The algebra and group deformations $I^m\,[SO(n) \otimes SO(m)] \Rightarrow SO(n,m)$, $I^m\,[U(n) \otimes U(m)] \Rightarrow U(n,m)$, and $I^m\,[Sp(n) \otimes Sp(m)] \Rightarrow Sp(n,m)$ for $1 \leqslant m \leqslant n$

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We discuss a class of deformations of the inhomogeneous classical algebras $i^m\ [k(n) \oplus k(m)]$ to $k(n,m)$ for $1 \leqslant m \leqslant n$. This generalizes the previously known expansions $i\,k(n) \Rightarrow k(n,1)$. As the title indicates, this is done explicitly for the orthogonal, unitary, and symplectic cases. We construct the corresponding deformed groups $K(n,m)$ as multiplier representations on the space of functions over the rank m coset space $K(n-m)\backslash K(n)$. This method allows us to build a principal series of unitary representations of $K(n,m)$. The contractions of the deformed algebras and groups are considered.

I. INTRODUCTION

The concepts of expansion and deformation of a Lie algebra are being actively developed in the mathematical physics literature. The motivation in studying this subject is twofold: first, it is, loosely speaking, the inverse of contraction and second, it allows one to build dynamical algebras for systems whose symmetry algebras are realized as a set of operators acting on a definite homogeneous space.

The first deformations treated were $iso(n) \Rightarrow so(n,1)$,[1] $i[u(n) \oplus u(1)] \Rightarrow u(n,1)$,[2] and $i[sp(n) \oplus sp(1)] \Rightarrow sp(n,1)$,[3] and their noncompact versions,[4] which constitute a family shown by Gilmore[5] to have a rank 1 coset space in the Cartan decomposition.[6] These deformations were constructed by the use of an algorithm in which the noncompact generators are produced by commuting the Casimir operator of a classical algebra with an element of a normal Abelian algebra transforming as a vector—a rank 1 tensor. A second family was treated in Ref. 7, which uses the algorithm with an Abelian algebra transforming as a second-rank symmetric tensor under the classical algebra. This produced the deformations of representations of $i_2so \Rightarrow sl(n,\mathbb{R})$, $i_2u(n) \oplus u(1) \Rightarrow sl(n,\mathbb{C}) \oplus u(1)$, and $i_2sp(n) \oplus sp(1) \Rightarrow sl(n,\mathbb{Q}) \oplus sp(1)$, which can only be realized on a rank 1 homogeneous space. As an example, for the hydrogen atom system where the symmetry algebra is $so(4)$ and the homogeneous space is the 3-sphere projected out of momentum space, the first kind of deformation yields $so(4,1)$[8] as a dynamical algebra while the second kind yields $sl(4,\mathbb{R})$.[9]

In Sec. II we show that the first family of deformations can be generalized by considering an Abelian ideal transforming as a set of m orthogonal vectors and gives rise to the deformations[10] $i^m[so(n) \oplus so(m)] \Rightarrow so(n,m)$, $i^m[u(n) \oplus u(m)] \Rightarrow u(n,m)$, and $i^m[sp(n) \oplus sp(m)] \Rightarrow sp(n,m)$. We will see that the set of vectors x_μ^α forming the Abelian ideal is also isomorphic to the rank m homogeneous spaces, respectively, $SO(n-m)\backslash SO(n)$, $U(n-m)\backslash U(n)$, and $Sp(n-m)\backslash Sp(n)$ for $1 \leqslant m \leqslant n$, and contains the first family mentioned for the special case $m = 1$. To the best of the authors' knowledge, this is the first application of the above deformation algorithm to coset spaces of rank greater than one. A quite different expansion, however, has been given by Mukunda.[11]

Corresponding to deformations of Lie algebra representations there are deformations of representations of the Lie groups[12] which gives rise to multiplier representations of the type developed by Bargmann[13] and Gel'fand and collaborators,[14] and generalized by Mackey.[15] The deformations $ISO(n) \Rightarrow SO(n,1)$[16] and $IU(n) \otimes U(1) \Rightarrow U(n,1)$[17] realized as multiplier representations on the real and complex spheres have been used to find the unitary irreducible representation (UIR) matrix elements of the principal series of the $SO(n,1)$ and $U(n,1)$ groups and have also been applied to the supplementary series[18] of the former. This method has been further used to develop a complete solution to the "missing label" problem in the noncanonical chain reduction $SO(n,1) \supset SO(1,1) \otimes SO(n-1)$.[19] The corresponding group representations for the second family of deformations has also been developed.[7]

In Sec. III we carry the deformation over to the corresponding groups and study the "nonrigid" action of the deformed group on the rank m homogeneous spaces, and show that one can thus obtain a principal series of UIRs of the deformed group. In Sec. IV we touch upon the inverse problem of contraction.[20]

II. DEFORMATIONS OF THE CLASSICAL ALGEBRAS

A. General construction

Consider the classical Lie algebra $k(n)$, which can be $so(n)$, $u(n)$, or $sp(n)$, the metric-preserving algebra of the sphere $S_{n-1}^{\mathbb{F}}$ on a field $\mathbb{F}$, i.e., the real $\mathbb{R}$, complex $\mathbb{C}$ or quaternionic $\mathbb{Q}$ fields, respectively, given by $x_\mu x_\mu^* = 1$, where the asterisk $*$ stands for the involutive automorphism of the field, identity for $\mathbb{R}$, complex conjugation for $\mathbb{C}$ and quaternionic conjugation for $\mathbb{Q}$. Summation over repeated indices is implied and all middle

Greek letters range from 1 to n. The dimension of $S_{n-1}^{\mathbb{F}}$ is $n \dim \mathbb{F} - 1$. Let the index ω range over the corresponding components of the field $\mathbb{F}$, i.e., $\omega \equiv 0$ for $\mathbb{R}$, $\omega = 0,1$ for $\mathbb{C}$, and $\omega = 0,1,2,3$ for $\mathbb{Q}$. The $n \dim \mathbb{F}$ quantities $x_\mu{}^\omega$ transform as the components of a vector under commutation with the elements $M_{\mu\nu}^\omega$ of $k(n)$ and can be adjoined to them to construct the inhomogeneous classical algebra $ik(n)$, semidirect sum of $k(n)$ with an $[n \dim \mathbb{F}]$-dimensional Abelian ideal. It has been known[1-4] that out of the $x_\mu{}^\omega$ and the second-order Casimir operator $\Psi_k = \frac{1}{2} M_{\mu\nu}^\omega M_{\mu\nu}^\omega$ of $k(n)$ one can build the $n \dim \mathbb{F}$ operators

$$M_{\mu,n+1}^\omega = \tfrac{1}{2}[\Psi_k, x_\mu^\omega] + \tau x_\mu^\omega, \quad \tau \in \mathbb{C}, \tag{2.1}$$

which are elements of the enveloping algebra of $ik(n)$. We can verify that together with the generators of $k(n)$, they close into a $k(n,1)$ algebra [$so(n,1)$, $u(n,1)$, and $sp(n,1)$, respectively, in the last two cases, though, one has to add[6] to the former set the commutator between two operators (2.1), producing $u(1)$ and $sp(1)$ subalgebras which commute with the original compact ones].

We now introduce the action of the algebra $k(n)$ on a set of mn-vectors $x_\mu{}^\alpha$ $(\alpha = 1, \ldots, m)$. Such vectors can be taken as orthonormal since $k(n)$ commutes with the vector space scalar product $x_\mu{}^\alpha x_\mu{}^{\beta *}$. We introduce the constraints

$$x_\mu{}^\alpha x_\mu{}^{\beta *} = \delta_{\alpha,\beta}. \tag{2.2}$$

Such a choice of vectors can be conveniently thought of as an $n \times m$ rectangular matrix which is a submatrix of the $n \times n$ matrix self-representation of the Lie group generated by $k(n)$. Equation (2.2) represents $\frac{1}{2}m(m-1) \times \dim \mathbb{F} + m$ restrictions since $x_\mu^\alpha x_\mu^{\beta *} = (x_\mu^\beta x_\mu^{\alpha *})^*$ and $x_\mu^\alpha x_\mu^{\alpha *}$ is real. The number of independent components of the matrix $\mathbf{x}$ is thus $m[(n - \frac{1}{2}\{m-1\}) \dim \mathbb{F} - 1]$. The $nm \dim \mathbb{F}$ quantities $x_\mu{}^{\alpha\omega}$, however, can form the generators of an Abelian algebra which, when added in semidirect sum to $k(n)$ produces what we shall call the $i^m k(n)$ algebra. Our algorithm now generalizes (2.1) in constructing the operators

$$M_{\mu,n+\alpha}^\omega = \tfrac{1}{2}[\Psi_k, x_\mu^{\alpha\omega}] + \tau x_\mu^{\alpha\omega}, \quad \tau \in \mathbb{C}. \tag{2.3}$$

Moreover, building the commutators $[M_{\mu,n+\alpha}^\omega, M_{\nu,n+\beta}^{\omega'}]$, we see that we still obtain some extra operators $M_{n+\alpha,n+\beta}^{\omega''}$ which close onto a $k(m)$ algebra commuting with the original $k(n)$, and all of these, together with (2.3) form a $k(n,m)$ algebra. The free parameter τ, it has to be noted, must be the same for all $M_{\mu,n+\alpha}^\omega$ (i.e., it cannot have indices μ, α, or ω), or the resulting operators will not close onto an algebra of finite dimension. We will now write the results for the classical groups considered, using for consistency the relations as presented in Ref. 7.

B. $i^m\,[so(n) \oplus so(m)] \Rightarrow so(n,m)$

The generators of $so(n)$ are $M_{\mu\nu}$ with the commutation relations[21]

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho} + g_{\mu\sigma} M_{\nu\rho}, \tag{2.4}$$

with $g_{\mu\nu} = \delta_{\mu\nu}$ for $\mu, \nu = 1, \ldots, n$. The generators of the normal Abelian subalgebra are $x_\mu{}^\alpha$ $(\alpha = 1, \ldots, m)$ satisfying (2.2) and

$$[M_{\mu\nu}, x_\rho{}^\alpha] = \delta_{\nu\rho} x_\mu{}^\alpha - \delta_{\mu\rho} x_\nu{}^\alpha. \tag{2.5}$$

Our deformation algorithm (2.3) now takes the form

$$M_{\mu,n+\alpha} = \tfrac{1}{2}[\Psi_{so}, x_\mu^\alpha] + \tau x_\mu^\alpha = x_\nu^\alpha M_{\nu\mu} + (-\tfrac{1}{2}[n-1] + \tau) x_\mu^\alpha. \tag{2.6}$$

Moreover, the commutator of two of the generators (2.6) will bring in the generators

$$M_{n+\alpha,n+\beta} = x_\mu{}^\alpha x_\nu{}^\beta M_{\mu\nu} \tag{2.7}$$

which close onto an $so(m)$ algebra and commute with the $M_{\mu\nu}$'s verifying that (2.6), (2.7), and the $M_{\mu\nu}$'s satisfy (2.4) with $g_{n+\alpha,n+\beta} = -\delta_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, m)$. Furthermore, one can show that the $so(m)$ subalgebra of generators (2.7) acts on the *column* indices of x_μ^α as

$$[M_{n+\alpha,n+\beta}, x_\mu{}^\gamma] = -(\delta_{\beta\gamma} x_\mu{}^\alpha - \delta_{\alpha\gamma} x_\mu{}^\beta), \tag{2.8}$$

i.e., as a vector with respect to the *upper* index. Notice that (2.5) and (2.8), however, have opposite signs. This will be shown in the next section to correspond to group actions from left and right.

C. $i^m\,[u(n) \oplus u(m)] \Rightarrow u(n,m)$

It is convenient to deal with the "complex" form of the generators of $u(n)$ given by $C_{\mu\nu}$ with the commutation relations

$$[C_{\mu\nu}, C_{\rho\sigma}] = g_{\nu\rho} C_{\mu\sigma} - g_{\mu\sigma} C_{\rho\nu}. \tag{2.9}$$

The Abelian generators are $z_\mu{}^\alpha$ and $z_\mu{}^{\alpha *}$ satisfying (2.2) in its form $z_\mu{}^\alpha z_\mu{}^{\beta *} = \delta_{\alpha\beta}$ and

$$[C_{\mu\nu}, z_\rho{}^\alpha] = \delta_{\nu\rho} z_\mu{}^\alpha, \tag{2.10}$$

$$[C_{\mu\nu}, z_\rho^{\alpha *}] = -\delta_{\mu\rho} z_\nu^{\alpha *}, \tag{2.11}$$

forming the $i^m u(n)$ algebra. The Casimir operator $\Psi_u = -2 C_{\mu\nu} C_{\nu\mu}$ now leads us to write (2.3) in the form

$$C_{\mu,n+\alpha} = \tfrac{1}{4}[\Psi_u, z_\mu^\alpha] + \tau' z_\mu^\alpha = -z_\nu^\alpha C_{\mu\nu} + (-\tfrac{1}{2}n + \tau') z_\mu^\alpha, \tag{2.12a}$$

$$C_{n+\alpha,\mu} = -\tfrac{1}{4}[\Psi_u, z_\mu^{\alpha *}] + \tau'^* z_\mu^{\alpha *} = -z_\nu^{\alpha *} C_{\nu\mu} + (\tfrac{1}{2}n + \tau'^*) z_\mu^{\alpha *}, \tag{2.12b}$$

and out of the commutators of (2.12) we find

$$C_{n+\alpha,n+\beta} = z_\mu^{\alpha *} z_\nu^\beta C_{\mu\nu} - (\tau' + \tau'^*) \delta_{\alpha\beta} \tag{2.13}$$

which together with (2.12) and the $C_{\mu\nu}$'s, close onto $u(n,m)$ with the commutator (2.9). Again we see that the z_μ^γ and $z_\mu^{\gamma *}$ transform as vectors with respect to the upper index under (2.13), i.e.,

$$[C_{n+\alpha,n+\beta}, z_\mu^\gamma] = \delta_{\alpha\gamma} z_\mu^\beta, \tag{2.14a}$$

$$[C_{n+\alpha,n+\beta}, z_\mu^{\gamma *}] = -\delta_{\beta\gamma} z_\mu^{\alpha *}. \tag{2.14b}$$

Writing $z_\mu^\alpha = x_\mu^{\alpha 0} + i x_\mu^{\alpha 1}$ and $z_\mu^{\alpha *} = x_\mu^{\alpha 0} - i x_\mu^{\alpha 1}$, the "real" form of the $u(n,m)$ generators can be written as

$$M_{ab}^0 = C_{ab} - C_{ba}, \tag{2.15a}$$

$$M_{ab}^1 = -i(C_{ab} + C_{ba}), \tag{2.15b}$$

for $a, b = 1, \ldots, n+m$. We can see that the results for the $so(n,m)$ algebra become a special case of those of the $u(n,m)$ algebra when we consider the subset of M^0's. Indeed, in the form (2.15) the $u(n,m)$ subalgebra of $sp(n,m)$ will become apparent in the next subsection.

D. $i^m\,[sp(n)\oplus sp(m)] \Rightarrow sp(n,m)$

As the symplectic algebra is not as well known as the other two classical algebras, we refer the reader to Ref. 7 where the explicit form of the generators is given. There are two isomorphic sets which we denote by $\{M^0_{\mu\nu}, M^{i+}_{\mu\nu}\}$ and $\{M^0_{\mu\nu}, M^{i-}_{\mu\nu}\}$ $(i=1,2,3)$, which correspond to left and right action with respect to quaternion multiplication. The commutation relations of the $sp(n)$ generators are

$$[M^0_{\mu\nu}, M^\omega_{\rho\sigma}] = g_{\nu\rho}M^\omega_{\mu\sigma} - g_{\mu\rho}M^\omega_{\nu\sigma} + g_{\nu\sigma}M^\omega_{\rho\mu} - g_{\mu\sigma}M^\omega_{\rho\nu}, \qquad (2.16a)$$

$$[M^i_{\mu\nu}, M^0_{\rho\sigma}] = g_{\nu\rho}M^i_{\mu\sigma} - g_{\mu\rho}M^i_{\nu\sigma} - g_{\nu\sigma}M^i_{\rho\mu} + g_{\mu\sigma}M^i_{\rho\nu}, \qquad (2.16b)$$

$$[M^i_{\mu\nu}, M^i_{\rho\sigma}] = -g_{\nu\rho}M^0_{\mu\sigma} - g_{\mu\rho}M^0_{\nu\sigma} + g_{\nu\sigma}M^0_{\rho\mu} + g_{\mu\sigma}M^0_{\rho\nu} \quad \text{(no sum)}, \qquad (2.16c)$$

$$[M^i_{\mu\nu}, M^j_{\rho\sigma}] = \epsilon_{ijk}(g_{\nu\rho}M^k_{\mu\sigma} + g_{\mu\rho}M^k_{\nu\sigma} + g_{\nu\sigma}M^k_{\rho\mu} + g_{\mu\sigma}M^k_{\rho\nu}). \qquad (2.16d)$$

The normal Abelian subalgebra is generated by $x^{\alpha\omega}_\mu$ satisfying (2.2) which, componentwise, yields $x^{\alpha\omega}_\mu\lambda^{\beta\omega}_\mu = \delta_{\alpha\beta}$ and $-x^{\alpha0}_\mu x^{\beta i}_\mu + x^{\alpha i}_\mu x^{\beta 0}_\mu - \epsilon_{ijk}x^{\alpha j}_\mu x^{\beta k}_\mu = 0$. The semidirect sum algebra $i^m sp(n)$ is given by (2.18) plus

$$[M^0_{\mu\nu}, x^{\alpha\omega}_\rho] = \delta_{\nu\rho}x^{\alpha\omega}_\mu - \delta_{\mu\rho}x^{\alpha\omega}_\nu, \qquad (2.17a)$$

$$[M^{i\pm}_{\mu\nu}, x^{\alpha0}_\rho] = \pm(\delta_{\nu\rho}x^{\alpha i}_\mu - \delta_{\mu\rho}x^{\alpha i}_\nu), \qquad (2.17b)$$

$$[M^{i\pm}_{\mu\nu}, x^{\alpha i}_\rho] = \mp(\delta_{\nu\rho}x^{\alpha 0}_\mu + \delta_{\mu\rho}x^{\alpha 0}_\nu) \quad \text{(no sum)}, \qquad (2.17c)$$

$$[M^{i\pm}_{\mu\nu}, x^{\alpha j}_\rho] = \epsilon_{ijk}(\delta_{\nu\rho}x^{\alpha k}_\mu + \delta_{\mu\rho}x^{\alpha k}_\nu). \qquad (2.17d)$$

For this case, the algorithm (2.3) takes the form

$$\begin{aligned} M^{0\pm}_{\mu,n+\alpha} &= \tfrac{1}{2}[\Psi^\pm_{sp}, x^{\alpha0}_\mu] + \tau x^{\alpha0}_\mu \\ &= x^{\alpha0}_\nu M^0_{\nu\mu} \pm x^{\alpha i}_\nu M^{i\pm}_{\nu\mu} + (-2n-1+\tau)x^{\alpha0}_\mu, \end{aligned} \qquad (2.18a)$$

$$\begin{aligned} M^{i\pm}_{\mu,n+\alpha} &= \tfrac{1}{2}[\Psi^\pm_{sp}, x^{\alpha i}_\mu] + \tau x^{\alpha i}_\mu \\ &= x^{\alpha i}_\nu M^0_{\nu\mu} \mp x^{\alpha0}_\nu M^{i\pm}_{\nu\mu} + \epsilon_{ijk}x^{\alpha j}_\nu M^{k\pm}_{\nu\mu} \\ &\quad + (-2n-1+\tau)x^{\alpha i}_\mu, \end{aligned} \qquad (2.18b)$$

and again, from their commutators we extract

$$\begin{aligned} M^0_{n+\alpha,n+\beta} &= x^{\alpha\omega}_\mu x^{\beta\omega}_\nu M^0_{\mu\nu} \\ &\quad + (\pm x^{\alpha0}_\mu x^{\beta i}_\nu \mp x^{\alpha i}_\mu x^{\beta0}_\nu - \epsilon_{ijk}x^{\alpha j}_\mu x^{\beta k}_\nu)M^{i\pm}_{\mu\nu}, \end{aligned} \qquad (2.19a)$$

$$\begin{aligned} M^{i\pm}_{n+\alpha,n+\beta} &= (x^{\alpha0}_\mu x^{\beta i}_\nu - x^{\alpha i}_\mu x^{\beta0}_\nu \mp \epsilon_{ijk}x^{\alpha j}_\mu x^{\beta k}_\nu)M^0_{\mu\nu} \\ &\quad \pm (x^{\alpha0}_\mu x^{\beta0}_\nu - x^{\alpha j}_\mu x^{\beta j}_\nu)M^{i\pm}_{\mu\nu} \\ &\quad + (\pm[x^{\alpha k}_\mu x^{\beta i}_\nu + x^{\alpha i}_\mu x^{\beta k}_\nu] - \epsilon_{ijk}[x^{\alpha0}_\mu x^{\beta j}_\nu + x^{\alpha j}_\mu x^{\beta0}_\nu])M^{k\pm}_{\mu\nu}, \end{aligned} \qquad (2.19b)$$

which after some calculation can be seen to close, together with (2.21) and the $M^\omega_{\mu\nu}$'s, onto $sp(n,m)$. Moreover, the $Sp(m)$ subalgebra generated by (2.19) transforms the $x^{\alpha\omega}_\mu$ as

$$[M^0_{n+\alpha,n+\beta}, x^{\gamma\omega}_\rho] = -(\delta_{\beta\gamma}x^{\alpha\omega}_\rho - \delta_{\alpha\gamma}x^{\beta\omega}_\rho), \qquad (2.20a)$$

$$[M^{i\pm}_{n+\alpha,n+\beta}, x^{\gamma0}_\rho] = \pm(\delta_{\beta\gamma}x^{\alpha i}_\rho - \delta_{\alpha\gamma}x^{\beta i}_\rho), \qquad (2.20b)$$

$$[M^{i\pm}_{n+\alpha,n+\beta}, x^{\gamma i}_\rho] = \mp(\delta_{\beta\gamma}x^{\alpha0}_\rho - \delta_{\alpha\gamma}x^{\beta0}_\rho) \quad \text{(no sum)}, \qquad (2.20c)$$

$$[M^{i\pm}_{n+\alpha,n+\beta}, x^{\gamma j}_\rho] = -\epsilon_{ijk}(\delta_{\beta\gamma}x^{\alpha k}_\rho - \delta_{\alpha\gamma}x^{\beta k}_\rho), \qquad (2.20d)$$

i.e., as the $M^\mp$'s on the column indices, acting from the opposite side, tensor- and quaternionwise, on the rectangular matrix x^α_μ.

It should be noticed that throughout this section we have never used any explicit realization of the algebra generators as operators on a homogeneous space. The only restriction has been that in each case we have only one continuous parameter τ, i.e., we can deform only along *one* direction.

III. MULTIPLIER REPRESENTATIONS

In this section we shall discuss representations of the groups $K(n,m)$ whose infinitesimal generators correspond to the Lie algebra representations presented in the last section. Rather than integrate directly these representations, we construct multiplier representations of $K(n,m)$ over the compact homogeneous space $X \equiv \{x^\alpha_\mu \in \mathbb{F} : x^\alpha_\mu x^\beta_\mu = \delta_{\alpha\beta}\}$ by generalizing the projective transformations on spheres used previously[16,17] for $SO(n,1)$ and $U(n,1)$ to projective transformations on X. Since such transformations will map X into itself, we are assured of the boundedness of the representations. Then by an appropriate choice of multiplier functions we obtain unitary representations. We then find the infinitesimal generators by the usual one-parameter subgroup method and it is seen that these correspond precisely to the formal representations of the Lie algebras $k(n,m)$ obtained through the deformation procedure of the previous section.

A. The group action

Given a realization of a compact classical group $K(n)$ of general element g by an $n\times n$ matrix $\mathbf{g} = \|g_{\mu\nu}\|$, $g_{\mu\nu} \in \mathbb{F}$ the action of $K(n)$ on the space of infinitely differentiable functions over the homogeneous space X can be written as

$$F(x^\alpha_\mu) \xrightarrow{g} F(g^{-1}_{\mu\nu}x^\alpha_\nu). \qquad (3.1a)$$

In the case when the field is the noncommutative quaternion field $\mathbb{Q}$, we have the possibility of a related though distinct action[7]

$$F(x^\alpha_\mu) \xrightarrow{g^{(*)}} F(x^\alpha_\nu g^{-1}_{\mu\nu}{}^*) \qquad (3.1b)$$

which is still from the left tensorwise, but from the right quaternionwise. The action (3.1a) for $Sp(n)$ is generated by the set of operators $\{M^0_{\mu\nu}, M^{i+}_{\mu\nu}\}$, while (3.1b) is generated by $\{M^0_{\mu\nu}, M^{i-}_{\mu\nu}\}$. For the $SO(n)$ groups (3.1a) and (3.1b) are the same and for $U(n)$ we have the complex conjugate representation of the group, an involutive automorphism of the algebra given by $M^i_{\mu\nu} \leftrightarrow -M^i_{\mu\nu}$. Only for $Sp(n)$ is it necessary to explicitly point out the difference.

In our case there are m orthonormal n-vectors forming an $n\times m$ rectagular matrix $\mathbf{x}$ satisfying (2.2). Consider first the vector x^1_μ. Equation (2.2) says it has to lie on the unit sphere $S^{\mathbb{F}}_{n-1}$. Now, x^2_μ is orthogonal to it, and thus constrained to lie on an $S^{\mathbb{F}}_{n-2}$ sphere orthogonal to x^1_μ. We follow the process up to x^m_μ and thus find that the space X is isomorphic with the product of the m spheres $S^{\mathbb{F}}_{n-1}\otimes S^{\mathbb{F}}_{n-2}\otimes\cdots\otimes S^{\mathbb{F}}_{n-m}$. This is also isomorphic to the homogeneous space $K(n-m)\backslash K(n)$ since a point $(x_0)^\alpha_\mu = \delta_{\mu\alpha}$ has $K(n-m)$ as its stability subgroup. The measure $d\mu(\mathbf{x})$ on X is induced by the Haar measure of $K(n)$ and is thus invariant under the action (3.1), the metric-preserving group of the manifold of $K(n)$. The transformations (3.1) can thus be called *rigid*.

B. Nonrigid transformations

The boost elements of the group $K(n,m)$, whose generators will be shown to be the noncompact operators (2.3) [concretely (2.6), (2.12), and (2.18)], are seen to produce *nonrigid* transformations of the space X, i.e., $d\mu(\mathbf{x})$ is not invariant under the general $K(n,m)$ action and a multiplier function is needed to obtain unitary representations.[13,14] Furthermore, the multiplier is generated by the inhomogeneous part of the $M^{\omega}_{\mu,n+\alpha}$'s, i.e., additive terms in $x^{\alpha\omega}_{\mu}$ with no derivative operators. Hence we obtain unitary representations of $K(n,m)$ in the form

$$F(\mathbf{x}) \xrightarrow{G} T^{\sigma}(G)F(\mathbf{x}) = \mu_{\sigma}(\mathbf{x}, \mathrm{G})F(\mathbf{x}'(\mathbf{x}, \mathrm{G})), \tag{3.2a}$$

where $G \in K(n,m)$ given by its $(n+m)\times(n+m)$ matrix representation G and, as will be shown in Secs. III.E to III.G.

$$\sigma = -\tfrac{1}{2}(n+m)\dim\mathbb{F} + 1 + i\rho, \quad \rho \text{ real}, \tag{3.2b}$$

and the action of the group on X, $\mathbf{x}'(\mathbf{x},\mathrm{G})$, will be given explicitly in Sec. III.D. Furthermore, as will be shown in Sec. III.F, the multiplier function $\mu_{\sigma}(\mathbf{x},\mathrm{G})$ enters into the Jacobian of the nonrigid transformation as

$$J \equiv \frac{d\mu(\mathbf{x}')}{d\mu(\mathbf{x})} = |\mu_{\sigma}(\mathbf{x},\mathrm{G})|^2. \tag{3.3}$$

C. The other rigid transformations in *K(n,m)*

The compact generators $M^{\omega}_{n+\alpha,n+\beta}$ produced out of commuting the operators (2.3) which close onto the $k(m)$ subalgebra of $k(n,m)$ [concretely (2.7), (2.13), and (2.19)] will be seen to correspond to the infinitesimal generators of a compact $K(m)$ subgroup of $K(n,m)$. It was shown in Sec. II that these generators transform the upper index of the x^{α}_{μ} in the same way (with opposite sign) as the original $k(n)$ algebra. We let the group action of $K(m)$ on the rectangular matrix $\mathbf{x}$ be from the right, i.e.,

$$F(x^{\alpha}_{\mu}) \xrightarrow{h} F(x^{\beta}_{\mu}h_{\beta\alpha}), \tag{3.4a}$$

where $\mathbf{h} = \|h_{\alpha\beta}\|$ is the $m\times m$ matrix realization of $h \in K(m)$ and, in the case when the x's and h's belong to the noncommutative field $\mathbb{Q}$, we have corresponding to the action (3.1b)

$$F(x^{\alpha}_{\mu}) \xrightarrow{h(*)} F(h^{*}_{\beta\alpha}x^{\beta}_{\mu}), \tag{3.4b}$$

for which the measure on X is again invariant and the transformations (3.4), therefore, rigid. Here we shall work only with the action (3.1a) and correspondingly (3.4a). The actions (3.1b)–(3.4b) do not bring in any fundamentally new features.

The space $X \approx K(n-m)\backslash K(n)$ is also isomorphic to a homogeneous space of the deformed group $K(n,m)$ which is elucidated in the Iwasawa decomposition[15] $K(n,m) \approx [K(n)\otimes K(m)]AN$, where A is an Abelian subgroup formed by m commuting boosts and N is a nilpotent subgroup. Now the stability subgroup of the point $(x_0)^{\alpha}_{\mu} = \delta_{\mu\alpha}$ can be shown[12] to be $H = \hat{K}AN$, where

$$\hat{\mathrm{K}} = \begin{pmatrix} \mathrm{K}(m) & 0 & 0 \\ 0 & \mathrm{K}'(n-m) & 0 \\ 0 & 0 & \mathrm{K}(m) \end{pmatrix}.$$

We can thus write $X \approx K(n-m)\backslash K(n) \approx H\backslash K(n,m)$.

D. General transformations of the homogeneous space

We shall now give explicitly the boost action (3.2) on the $n\times m$ rectangular matrix space X, generalizing the projective transformations used in Refs. 16 and 17 to matrix form. Split the matrix realization of $G\in K(n,m)$ as

$$\mathrm{G} = \begin{pmatrix} \mathbf{g} & \mathbf{b} \\ \tilde{\mathbf{b}} & \mathbf{h} \end{pmatrix}, \tag{3.5}$$

where $\mathbf{g}$ and $\mathbf{h}$ are $n\times n$ and $m\times m$ submatrices which contain the $K(n)$ and $K(m)$ subgroups of $K(n,m)$ and $\mathbf{b}$ and $\tilde{\mathbf{b}}$ are rectangular $n\times m$ and $m\times n$ matrices. We propose the action of G on X to be given by

$$\mathbf{x} \xrightarrow{G} \mathbf{x}' = (\mathbf{g}^{(-1)}\mathbf{x} + \mathbf{b}^{(-1)})(\tilde{\mathbf{b}}^{(-1)}\mathbf{x} + \mathbf{h}^{(-1)})^{-1} \tag{3.6}$$

where $\mathbf{g}^{(-1)}, \ldots, \mathbf{h}^{(-1)}$ are the *submatrices* of G^{-1} in the decomposition (3.5). We emphasize that $\mathbf{g}^{(-1)}, \ldots, \mathbf{h}^{(-1)}$ are *not* the inverses of the submatrices $\mathbf{g}, \ldots, \mathbf{h}$.

The action (3.6) is seen to give the correct composition law $F(\mathbf{x}''(\mathbf{x}'(\mathbf{x},\mathrm{G}_1),\mathrm{G}_2)) = F(\mathbf{x}'(\mathbf{x},\mathrm{G}_1\mathrm{G}_2))$ and reduce to the action (3.1a) and (3.4a) when $G\in K(n)$ and $K(m)$, respectively. Moreover, it can be verified that (3.6) preserves the restrictions (2.2) and hence maps the space X onto itself. Now consider an infinitesimal transformation through

$$\mathrm{G} \approx \mathbf{1} + \epsilon\Gamma, \quad \epsilon \ll 1, \quad \text{where } \Gamma = \begin{pmatrix} \gamma & \beta \\ \tilde{\beta} & \eta \end{pmatrix}.$$

The action on X is then given by

$$\mathbf{x}'(\mathbf{x}, 1+\epsilon\Gamma) \approx \mathbf{x} - \epsilon(\gamma\mathbf{x} + \beta - \mathbf{x}\eta - \mathbf{x}\tilde{\beta}\mathbf{x}). \tag{3.7}$$

It can be finally verified, after some computation, that the infinitesimal generators M in $F(\mathbf{x}'(\mathbf{x}, 1+\epsilon\Gamma)) = (\mathbb{1} - \epsilon M)F(\mathbf{x})$ are exactly the homogeneous part of the generators of $K(n,m)$ found by the deformation process in Sec. II when we use the explicit forms[7]

$$M^{0}_{\mu\nu} = x^{\alpha\omega}_{\mu}\partial^{\alpha\omega}_{\nu} - x^{\alpha\omega}_{\nu}\partial^{\alpha\omega}_{\mu}, \tag{3.8a}$$

$$M^{i\pm}_{\mu\nu} = \pm(x^{\alpha i}_{\mu}\partial^{\alpha 0}_{\nu} + x^{\alpha i}_{\nu}\partial^{\alpha 0}_{\mu} - x^{\alpha 0}_{\mu}\partial^{\alpha i}_{\nu} - x^{\alpha 0}_{\nu}\partial^{\alpha i}_{\mu}) - \epsilon_{ijk}(x^{\alpha j}_{\mu}\partial^{\alpha k}_{\nu} + x^{\alpha j}_{\nu}\partial^{\alpha k}_{\mu}), \tag{3.8b}$$

where $\partial^{\alpha\omega}_{\mu} \equiv \partial/\partial x^{\alpha\omega}_{\mu}$. We have the freedom to add to the generators (3.8) a spin part induced by the subgroup $K(n-m)$, the centralizer of the boosts. These would arise if we consider tensor-valued functions over the coset space X.

E. The multiplier function

The inhomogeneous part of the boost generators is obtained when we consider the full representation (3.2) with the multiplier function

$$\mu_{\sigma}(\mathbf{x},\mathrm{G}) = [\mathrm{DET}(\tilde{\mathbf{b}}^{(-1)}\mathbf{x} + \mathbf{h}^{(-1)})]^{\sigma/2}, \tag{3.9}$$

where the determinant symbol DET of an $m\times m$ matrix A of in general noncommuting quaternionic elements $A_{\alpha\beta}$ is to be taken as the ordinary determinant of a $2m\times 2m$ matrix constructed representing the quaternions involved as 2×2 submatrices $A^{0}_{\mu\nu}\sigma_{\sigma} - iA^{k}_{\mu\nu}\sigma_k$, where σ_k are the Pauli matrices and $\sigma_0 = \mathbf{1}$. This construction can also be used for the real and complex cases where we have $A^{\omega}_{\mu\nu}$ for $\omega \equiv 0$ and $\omega = 0, 1$. In these two cases DET A

$=|\det \mathbf{A}|^2$. Notice that the trace of such a matrix is $\mathrm{TR}\ \mathbf{A}=2A^0_{\mu\mu}$.

In order to show that (3.9) is the correct multiplier, one can verify the corresponding composition law $\mu(\mathbf{x},G_1)\mu(\mathbf{x}'(\mathbf{x},G_1),G_2)=\mu(\mathbf{x},G_1G_2)$ while $\mu(\mathbf{x},G_0)=1$ when $G_0\in K(n)\otimes K(m)$ and thus for the group identity. Notice also that due to (3.6), (3.9) cannot be zero. Again, the consideration of infinitesimal transformations yields for the multiplier function in (3.2)

$$\mu_\sigma(\mathbf{x},1+\epsilon\Gamma)\approx[\mathrm{DET}(1-\epsilon[\tilde{\beta}\mathbf{x}+\eta])]^{\sigma/2}$$
$$\approx 1-\tfrac{1}{2}\sigma\epsilon\mathrm{TR}(\tilde{\beta}\mathbf{x}+\eta)=1-\sigma\epsilon(\tilde{\beta}\mathbf{x})^0_{\alpha\alpha}. \quad (3.10)$$

Consideration of a particular infinitesimal boost given by one $\tilde{\beta}_{\alpha\mu}=1$ and all others zero, shows that the coefficient of ϵ is σx^α_μ, which is precisely the inhomogeneous part of all boost generators seen in the last section with

$$\sigma=-\tfrac{1}{2}(n+1)\dim \mathbb{F}+1+\tau. \quad (3.11)$$

F. The transformation Jacobian

Having found the multiplier function $\mu_\sigma(\mathbf{x},G)$ in (3.2), we shall show that the Jacobian function (3.3) is closely related to it. Instead of starting directly with the transformation (3.6), it will prove easiest to show that the infinitesimal Jacobian has a form related to (3.10).

First notice that not all $x^{\alpha\omega}_\mu$'s are independent, but obey the restrictions (2.2). We start, therefore, with $nm \dim\mathbb{F}$ *independent* quantities $y^{\alpha\omega}_\mu$ which are made to undergo the transformation (3.6) induced by the x's. We shall show that the transformation Jacobian $J'\equiv\partial(y')/\partial(y)$ is equal to (3.3). Indeed, parametrize $y^{\alpha\omega}_\mu$ through (i) the in general quaternionic quantities $r_{\alpha\beta}\equiv y^\alpha_\mu y^{\beta *}_\mu$ of which there are $\frac{1}{2}m(m-1)\dim\mathbb{F}+m$ independent components $r^\omega_{\alpha\beta}$. Since $r_{\alpha\beta}=r^*_{\beta\alpha}$ while $r_{\alpha\alpha}$ is real, (ii) the independent parameters in $x^{\alpha\omega}_\mu$ chosen so that they satisfy (2.2) and which can be written in terms of the quaternionic Euler angles.[22] Now, the Jacobian J' is independent of the $r_{\alpha\beta}$ since they are invariant under $K(n,m)$ transformations. Hence J' only depends on the $x^{\alpha\omega}_\mu$ and is thus the Jacobian J in (3.3).

The explicit calculation of the infinitesimal $J'=J$ proceeds rather easily: From (3.6) we find $\mathbf{y}'(\mathbf{y},1+\epsilon\Gamma)$ and of these we need only the diagonal elements $\partial y'^{\alpha\omega}_\mu/\partial y^{\alpha\omega}_\mu$ (no sum). The Jacobian then reduces to

$$J\approx 1-\epsilon[(n+m)\dim\mathbb{F}-2]\mathrm{TR}(\tilde{\beta}\mathbf{x})^0 \quad (3.12)$$

which is directly comparable with (3.10) and (3.11), and assures us the form (3.3).

G. Unitary representations of *K(n,m)* on *X*

We can obtain unitary representations of the group $K(n,m)$ on the space of infinitely differentiable functions[23] over $X\approx K(n-m)\backslash K(n)$ completing then with respect to the norm induced by the inner product

$$(F_1,F_2)_X=\int_X d\mu(\mathbf{x})F_1(\mathbf{x})^*F_2(\mathbf{x}) \quad (3.13)$$

when we introduce the group action through the operators $T^\sigma(G)$ as in (3.2). The choice of a complete and orthonormal set of functions $\{\Phi_n(\mathbf{x})\}$ on X allows the construction of the representation matrix elements as

$$D^\sigma_{n'n}(G)=(\Phi_{n'},T^\sigma(G)\Phi_n)_X. \quad (3.14)$$

Due to the relation between the multiplier function and the transformation Jacobian, for

$$\tau=-\tfrac{1}{2}(m-1)\dim\mathbb{F}+i\rho,\quad \rho \text{ real}, \quad (3.15)$$

the representations are unitary, i.e., $(T^\sigma(G)F_1, T^\sigma(G)F_2)_X=(F_1,F_2)_X$. They correspond to a principal degenerate series of representations[22,24] of $K(n,m)$ characterized by the value of σ given by (3.2b). Had we used the freedom allowed by the addition of a "spin" part to the generators, our functions F would be tensor-valued and the inner product (3.13) would include an inner product in an additional finite-dimensional vector space. In this way we can describe less degenerate representations where the additional labels are induced by the subgroup $K(n-m)$. This in no way hinders our construction since $K(n-m)$ is the boosts' centralizer in $K(n)\otimes K(m)$.

IV. CONTRACTIONS

A. Of the algebra

The representations of the algebras $k(n,m)$ in Sec. II can be labelled by $k(n,m)_\tau$. By a contraction of these representations we mean to divide some of the generators by τ and let $|\tau|\to\infty$. It is then seen that we can effect essentially two kinds of contractions, one with respect to the $k(n)$ subalgebra considering $M^\alpha_{\mu\nu}$, $\tau^{-1}M^\omega_{\mu,n+\alpha}$, and $\tau^{-1}M^\omega_{n+\alpha,n+\beta}$ and letting $|\tau|\to\infty$, thereby contracting $k(n,m)$ back to $i^m[k(n)\oplus 0]$, where 0 denotes the identity representation of $k(m)$. Another contraction considers $M^\omega_{\mu\nu}$, $\tau^{-1}M^\omega_{\mu,n+\alpha}$, and $M^\omega_{n+\alpha,n+\beta}$ for $|\tau|\to\infty$ [i.e., with respect to the $k(n)\oplus k(m)$ subalgebra]. In this case we obtain an $i^m[k(n)\oplus k(m)]$ algebra where the boost generators x^α_μ transform as vectors with respect to the lower index under $k(n)$ and with respect to the upper index under $k(m)$.

B. Of the group

The contraction of the representations of $K(n,m)$ given by (3.2) proceeds through considering group transformations $G(\epsilon)$ approaching the identity for $\epsilon\to 0$ in the boost elements and letting the representation parameter $\rho\to\infty$ such that $\epsilon\rho=\xi$, a finite number. As regards the boost generators, the group action (3.6) colapses to the identity while the multiplier function in (3.2) becomes, using (3.10),

$$\lim_{\epsilon\to 0}\mu_\sigma(\mathbf{x},G(\epsilon))\approx\lim_{\epsilon\to 0}[1-2\epsilon(\tilde{\beta}\mathbf{x})^0_{\alpha\alpha}]^{i\rho/2}=\exp[-i\xi(\tilde{\beta}\mathbf{x})^0_{\alpha\alpha}], \quad (4.1)$$

thus the boost action in the direction (μ,α) becomes multiplication by $\exp(-i\xi x^\alpha_\mu)$ and the group contracts to $I^m[K(n)\otimes K(m)]$.

[1] M.A. Melvin, Bull. Am. Phys. Soc. 7, 493 (1962); 8, 356 (1963); A. Sankaranarayanan, Nuovo Cimento 38, 1441 (1965); A. Sankaranarayanan and R.H. Good Jr., Phys. Rev. 140, B509 (1965); A. Bohm, Phys. Rev. B145, 1212 (1966).

[2] J. Rosen and P. Roman, J. Math. Phys. 7, 2072 (1966); J. Rosen, Nuovo Cimento 46B, 1 (1966); J. Rosen, J. Math. Phys. 9, 1305 (1968); A. Chakrabarti, J. Math. Phys. 9, 2087 (1968).

[3] J.G. Nagel and K.T. Shah, J. Math. Phys. 11, 1483 (1970); J.G. Nagel, J. Math. Phys. 11, 1779 (1970).

[4]J.G. Nagel, Ann. Inst. H. Poincaré 13, 1 (1970).
[5]R. Gilmore, J. Math. Phys. 13, 883 (1972).
[6]A remark is in order here concerning the difference between expansions and deformations. Definitions for both terms are given in Ref. 7. We prefer to speak of deformations instead of expansions since we wish to emphasize the close connection with contractions. Expansions on the other hand have in general case little to do with contractions. Thus $u(n,1)$ is a deformation of $i[u(n)\otimes u(1)]$ and an expansion—but not a deformation—of $iu(n)$, where similar remarks hold for the more general case, i.e., $k(n,m)$ is an expansion of $i^m[k(n)\otimes k(m)]$.
[7]C.P. Boyer and K.B. Wolf, J. Math. Phys. 14, 1853 (1973).
[8]M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966).
[9]C.P. Boyer and K.B. Wolf, Lett. Nuovo Cimento 8, 458 (1973).
[10]The first of these deformations was suggested as a private communication to one of the authors (K.B.W.) by Prof. Yuval Ne'eman of Tel-Aviv University, Israel, and has remained unpublished but for a footnote (p. 1045) in K.B. Wolf, Suppl. Nuovo Cimento 5, 1041 (1967).
[11]N. Mukunda, J. Math. Phys. 10, 897 (1969).
[12]C.P. Boyer, Rev. Mex. Física 23, 99 (1974).
[13]V. Bargmann, Ann. Math. 48, 568 (1947).
[14]I.M. Gel'fand and M.A. Naĭmark, *Unitäre Darstellungen der Klassischen Gruppen* (Academie Verlag, Berlin, 1957); I.M. Gel'fand and M.I. Graev, Am. Math. Soc. Transl. (Ser. 2) 37, 351 (1964).
[15]G.W. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New York, 1967).
[16]K.B. Wolf, J. Math. Phys. 12, 197 (1971); C.P. Boyer, J. Math. Phys. 12, 1599 (1971); C.P. Boyer and F. Ardalan, J. Math. Phys. 12, 2070 (1971).
[17]K.B. Wolf, J. Math. Phys. 13, 1634 (1972).
[18]C.P. Boyer, J. Math. Phys. 14, 609 (1973).
[19]C.P. Boyer and K.B. Wolf, J. Math Phys. 15, 560 (1974).
[20]E. Inönü and E.P. Wigner, Proc. Natl. Acad. Sci. 39, 510 (1953); E.J. Saletan, J. Math. Phys. 2, 1 (1961); M. Lévy-Nahas, J. Math. Phys. 8, 1211 (1967); M. Lévy-Nahas and R. Seneor, Commun. Math. Phys. 9, 242 (1968); R. Hermann, Commun. Math. Phys. 5, 131 (1967); 6, 157 (1967).
[21]M. Gourdin, *Unitary Symmetries* (North-Holland, Amsterdam, 1967).
[22]P. Pajas and R. Rączka, J. Math. Phys. 9, 1188 (1968); P. Pajas, J. Math. Phys. 10, 1777 (1969).
[23]R.L. Anderson and K.B. Wolf, J. Math. Phys. 11, 3176 (1970).
[24]For $SO(n,m)$ see J. Fischer, J. Niederle, and R. Rączka, J. Math. Phys. 7, 816 (1966); R. Rączka, N. Limić, and J. Niederle, J. Math. Phys. 7, 1861 (1966); N. Limić, J. Niederle, and R. Rączka, J. Math. Phys. 7, 2026 (1966); N. Limić, J. Niederle, and R. Rączka, J. Math. Phys. 8, 1079 (1967); and for $U(n,m)$ see J. Fischer and R. Rączka, Commun. Math. Phys. 3, 233 (1966); 4, 8 (1967); M.I. Graev, Am. Math. Soc. Transl. 66, 1 (1968); T. Hirai, J. Math. Soc. Japan 22, 134 (1970).