c \tag{1.4}$$

has proven to be of great importance, as developed by Bargmann[3,4] and applied to the coherent-state formulation of quantum optics.[5] Equation (1.1) with (1.4) gives to $\hat{\eta}$ and $i\hat{\xi}$ (notice that Bargmann's $\hat{\xi}$ is here $i\hat{\xi}$) the meaning of creation and annihilation operators with respect to the harmonic oscillator states. Hermitian conjugation in $\mathcal{H}$ induces the properties $\hat{\eta}^+ = i\hat{\xi}$ and $(i\hat{\xi})^+ = \hat{\eta}$. In order to find a Hermitian form where the Schrödinger realization for $\hat{\eta}$ and $\hat{\xi}$ can be implemented, Bargmann introduced a space $\mathcal{F}$ of entire analytic functions $\bar{f}$ in $\eta \in \mathbb{C}$—the complex field—restricted by the condition $|\bar{f}(\eta)| \le \gamma \exp(\frac{1}{2}\alpha\eta^*\eta)$ for finite $\gamma > 0$ and $0 < \alpha < 1$, with a scalar product given by

$$(\bar{f}, \bar{g}) = \int_{\mathbb{C}} d\mu(\eta) \bar{f}(\eta)^* \bar{g}(\eta), \tag{1.5a}$$

$$d\mu(\eta) = \nu(\eta, \eta^*)\, d\mathrm{Re}\,\eta\, d\mathrm{Im}\,\eta \tag{1.5b}$$

for $\bar{f}, \bar{g} \in \mathcal{F}$, where the integration is extended over the complex η-plane (with a definite limiting procedure, see Ref. 3) and, in Bargmann's case, the weight $\nu(\eta, \eta^*) = \pi^{-1}\exp(-\eta^*\eta)$. It was also shown in Ref. 3 that $\mathcal{F}$ completed with respect to the norm induced by (1.5) is a Hilbert space and, moreover, a unitary mapping $\mathcal{A}: \mathcal{H} = \mathcal{F}$ can be implemented through the transform pair

$$\bar{f}(\eta) = \int_{\mathbb{R}} dx A(\eta, x) f(x), \tag{1.6a}$$

$$f(x) = \int_{\mathbb{C}} d\mu(\eta) A(\eta, x)^* \bar{f}(\eta), \tag{1.6b}$$

with the kernel $A(\eta, x) = \pi^{-1/4}\exp[-\frac{1}{2}(x^2 + \eta^2) + 2^{1/2}x\eta]$.

In a recent work, Kramer, Moshinsky, and Seligman[6] have considered a class of complex linear transformations of the type (1.1) and applied them to the study of clustering in nuclei, thereby achieving significant conceptual and calculational simplifications. We have taken their motivation to study the general case of complex linear transformations and set up a continuum of spaces $\mathcal{F}$ of entire analytic functions with different growth restrictions in the complex η-variable and a scalar product of the general type (1.5) with appropriate measures $\nu(\eta, \eta^*)$, where the Schrödinger representation is realized. As in Bargmann's case, completion with respect to the norm induced by (1.5) shows that the $\mathcal{F}$'s are Hilbert spaces and that unitary maps $\mathcal{A}: \mathcal{H} = \mathcal{F}$ can be implemented through transforms of the type (1.6). We shall call these *canonical* transforms.

In Sec. II we construct and characterize the spaces $\mathcal{J}$ and find the transform kernels in Sec. III. In Sec. IV we determine the behavior of the transforms in the limit where the parameters a, b, c, d become real. The scalar product (1.5) is shown to collapse to an integral over $\mathbb{R}$, so that the Moshinsky—Quesne transform is regained. As the composition of two canonical transformations is of the same type, the composition of the corresponding transforms is developed in Sec. V. In Sec. VI, the transform kernels are shown to provide, when bounded, representations of a semigroup $HSL(2, \mathbb{C})$ of the group $SL(2, \mathbb{C})$ of canonical transformations (1.1). In Appendix A we give a realization of canonical transforms through hyperdifferential operators, while in Appendix B results for general n-dimensional spaces are presented.

In a future series of articles we intend to explore the consequences of more general complex canonical transformations in quantum mechanics. In Ref. 6 it was shown that a transformation in the radial coordinate[7] of a higher-dimensional space undergoing a linear transformation is related with the Barut—Girardello transform.[8] Among the classes of canonical transformations where classical and quantum mechanics follow each other[9] are point transformations followed by linear ones. This has been used to relate[10] the representation of the algebra $so(2,1)$ given by the dynamical algebra of a harmonic oscillator (with the addition of an inverse-square potential) and its exponentiation to the discrete series representations of the group $SO(2,1)$, with Bargmann's realization[11] of the same series. Finally, many-sheeted canonical mappings of phase space into itself such as those considered in Ref. 12 can be implemented with the help of the representations of the group of automorphisms and an associated transform.[6]

II. THE SPACE $\mathcal{J}$

Consider the complex unimodular linear transformation (1.1) written in matrix form as

$$\mathrm{Z} \equiv \begin{pmatrix} \hat{\eta} \\ \hat{\xi} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \equiv M\mathrm{Z}_0, \quad \det M = 1 \tag{2.1}$$

(i.e., $M \in SL(2,\mathbb{C})$). The corresponding adjoint operators, relative to the scalar product in $\mathcal{H}$, where $\hat{x}$ and $\hat{p}$ are Hermitian, can be then written in terms of the original ones as

$$\mathrm{Z}^+ \equiv \begin{pmatrix} \hat{\eta}^+ \\ \hat{\xi}^+ \end{pmatrix} = M^*\mathrm{Z}_0 = M^*M^{-1}\mathrm{Z} \equiv C\mathrm{Z} \tag{2.2a}$$

where the *conjugation* matrix

$$C = \begin{pmatrix} u & iv \\ iw & u^* \end{pmatrix} \tag{2.2b}$$

is such that $\det C = 1$, $CC^* = 1$ and its elements are given and restricted by

$$u = a^*d - b^*c \in \mathbb{C}, \tag{2.3a}$$

$$v = 2\,\mathrm{Im}\, b^*a, \quad w = 2\,\mathrm{Im}\, c^*d \in \mathbb{R}, \tag{2.3b}$$

$$|u|^2 + vw = 1. \tag{2.3c}$$

For every $M \in SL(2,\mathbb{C})$ we have thus a conjugation matrix $C(M)$. In particular, if $R \in SL(2,\mathbb{R})$, then $C(R) = \mathbb{1}$ and $\hat{\eta}$ and $\hat{\xi}$ are Hermitian, and $C(MR) = C(M)$. Bargmann's case (1.4) corresponds to the imaginary antidiagonal matrix with $u=0$, $v=1=w$. Since from (1.2), $\hat{\eta}$ and $\hat{\xi}$ are canonically conjugate, we want to implement the Schrödinger representation

$$\hat{\eta}\bar{f}(\eta) = \eta\bar{f}(\eta), \tag{2.4a}$$

$$\hat{\xi}\bar{f}(\eta) = -i\frac{d}{d\eta}\bar{f}(\eta) \tag{2.4b}$$

on any suitable function $\bar{f}$ of the complex variable η. In order that the total derivative in (2.4b) be well defined, the function $\bar{f}$ must be analytic in η. The conditions we are asking for a scalar product to satisfy can then be formulated, through (2.2), as

$$(\hat{\eta}\bar{f}, \bar{g}) = (\bar{f}, [u\hat{\eta} + iv\hat{\xi}]\bar{g}), \tag{2.5a}$$

$$(\hat{\xi}\bar{f}, \bar{g}) = (\bar{f}, [iw\hat{\eta} + u^*\hat{\xi}]\bar{g}). \tag{2.5b}$$

We can see that an ordinary scalar product of the type (1.3) cannot fulfill this requirement. One must look for a more general kind of scalar product. Proposing the form (1.5) we can turn Eqs. (2.5) into differential equations for the weight function $\nu(\eta, \eta^*)$. Using (1.5), (2.4) and performing an integration by parts [provided that the boundary value of $\bar{f}(\eta)^*\nu(\eta,\eta^*)\bar{g}(\eta)$ at infinity be zero], the conditions (2.5) can be given as

$$\eta^*\nu(\eta,\eta^*) = \left(u\eta - v\frac{\partial}{\partial\eta}\right)\nu(\eta,\eta^*), \tag{2.6a}$$

$$\frac{\partial}{\partial\eta^*}\nu(\eta,\eta^*) = -\left(w\eta + u^*\frac{\partial}{\partial\eta}\right)\nu(\eta,\eta^*). \tag{2.6b}$$

The solution of (2.6) with a specific choice of normalization is

$$\nu(\eta,\eta^*) = 2(2\pi v)^{-1/2}\exp\left\{\frac{1}{2v}[u\eta^2 - 2\eta\eta^* + u^*\eta^{*2}]\right\} = \nu(\eta^*,\eta)^*. \tag{2.7a}$$

A convenient representation is obtained when we write in polar form $\eta = \rho e^{i\theta}$, $u = \omega e^{i\phi}$, whereupon (2.7a) becomes

$$\nu(\eta,\eta^*) \equiv \nu[\rho,\theta] = 2(2\pi v)^{-1/2}\exp\left\{-\frac{\rho^2}{v}[1 - \omega\cos(\varphi + 2\theta)]\right\}. \tag{2.7b}$$

The boundary condition on $\bar{f}(\eta)^*\nu(\eta,\eta^*)\bar{g}(\eta)$ can now be made explicit: we write $\bar{f}(\eta) = f_b(\eta v^{-1/2})\exp[(-u/2v)\eta^2]$, imposing the condition $v > 0$, then the scalar product (1.5)—(2.7) becomes the Bargmann scalar product[3] between $f_b(\eta')$ and $g_b(\eta')$ for $\eta' = \eta v^{-1/2}$. The growth conditions imposed on these functions imply then that $\bar{f}$ and $\bar{g}$ must satisfy

$$|\bar{f}(\rho e^{i\theta})| \leq \gamma\exp\left\{\frac{\rho^2}{v}[\alpha - \omega\cos(\varphi + 2\theta)]\right\}, \tag{2.8}$$

for finite $\gamma > 0$ and $0 < \alpha < 1$, which is dependent on the direction θ in the complex η-plane. This is sufficient to characterize the space $\mathcal{J}$ of entire analytic functions for which the scalar product (1.5) is finite. Bargmann's analysis[3] can now be translated to state that for $v > 0$, the space $\mathcal{J}$ with the scalar product (1.5) is a Hilbert space, unitarily equivalent to $\mathcal{H}$ through a transform of the kind (1.6). It should be noticed that Bargmann's transform is indeed regained in the particular case (1.4), allowing for the choice in the measure normalization: here it is chosen as $2(2\pi v)^{-1/2}$ so that it go over smoothly to the Moshinsky–Quesne transform (Sec. IV), while in the original work[3] it is set as π^{-1}. For every matrix $M \in SL(2,\mathbb{C})$ such that $C(M)$ satisfies $v>0$ we have thus a Hilbert space $\mathcal{J}$.

A dense orthonormal basis for $\mathcal{J}$ can now be constructed as

$$\bar{u}_n(\eta) = [(2\pi v)^{1/2} n!]^{-1/2} \exp\left(-\frac{u}{2v}\eta^2\right)(\eta v^{-1/2})^n, \qquad n=0,1,2,\ldots. \tag{2.9}$$

These functions satisfy the following recursion relations:

$$[v^{-1/2}\eta]\bar{U}_n(\eta) = (n+1)^{1/2}\bar{U}_{n+1}(\eta), \tag{2.10a}$$

$$\left[uv^{-1/2}\eta + v^{1/2}\frac{d}{d\eta}\right]\bar{U}_n(\eta) = n^{1/2}\bar{U}_{n-1}(\eta), \tag{2.10b}$$

and, in particular,

$$\left[u\eta + v\frac{d}{d\eta}\right]\bar{U}_0(\eta) = 0. \tag{2.10c}$$

They are, thus, eigenfunctions of a number operator

$$\hat{N}_u \bar{U}_n(\eta) \equiv \left[uv^{-1}\eta^2 + \eta\frac{d}{d\eta}\right]\bar{U}_n(\eta) = \frac{1}{v}\hat{\eta}\hat{\eta}^\dagger \bar{U}_n(\eta) = n\bar{U}_n(\eta). \tag{2.11}$$

From the orthonormal basis (2.9) we can build the generating function

$$K(\eta,\eta') \equiv \sum_{n=0}^{\infty} \bar{U}_n(\eta)\bar{U}_n(\eta')^* = (2\pi v)^{-1/2} \times \exp\left\{-\frac{1}{2v}[u\eta^2 - 2\eta\eta'^* + u^*\eta'^{*2}]\right\} = K(\eta',\eta)^*, \tag{2.12a}$$

which acts as the reproducing kernel under the scalar product (1.5):

$$(K(\cdot,\eta'),\bar{f}) = \int_{\mathbb{C}} d\mu(\eta) K(\eta,\eta')^* \bar{f}(\eta) = \bar{f}(\eta'). \tag{2.12b}$$

III. THE TRANSFORMATION KERNEL AND PAIRS OF TRANSFORM BASES

We want to establish a mapping between the elements f of the Hilbert space $\mathcal{H}$ and the elements $\bar{f}$ in $\mathcal{J}$, as given by (1.6) in such a way that if $f(x)$ is mapped into $\bar{f}(\eta)$, then $\hat{\eta} f(x)$ maps into $\eta\bar{f}(\eta)$ and $\hat{\xi} f(x)$ into $-i(d/d\eta)\bar{f}(\eta)$. Through (2.1), this means

$$\eta\bar{f}(\eta) = \int_{\mathbb{R}} dx A(\eta,x)\hat{\eta} f(x) = \int_{\mathbb{R}} dx\left(\left[ax + ib\frac{\partial}{\partial x}\right]A(\eta,x)\right)f(x), \tag{3.1a}$$

$$-i\frac{d}{d\eta}\bar{f}(\eta) = \int_{\mathbb{R}} dx A(\eta,x)\hat{\xi} f(x) = \int_{\mathbb{R}} dx\left(\left[cx + id\frac{\partial}{\partial x}\right]A(\eta,x)\right)f(x), \tag{3.1b}$$

and hence the transformation kernel $A(\eta,x)$ must satisfy the differential equations

$$\eta A(\eta,x) = \left[ax + ib\frac{\partial}{\partial x}\right]A(\eta,x), \tag{3.2a}$$

$$-i\frac{\partial}{\partial\eta}A(\eta,x) = \left[cx + id\frac{\partial}{\partial x}\right]A(\eta,x). \tag{3.2b}$$

The solution, with proper normalization, is

$$A(\eta,x) = \varphi_A (2\pi|b|)^{-1/2} \exp\left\{\frac{i}{2b}[ax^2 - 2x\eta + d\eta^2]\right\}, \tag{3.3a}$$

where we choose the phase factor to be

$$\varphi_A = \exp\left(-\frac{i}{2}\left[\frac{\pi}{2} + \Phi(b)\right]\right), \tag{3.3b}$$

where $\Phi(b) \equiv$ phase of $b \in [-\pi, \pi)$. This choice of phase has been made so that the representation properties of the $A(\eta,x)$ be simple (Sec. VI) and for $M \in SL(2,\mathbb{R})$ they agree with Ref. 2. The integrability condition in (1.6a) requires that $\mathrm{Im}(a/b) \geq 0$ (i.e., $v \geq 0$) and that if $a=0$, then b should be real. The integrability of Eq. (1.6b) can then be seen to hold through the identity $ub = -ivd + b^*$ since this implies that $|id/2b| \leq |1-\omega|/2v$. The normalization makes the transforms (1.6) be inverse to each other, as

$$\int_{\mathbb{R}} dx A(\eta,x)A(\eta',x)^* = K(\eta,\eta'), \tag{3.4a}$$

$$\int_{\mathbb{C}} d\mu(\eta) A(\eta,x)^* A(\eta,x') = \delta(x-x'). \tag{3.4b}$$

Equation (3.4a) can be verified directly, while Eq. (3.4b) will be shown to hold when we will write the transform kernel $A(\eta,x)$ as the generating function linking two orthonormal bases, one in $\mathcal{H}$ and one in $\mathcal{J}$.

We have constructed an orthonormal basis of functions $\{\bar{U}_n(\eta)\}$ for $\mathcal{J}$ in (2.9). In searching for a corresponding basis $\{U_n(x)\}$ for $\mathcal{H}$ we can go directly through the transform definition (1.6b) or, preferably, use the independent method of using the raising operators (2.10) for $\{\bar{U}_n(\eta)\}$ translated to operators in x and d/dx through (2.1). The extremum $U_0(x)$ of the ladder is found from (2.10c) as

$$U_0(x) = \varphi_A^{-1}\left(\pi/\mathrm{Im}\,\frac{a}{b}\right)^{-1/4} \exp\left(-i\frac{a^*}{2b^*}x^2\right) \tag{3.5a}$$

normalized with respect to the scalar product in $\mathcal{H}$, with φ_A given by (3.3b). From $U_0(x)$ and the raising operator (2.10a) we find

$$U_n(x) = [v^n n!]^{-1/2}\left[ax - ib\frac{d}{dx}\right]^n U_0(x) = \varphi_A^{-1}\varphi_n\left[2^n n!\left(\pi/\mathrm{Im}\,\frac{a}{b}\right)^{1/2}\right]^{-1/2} \times \exp\left(-i\frac{a^*}{2b^*}x^2\right)H_n\left(\left[\mathrm{Im}\,\frac{a}{b}\right]^{1/2}x\right). \tag{3.5b}$$

with

$$\varphi_n = \exp\left[in\left(\frac{\pi}{2} + \Phi(b)\right)\right]. \tag{3.5c}$$

The basis $\{U_n(x)\}$ can be checked to be indeed orthonormal under the scalar product in $\mathcal{H}$ and we can verify directly that the transformation kernel is indeed the generating function between the bases:

$$A(\eta, x) = \sum_{n=0}^{\infty} \bar{U}_n(\eta) U_n(x)^*. \tag{3.6}$$

In particular, notice that for Bargmann's case (1.4), $\{\bar{U}_n(\eta)\}$ is the basis of monomials in η while $\{U_n(x)\}$ are the harmonic oscillator wavefunctions $\psi_n(x)$.

There are reasons for not being satisfied with the basis $\{\bar{U}_n(\eta)\}$ alone. There is the problem of not having a manifest limit as $v \to 0$ (when the transformation matrix M becomes real) and that of being eigenfunctions of the number operator (2.11) which in $\mathcal{H}$ reads $v^{-1}(ax - ib\, d/dx) \times (a^* x - ib^*\, d/dx)$. Thus, we introduce the well-known harmonic oscillator wavefunction basis (with the usual phase convention)

$$\psi_n(x) = [2^n n!\, \pi^{1/2}]^{-1/2} \exp(-\tfrac{1}{2}x^2) H_n(x), \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

The raising, lowering, and number operators are simple and can be translated to operators in η and $d/d\eta$ through (2.1) in order to find the transform basis. The differential equation for the ground function yields

$$\bar{\psi}_0(\eta) = [\pi^{1/2}(a + ib)]^{-1/2} \exp\left(-\frac{d - ic}{a + ib}\frac{\eta^2}{2}\right), \tag{3.8a}$$

where we must take the sheet given by $(a+b)^{-1/2} = |a + ib|^{-1/2} \exp - \tfrac{1}{2} i\, \Phi(a + ib)$, and the rest of the basis can be generated through the application of the raising operator, i.e.,

$$\bar{\psi}_n(\eta) = [2^n n!]^{-1/2} \left[(d + ic)\eta + (-a + ib)\frac{d}{d\eta}\right]^n \bar{\psi}_0(\eta)$$

$$= \left[\left(2\frac{a + ib}{a - ib}\right)^n n!\, \pi^{1/2}(a + ib)\right]^{-1/2}$$

$$\times \exp\left[-\frac{d - ic}{a + ib}\frac{\eta^2}{2}\right] H_n\left([a^2 + b^2]^{-1/2}\eta\right), \tag{3.8b}$$

which reduces to (3.7) when M becomes $\mathbb{1}$. It is also interesting to notice that Bargmann's case (1.4) gives back the basis $\{\bar{U}_n(\eta)\}$ with the proper normalization. (Notice that only the leading term of the Hermite polynomial survives). As a final check of the calculation we can verify that the transformation kernel $A(\eta, x)$ in (3.3) is the generating function between the bases $\{\psi_n(x)\}$ and $\{\bar{\psi}_n(\eta)\}$, i.e.,

$$A(\eta, x) = \sum_{n=0}^{\infty} \bar{\psi}_n(\eta) \psi_n(x)^* \tag{3.9}$$

implemented through the use of an integral representation for one of the Hermite functions.[13]

IV. THE LIMIT OF REAL TRANSFORMATIONS

We now want to examine the behavior of our construction when the parameters $a, b, c, d \in \mathbb{C}$ in (2.1) become real. Notice that the basis functions $\{\bar{\psi}_n(\eta)\}$ present no peculiar behavior and indeed go smoothly into $\{\psi_n(x)\}$ when $M \to \mathbb{1}$. The transformation kernel $A(\eta, x)$ in (3.3) is uneventful when a, b, c, d become real and only when b approaches zero does the expression become indeterminate at first sight. The analysis in Ref. 2 leads us to expect that the kernel will become a Dirac δ in $\eta - x$. This has to be examined further. Indeed, we intend to show that the scalar product (1.5) collapses to a line integral as $v \to 0$.

Consider the measure (1.5b) parametrized in its polar decomposition (2.7b) as $d\mu(\eta) = \nu[\rho, \theta]\rho\, d\rho\, d\theta$. When $v \to 0$, $\omega = |1 - vw|^{1/2} \approx 1 - \tfrac{1}{2}vw \to 1$. Recalling that for real, positive $\epsilon \to 0$, $\text{l.i.m.}\, \epsilon^{-1/2} \exp(-q^2/\epsilon) = \pi^{1/2}\delta(q)$, we can write

$$\underset{\nu \to 0}{\text{l.i.m.}}\ \nu[\rho, \theta] = \underset{\nu \to 0}{\text{l.i.m.}}\ 2(2\pi v)^{-1/2} \exp\left\{-\frac{\rho^2}{v}[1 - (1 - \tfrac{1}{2}vw) \times \cos(\varphi + 2\theta)]\right\}$$

$$= 2^{1/2}\delta(\rho[1 - \cos(\varphi + 2\theta)]^{1/2})$$

$$\times \exp[-\tfrac{1}{2}\rho^2 w \cos(\varphi + 2\theta)]$$

$$= \rho^{-1}\delta(\sin(\tfrac{1}{2}\varphi + \theta)) \exp[-\tfrac{1}{2}\rho^2 w \cos(\varphi + 2\theta)]$$

$$= \rho^{-1}[\delta(\theta + \tfrac{1}{2}\varphi) + \delta(\theta + \tfrac{1}{2}\varphi - \pi)] \exp[-\tfrac{1}{2}\rho^2 w]. \tag{4.1}$$

All of these steps should be done remembering that the functions are under the double integral $\int_0^\infty \rho\, d\rho \int_0^{2\pi} d\theta$, in particular, the third step takes into account the fact that the point $\rho = 0$ is immaterial for the δ as it is cancelled by the measure in ρ, and the last step makes use of the consequence that the δ will act only in picking out values in the integration over θ. The growth condition (2.8) on the function space is such that the scalar product is finite and for the line $\theta_0 \equiv -\tfrac{1}{2}\varphi$, $\pi - \tfrac{1}{2}\varphi$ is

$$|\bar{f}(\rho e^{i\theta_0})| \leq \gamma \exp\left(\frac{\rho^2}{2v}[\alpha - \omega]\right) < \gamma \exp(\tfrac{1}{4}w\rho^2) \tag{4.2}$$

when we write $\omega \approx 1 - \tfrac{1}{2}vw$, $\alpha = 1 - A(v)$ and let $A(v)$ be any function of v which decreases faster than v as $v \to 0$. Similarly for $\bar{g}$. If we now define for $\bar{f}(\eta) = \bar{f}[\rho, \theta]$, $\bar{f}(x) \equiv \bar{f}[x, -\tfrac{1}{2}\varphi]$ and $\bar{f}(-x) \equiv \bar{f}[x, \pi - \tfrac{1}{2}\varphi]$ for $x \geq 0$, the limit indicated follows, i.e.,

$$\lim_{v \to 0} \int_{\mathbb{C}} d\mu(\eta) \bar{f}(\eta)^* \bar{g}(\eta) = \int_{\mathbb{R}e^{-i\varphi/2}} dx\, e^{-w|x|^2/2} \bar{f}(x)^* \bar{g}(x) \tag{4.3}$$

with the condition, in effect, that $\bar{f}$ be such that $\bar{f}(x) \times \exp(-\tfrac{1}{4}wx^2)$ is square integrable over $\mathbf{R}$, and similarly for $\bar{g}$.

As can be seen, as $v \to 0$ the integral over $\eta \in \mathbb{C}$ becomes an integral over a straight line passing through the origin and with a phase $-\tfrac{1}{2}\varphi = -\tfrac{1}{2}\Phi(u) = \Phi(a)$. When the transformation matrix M is real, $u = 1$ and the integration path becomes the real axis. By a similar argument, the reproducing kernel $K(\eta, \eta')$ in (2.12) becomes the Dirac $\delta(x - x')$. The behavior of the transformation kernel $A(\eta, x)$ at the limit $b \to 0$ can be analized when this takes place from any direction in the complex plane. Using (1.1b),

$$A(x', x) = (2\pi)^{-1/2}\varphi_A |b|^{-1/2} \exp\{-|b|^{-1}[\bar{\varphi}_A(2/a)^{-1/2}x - \varphi_A(2a)^{-1/2}x']^2\} \exp\left(\frac{ic}{2a}x'^2\right)$$

$$\underset{|b| \to 0}{\to} a^{-1/2}\delta(x - a^{-1}x') \exp\left(\frac{ic}{2a}x'^2\right) \tag{4.4}$$

and the phase of the direction in which the inverse transform takes place, $\Phi(a)=-\frac{1}{2}\Phi(u)$, is the appropriate one which will make use of the Dirac δ.

We can make explicit the condition that a transformation M in (2.1) lead to a transform involving only a line integral. Notice first that $C(M)=\mathbb{1}$ if and only if $M\in SL(2,\mathbb{R})$, the measure in the transform space being simply dx. Next, we can examine the cases when $C(M)$ is a lower triangular matrix ($v=0$). We consider the case $u=1$ so that the integral be along the real axis. Analysis of the conditions (2.3) leads us to the restrictions: a, b real. An important subclass is that considered in Ref. 6, namely $b=0$, $a=d^{-1}$ real, where (4.4) simulates the matrix elements of a Gaussian potential for $c=iq$, $q>0$.

Transforms involving line integrals along a path tilted by a phase α can be obtained multiplying the transformation matrix M on the left by a diagonal matrix with elements $\exp(i\alpha)$, $\exp(-i\alpha)$ as then $u=\exp(-2i\alpha)$. In particular, for $b=i=-c^{-1}$, $d=0$ ($\alpha=\pi/2$) we obtain a Laplace transform with kernel (3.3) given by $-i(2\pi)^{-1/2}\times\exp(-xx')$, which is off by a factor and a phase from the usual Laplace transform. The condition "b real when $a=0$" for the kernel (3.3) is now violated, so it is not surprising that the integral in (1.6a) can diverge for $f\in\mathcal{H}$. A restriction on $\mathcal{H}$ [for instance $f(x)=0$ for $x<0$] may make the transform meaningful. The inverse transform is an integral over a Bromwich contour up along the imaginary axis.

V. COMPOSITION OF TRANSFORMS

For every matrix $M\in SL(2,\mathbb{C})$ in (2.1) satisfying $\mathrm{Im}(a/b)\geq 0$ we have associated a canonical transform (1.6) from the Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{F}$ characterized by (1.5), (2.7), and (2.8). Take now two such spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ associated to the transformations $\mathbf{z}_1=M_1\mathbf{z}_0$ and $\mathbf{z}_2=M_2\mathbf{z}_0$, with transformation kernels $A_1(\eta,x)$ and $A_2(\eta,x)$. Then, since $\mathbf{z}_2=M_2M_1^{-1}\mathbf{z}_1\equiv M_{21}\mathbf{z}_1$, we want to find the unitary mapping between $\mathcal{F}_1$ and $\mathcal{F}_2$. Labelling $\bar{f}^{(k)}(\eta)\in\mathcal{F}_k$ and the corresponding measures $d\mu_k(\eta)$, we obtain from (1.6),

$$\bar{f}^{(2)}(\eta)=\int_{\mathbb{C}}d\mu_1(\eta')A_{21}(\eta,\eta')\bar{f}^{(1)}(\eta'), \tag{5.1a}$$

$$\bar{f}^{(1)}(\eta')=\int_{\mathbb{C}}d\mu_2(\eta)A_{21}(\eta,\eta')^*\bar{f}^{(2)}(\eta), \tag{5.1b}$$

where the transform kernel $A_{21}(\eta,\eta')$ from $\mathcal{F}_1$ to $\mathcal{F}_2$ is

$$A_{21}(\eta,\eta')=\int_{\mathbb{R}}dx\,A_2(\eta,x)A_2(\eta',x)^*=A_{12}(\eta',\eta)^*. \tag{5.1c}$$

Explicitly, it is

$$A_{21}(\eta,\eta')=\Phi(b_2,-b_1^*;b)\exp[-\tfrac{1}{2}i(\pi/2+\Phi(b))](2\pi|b|^{-1/2}\times\exp\{(i/2b)[a\eta'^{*2}-2\eta'^*\eta+d\eta^2]\}, \tag{5.2a}$$

where

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}^{*-1}, \tag{5.2b}$$

and

$$\Phi(b',b'';b)\equiv\exp\{-\tfrac{1}{2}i[\Phi(b')+\Phi(b'')-\Phi(b)-\Phi(b'b''/b)]\}=\pm 1 \tag{5.2c}$$

[compare with Eqs. (3.3)], and can be written as a generating function

$$A_{21}(\eta,\eta')=\sum_{n=0}^{\infty}\bar{\psi}_n^{(2)}(\eta)\bar{\psi}_n^{(1)}(\eta')^*. \tag{5.3}$$

In particular, this allows us to define $A_k(\eta,x)\equiv A_{k0}(\eta,x)$, $A_{0k}(x,\eta)\equiv A_k(\eta,x)^*$ and the reproducing kernel in each space as $K_k(\eta,\eta')=A_{kk}(\eta,\eta')$. The composition of transforms can then be effected through any (allowed) space $\mathcal{F}_3$ as

$$A_{21}(\eta,\eta'')=\int_{\mathbb{C}}d\mu_3(\eta')A_{23}(\eta,\eta')A_{31}(\eta',\eta''), \tag{5.4}$$

which generalizes (5.1c) when we understand that $\int_{\mathbb{C}}d\mu_0(\eta)\cdots=\int_{\mathbb{R}}dx\cdots$ and $\mathcal{H}\equiv\mathcal{F}_0$, it corresponds to $M_{21}=M_{23}M_{31}$ for $M_{31}=M_3M_1^{-1}$, etc. with the explicit forms as obtained from (5.2). Notice that when M_1 and M_2 belong to the class $v=0$, the transform (5.1) involves only line integrals although $M_2M_1^{-1}$ may not belong to this class. Similarly, the condition $\mathrm{Im}(a/b)\geq 0$ which must hold for M_1 and M_2 may not hold for their composition $M_2M_1^{-1}$. The existence of the transform (5.1) is assured, however, as $A_{21}(\eta,\eta')$ belongs to $\mathcal{F}_1$ as a function of its second argument and to $\mathcal{F}_2$ as a function of the first. Square integrability is only demanded in $\mathcal{H}$ or its isomorphic spaces.

VI. LINEAR OPERATORS AND REPRESENTATIONS OF HSL (2,C)

Let $\mathcal{P}$ be a bounded operator mapping $\mathcal{H}$ onto itself, represented by an integral kernel $P(x,x')$ through

$$f'(x)=\int_{\mathbb{R}}dx'P(x,x')f(x'). \tag{6.1}$$

It then follows from (1.6) that $\mathcal{P}$ will also map $\mathcal{F}$ onto $\mathcal{F}$ through

$$\bar{f}'(\eta)=\int_{\mathbb{C}}d\mu(\eta')\bar{P}(\eta,\eta')\bar{f}(\eta'), \tag{6.2}$$

represented by the integral kernel

$$\bar{P}(\eta,\eta')=\int\int_{\mathbb{R}}dx\,dx'A(\eta,x)P(x,x')A(\eta',x')^*. \tag{6.3}$$

To a product $\mathcal{R}=\mathcal{P}\mathcal{Q}$ of such bounded operator then corresponds

$$R(x,x'')=\int_{\mathbb{R}}dxP(x,x')Q(x',x'') \tag{6.4}$$

which is also bounded and

$$\bar{R}(\eta,\eta'')=\int_{\mathbb{C}}d\mu(\eta')\bar{P}(\eta,\eta')\bar{Q}(\eta',\eta''). \tag{6.5}$$

In particular, to the unit operator, whose representative in $\mathcal{H}$ is $\delta(x-x')$, will correspond through (3.4a) the reproducing kernel $K(\eta,\eta')$ in $\mathcal{F}$.

Now, for every $M\in SL(2,\mathbb{C})$, consider the operator $\mathcal{A}(M)$ with the integral kernel given by (3.3), when we restrict η to the real line. These are now operators mapping $\mathcal{H}$ onto $\mathcal{H}$, and can be seen as *passive* $SL(2,\mathbb{C})$ transformations, as opposed to the active transformations seen in the last section, which mapped $\mathcal{H}$ onto $\mathcal{F}$. We shall denote this integral kernel by

$$D^{(0)}_{xx'}(M)=A_M(x,x')=\exp[-\tfrac{1}{2}i(\pi/2+\Phi(b))](2\pi|b|)^{-1/2}\times\exp\{(i/2b)[ax'^2-2x'x+dx^2]\}. \tag{6.6}$$

When integration is possible, these kernels satisfy

$$\int_{\mathbb{R}}dx'D^{(0)}_{xx'}(M_1)D^{(0)}_{x'x''}(M_2)=\Phi(b_1,b_2;b_{12})D^{(0)}_{xx''}(M_1M_2) \tag{6.7}$$

and hence form a ray representation of a subset of

$SL(2,\mathbb{C})$: the subset for which the operators $\mathcal{A}(M)$ are bounded. As the product of two bounded operators is bounded, such a set must be a semigroup contained in $SL(2,\mathbb{C})$.

Notice first that the kernels representing $\mathcal{A}(M)$ with $M\in SL(2,\mathbb{R})$ are bounded. This is obvious when we examine the transform normalized basis (3.8), as here $\mathcal{J}\equiv\mathcal{H}$, $(\mathcal{A}(M)\psi_n,\mathcal{A}(M)\psi_n)_0=(\bar{\psi}_n,\bar{\psi}_n)_0=1$ and $\{\psi_n(x)\}$ is dense in $\mathcal{H}$ and $\mathcal{J}$. For $M\in SL(2,\mathbb{C})$ the operators $\mathcal{A}(M)$ will be Hilbert—Schmidt operators when the kernels (6.6) satisfy $\iint dx\,dx'|D^{(0)}_{xx'}(M)|^2<\infty$. In performing the integrals, we see that we obtain the conditions

$$\mathrm{Im}\,b^*a>0:v>0, \tag{6.8a}$$

$$\mathrm{Im}\,b^*a\,\mathrm{Im}\,b^*d>\mathrm{Im}^2 b. \tag{6.8b}$$

Now, the product of a Hilbert—Schmidt and a bounded one is a Hilbert—Schmidt operator, hence the set of matrices

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\cosh\zeta & -i\sinh\zeta\\ i\sinh\zeta & \cosh\zeta\end{pmatrix}\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} \tag{6.9}$$

($\alpha,\alpha',\beta,\ldots,\delta'$ real) will be represented by Hilbert—Schmidt operators for $\zeta>0$, as can be verified directly from (6.8). This is a semigroup which does not contain the identity. If we add to (6.9) the point $\zeta=0$, thereby making (6.9) contain $SL(2,\mathbb{R})$, we will have a set of bounded operators representing the semigroup denoted by $HSL(2,\mathbb{C})$ in Ref. 6. Notice that the matrix (1.4) corresponding to the Bargmann transform does not belong to this set.

An important subset of $HSL(2,\mathbb{C})$ is the set of matrices which we write and decompose as

$$\begin{pmatrix}\alpha'' & -i\beta''\\ i\gamma'' & \delta''\end{pmatrix}=\begin{pmatrix}1 & 0\\ iq & 1\end{pmatrix}\begin{pmatrix}D & 0\\ 0 & D^{-1}\end{pmatrix}\begin{pmatrix}1 & -iq'\\ 0 & 1\end{pmatrix} \tag{6.10}$$

with $\alpha'',\ldots,\delta''$, $q,q'\geq 0$, $D>0$, which are bounded, but not Hilbert—Schmidt operators [as conditions (6.8) may be violated]. The set (6.10) manifestly forms a semigroup denoted by $HSL(2,\mathbb{R})$ in Ref. 6, since it is related through a similarity transformation [by a diagonal matrix with elements $\exp(-i\pi/4)$, $\exp(i\pi/4)$] with the set of $SL(2,\mathbb{R})$ matrices with nonnegative elements. The parametrization (6.10) furthermore allows us to reach the special cases $\beta''=0$ [Eq. (4.4) which simulates the Gaussian potential] for which the decomposition (6.9) fails.

From the representation (6.6) we can build through (6.3) a continuum of representations of $HSL(2,\mathbb{C})$ through (5.1c) as

$$D^{(k)}_{\eta\eta'}(M)=D^{(0)}_{\eta\eta'}(M_kMM_k^{*-1}) \tag{6.11}$$

where $M_k\in SL(2,\mathbb{C})$ satisfying the conditions for the existence of a transform. Notice that the variable η' in (6.11) appears as η'^* in the explicit form (6.6). These D's will exhibit the composition

$$\int_{\mathbb{C}} d\mu_k(\eta')D^{(k)}_{\eta\eta'}(M_1)D^{(k)}_{\eta'\eta''}(M_2)=\varphi D^{(k)}_{\eta\eta''}(M_1M_2) \tag{6.12}$$

and the property

$$D^{(k)}_{\eta\eta'}(M)=D^{(k)}_{\eta'\eta}(M^{*-1})^* \tag{6.13}$$

so that the representation is unitary for $M\in SL(2,\mathbb{R})$.

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APPENDIX A: REALIZATION THROUGH HYPERDIFFERENTIAL OPERATORS

In this Appendix we want to introduce a Lie algebra structure for the set of canonical transforms as

$$\bar{f}(x)=[U_\tau f](x)=\exp(i\tau H)f(x)=\int_{\mathbb{R}} dx'A_\tau(x,x')f(x') \tag{A1}$$

where τ labels the elements of a one-parameter subgroup (or subsemigroup) of $SL(2,\mathbb{C})$. For our purposes it is sufficient to ask that the integral in (A1) to exist, so that we can disregard the Hilbert space structure of the functions involved, and the operator U_τ need not be bounded.[14,15]

We want to find a differential operator H which generates the transform (A1), i.e.,

$$H\left(x,\frac{d}{dx}\right)f(x)=-i\int_{\mathbb{R}}dx'\left[\frac{\partial}{\partial\tau}A_\tau(x,x')\bigg|_{\tau=0}f(x')\right] \tag{A2a}$$

with the boundary condition

$$A_\tau(x,x')|_{\tau=0}=\delta(x-x'). \tag{A2b}$$

If we knew H and solved for $A_\tau(x,x')$, this would be a Green's function problem,[16] where $A_\tau(x,x')$ is the Green's function of $\exp(+i\tau H)$. Here we know $A_\tau(x,x')$ as given by (6.6) and [and (4.4)], so that we can build the operator $H(x,d/dx)$ by inspection of (A2a), for various one-parameter subgroups of $SL(2,\mathbb{C})$, viz.:

$$\exp[ic(\tfrac{1}{2}\hat{x}^2)]:\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}, \tag{A3a}$$

$$\exp[ib(\tfrac{1}{2}\hat{p}^2)]:\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}, \tag{A3b}$$

$$\exp[i\alpha\tfrac{1}{4}(\hat{p}^2-\hat{x}^2)]:\begin{pmatrix}\cosh\frac{1}{2}\alpha & -\sinh\frac{1}{2}\alpha\\ -\sinh\frac{1}{2}\alpha & \cosh\frac{1}{2}\alpha\end{pmatrix}, \tag{A3c}$$

$$\exp[i\beta\tfrac{1}{4}(\hat{x}\hat{p}+\hat{p}\hat{x})]:\begin{pmatrix}\exp(-\frac{1}{2}\beta) & 0\\ 0 & \exp(\frac{1}{2}\beta)\end{pmatrix}, \tag{A3d}$$

$$\exp[i\gamma\tfrac{1}{4}(\hat{p}^2+\hat{x}^2)]:\begin{pmatrix}\cos\frac{1}{2}\gamma & -\sin\frac{1}{2}\gamma\\ \sin\frac{1}{2}\gamma & \cos\frac{1}{2}\gamma\end{pmatrix}. \tag{A3e}$$

The last three generators can be seen to constitute the well-known $su(1,1)$ dynamical algebra of the harmonic oscillator,[2] (A3d) being a scale operator, i.e.,

$$\bar{f}(x)=\exp[i\beta\tfrac{1}{4}(\hat{x}\hat{p}+\hat{p}\hat{x})]f(x)=\exp(\tfrac{1}{4}\beta)f[\exp(\tfrac{1}{2}\beta)x] \tag{A4}$$

while Eq. (A3e), $\frac{1}{2}(\hat{x}^2+\hat{p}^2)$ being the oscillator Hamiltonian, gives the development in time $t=\frac{1}{2}\gamma$ of the system.

The association of hyperdifferential operators in (A3) with 2×2 matrices can yield a host of Baker—Campbell—Hausdorff relations between second order differential operators,[17] as

$$\begin{pmatrix}\cosh\theta & -\sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix} = \begin{pmatrix}1 & -\tanh\theta\\ 0 & 1\end{pmatrix}\begin{pmatrix}1/\cosh\theta & 0\\ 0 & \cosh\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ -\tanh\theta & 1\end{pmatrix} \quad \text{(A5a)}$$

which gives

$$\exp\left[-\tfrac{1}{2}i\theta\left(\frac{d^2}{dx^2}+x^2\right)\right] = \exp\left[-\tfrac{1}{2}i\tanh\theta\frac{d^2}{dx^2}\right]\exp\left[\tfrac{1}{2}\ln\cosh\theta\left(x\frac{d}{dx}+\frac{d}{dx}x\right)\right] \times\exp[-\tfrac{1}{2}i\tanh\theta x^2]. \quad \text{(A5b)}$$

Further, when allowed to act on specific functions f whose canonical transforms $\bar{f}$ are known, (A3) yield special function relations. For $\theta = i\pi/4$, (A5a) becomes the Bargmann transform matrix (1.4), thus

$$\bar{f}(x) = \exp\left[\tfrac{1}{8}\pi\left(\frac{d^2}{dx^2}+x^2\right)\right]f(x) = 2^{-1/4}\exp\left(\frac{1}{2}\frac{d^2}{dx^2}\right)\exp(\tfrac{1}{4}x^2)f(2^{-1/2}x). \quad \text{(A6)}$$

In particular, letting f be one of the harmonic oscillator wavefunctions $\psi_n(x)$ given by (3.7), $\bar{f}$ will be (2.9) for $u=0$, $v=1$. Eq. (A6) with a change of scale gives immediately

$$x^n = 2^{-n}\exp\left(\frac{1}{4}\frac{d^2}{dx^2}\right)H_n(x) \quad \text{(A7a)}$$

and its inverse

$$H_n(x) = \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)(2x)^n \quad \text{(A7b)}$$

which are formulas that do not commonly appear in special function tables.[14,18]

APPENDIX B; EXTENSION TO n DIMENSIONS

We shall sketch here some of the results for the case of n-dimensional spaces $\mathcal{F}^n$. The most general complex linear canonical transformation (2.1) now reads

$$\begin{pmatrix}\hat{\eta}\\ \hat{\xi}\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{p}}\end{pmatrix} \quad \text{(B1)}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$, $\hat{\eta}$, and $\hat{\xi}$ are n-component column vectors and $A, \ldots, D$ are $n\times n$ matrices satisfying[2] $A\tilde{B} = B\tilde{A}$, $C\tilde{D} = D\tilde{C}$, and $A\tilde{D} - B\tilde{C} = \mathbb{1}$ (the tilde means matrix transposition). Hermitian conjugation is achieved as

$$\begin{pmatrix}\hat{\eta}^+\\ \hat{\xi}^+\end{pmatrix} = \begin{pmatrix}A^* & B^*\\ C^* & D^*\end{pmatrix}\begin{pmatrix}\tilde{D} & -\tilde{B}\\ -\tilde{C} & \tilde{A}\end{pmatrix}\begin{pmatrix}\hat{\eta}\\ \hat{\xi}\end{pmatrix} = \begin{pmatrix}U & iV\\ iW & \tilde{U}^*\end{pmatrix}\begin{pmatrix}\hat{\eta}\\ \hat{\xi}\end{pmatrix}, \quad \text{(B2)}$$

where $U = A^*\tilde{D} - B^*\tilde{C}$, $V = 2\,\mathrm{Ah}(B^*\tilde{A})$, and $W = 2\,\mathrm{Ah}(C^*\tilde{D})$, the symbol $\mathrm{Ah}M = (2i)^{-1}(M - \tilde{M}^*)$ denotes the anti-Hermitian part of a matrix, so that V and W are Hermitian and their determinants are real. An analysis parallel to (2.4)–(2.7) yields a Hermitian form for the space $\mathcal{F}^n$ given by

$$(\bar{f}, \bar{g}) = \int_{\mathbb{C}^n}\nu(\eta,\eta^*)d^n\mathrm{Re}\,\eta\, d^n\mathrm{Im}\,\eta f(\eta)^* g(\eta) \quad \text{(B3)}$$

with the weight

$$\nu(\eta,\eta^*) = ([\tfrac{1}{2}\pi]^n\det V)^{-1/2}\exp\{\tfrac{1}{2}\tilde{\eta}V^{-1}U\eta - \tilde{\eta}V^{-1}\eta^* + \tfrac{1}{2}\tilde{\eta}^*V^{*-1}U^*\eta^*\} \quad \text{(B4)}$$

the growth restrictions on $\bar{f}\in\mathcal{F}^n$ can be seen writing $\bar{f}(\eta) = f_b(V^{*-1/2}\eta)\exp\{-\tfrac{1}{2}\tilde{\eta}V^{-1}U\eta\}$ where $(V^{1/2})^2 = V$. As V is Hermitian, when we ask it to be positive definite, its positive definite square root is uniquely defined and f_b can be asked to be in the n-dimensional Bargmann space. The restrictions are then

$$|\bar{f}(\eta)| \le \gamma\exp\{\tfrac{1}{2}\alpha\tilde{\eta}V^{-1}\eta^* - \tfrac{1}{2}\mathrm{Re}[\tilde{\eta}V^{-1}U\eta]\}, \quad \alpha < 1. \quad \text{(B5)}$$

The transform kernel between $\mathcal{H}^n$ and $\mathcal{F}^n$ will be, in terms of the submatrices in (B1), up to a phase φ,

$$A(\eta, x) = \varphi([2\pi]^n|\det B|)^{-1/2}\exp i\{\tfrac{1}{2}\tilde{\mathbf{x}}B^{-1}A\mathbf{x} - \tilde{\mathbf{x}}B^{-1}\eta + \tfrac{1}{2}\tilde{\eta}DB^{-1}\eta\} \quad \text{(B6)}$$

out of an analysis parallel to (3.1)–(3.3).

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