The group chain $SO_{n,1} \supset SO_{1,1} \otimes SO_{n-1}$, a complete solution to the "missing label" problem

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We discuss the decomposition $SO_{n,1} \supset SO_{1,1} \otimes SO_{n-1}$ by constructing multiplier representations over the group manifold of SO_n. Explicit orthogonal and complete bases in terms of functions diagonal with respect to the canonical ($SO_{n,1} \supset SO_n$) and noncanonical ($SO_{n,1} \supset SO_{1,1} \otimes SO_{n-1}$) chains are provided which give a complete solution to the "missing label" and multiplicity problems occuring in the latter decomposition. Moreover, an integral representation for the overlap functions between the two chains is given, for which the singularity structure can be immediately ascertained. Expressions for the cases $n = 3$ and 4 are given.

1. INTRODUCTION

The orthogonal and unitary groups SO_n and U_n exhibit canonical chains, i.e., chains of subgroups whose unitary irreducible representation (UIR) labels can be used to characterize uniquely their basis vectors. These are well known to be $SO_n \supset SO_{n-1} \supset \cdots \supset SO_2$ and $U_n \supset U_{n-1} \supset \cdots \supset U_1$. Many physical systems, however, require different chains. The foremost example of the latter is the Elliott chain $U_3 \supset SO_3$ in nuclear shell theory as studied by Beidenharn and Moshinsky[1] which has been the subject of extensive research. More in line with the problem we shall treat here are the reductions of the type $SO_{m+n} \supset SO_m \otimes SO_n$ and $U_{m+n} \supset U_m \otimes U_n$. The latter ones are of relevance in some elementary particle classification schemes[2] while the former, in its noncompact version $SO_{3,1} \supset SO_{1,1} \otimes SO_2$ has been used to work with helicity bases for the Lorentz group[3] and applied to the construction of solutions to the Dirac and Proca free field equations.[4]

The Elliott chain as well as the other examples mentioned (except the last one) exhibit the multiplicity and what is called the "missing label" problems,[2,5,6] that is, the UIR labels obtained from the subgroup do not specify completely the UIR bases of the group. The problem has been solved by constructive methods[2] whereby one starts with the highest or lowest weight state of a multiplet and, applying convenient lowering or raising operators, generates all the states of that multiplet labelling thus the states in the process. This is a valid procedure for finite multiplets or for "half-infinite" multiplets belonging to some discrete series of UIRs which have an extremum state.

Most of these procedures, however, are *algebraic* in the sense of making use of the enveloping algebra of the group. It is our contention that the labelling problem can be solved through the methods of harmonic analysis on the *group manifold*.[7] The procedure is essentially that of building complete and orthogonal sets of functions on a subgroup manifold and defining a multiplier representation of the group in both canonical and noncanonical chains. In the process of constructing such a set of functions we provide an explicit basis classified by labels, some of which are eigenvalues of operators which are *not* elements of the enveloping algebra.

In Sec. 2 we review the mathematical framework needed: Gel'fand states, Euler angles and their labels, the Wigner D-functions and generalizations of the spherical harmonics which we call E functions, and the relevant orthogonality and completeness relations. The multiplier representation of $SO_{n,1}$ in the canonical and noncanonical chains are set up in Secs. 3 and 4 and the overlap functions are found in Sec. 5 together with its pole structure and asymptotic behavior. In two appendixes we treat the cases $SO_{3,1}$, known from the work of Kuznetzov et al.[3] and Kalnins,[4] and $SO_{4,1}$.

2. THE BASIS FUNCTIONS

The Gel'fand kets and Euler angles for SO_n are very well known (see, e.g., Ref. 8). Only some points on notation will therefore be repeated. The SO_n single-valued UIRs are labeled by a set of $[\frac{1}{2}n]$ integer numbers ($[\frac{1}{2}n]$ stands for the largest integer $\leqslant \frac{1}{2}n$), $\{J_{n,1}, J_{n,2}, \cdots, J_{n,[n/2]}\} \equiv J_n$ and its bases for UIRs are completely classified by the canonical chain $SO_n \supset SO_{n-1} \supset \cdots \supset SO_2$, whereby the basis vectors have their rows labeled by $\{J_{n-1}, J_{n-2}, \cdots, J_2\} \equiv \overline{J_{n-1}}$. The Gel'fand ket

$$\left|\begin{matrix} J_{n,1} J_{n,2} \cdots J_{n,[n/2]} \\ J_{n-1,1} J_{n-1,2} \cdots J_{n-1,[1/2(n-1)]} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ J_{41} J_{42} \\ J_{31} \\ J_{21} \end{matrix}\right\rangle \equiv \left|\begin{matrix} J_n \\ \overline{J_{n-1}} \end{matrix}\right\rangle \tag{2.1}$$

containing $\frac{1}{2}[\frac{1}{2}n^2]$ labels thus transforms as the J_p UIR of SO_p ($p = n, n-1, \ldots, 2$) and the SO_p content of SO_n is found through the orthogonal group branching relations.[8] In order to reduce the indexing to a bare minimum, we shall always denote the UIR labels of SO_n (resp. SO_{n-1} and SO_{n-2}) by the letters J (resp. L and M) and their row labels by $\underline{L} \equiv L, \overline{M}$ (resp. $\overline{M} \equiv \underline{M}, \overline{N}$ and $\overline{N}$) and write the ket (2.1) horizontally as $|J\underline{L}\rangle$.

The elements r of SO_n can be parametrized by the set of $\frac{1}{2}n(n-1)$ Euler angles $\theta_{jk}^{(p)}$ ($1 \leqslant j < k \leqslant p \leqslant n$) which represent rotations in the j-k plane, written as

$$R_p(\{\theta\}^{(p)}) = R_{p-1}(\{\theta\}^{(p-1)}) S_{p-1}(\{\theta^{(p-1)}\}), \tag{2.2a}$$

$$S_{p-1}(\{\theta^{(p)}\}) = r(\theta_{p-1,p}^{(p)}) S_{p-2}(\{\theta^{(p)}\}), \; n \geqslant p > 2, \tag{2.2b}$$

$$R_2(\{\theta\}^{(2)}) = S_1(\{\theta^{(2)}\}) = r(\theta_{12}^{(2)}), \tag{2.2c}$$

where R_p is thus an element of SO_p and S_{p-1} a representative of the coset space SO_{p-1}/SO_p isomorphic to the $(p-1)$-dimensional sphere S_{p-1}. The Haar measure can be similarly split as

$$dR_n = \omega(Rn) d\{\theta\}^{(n)} = dR_{n-1} dS_{n-1}, \tag{2.3a}$$

$$dS_{n-1} = sin^{n-2}\theta_{n-1,n} d\theta_{n-1,n} dS_{n-2}, \tag{2.3b}$$

$$dR_2 = dS_1 = d\theta_{12}, \tag{2.3c}$$

where $\omega(R_n)$ is the Haar weight of SO_n. The ranges of $\theta^{(p)}_{jk}$ are $[0,\pi]$ for $j,k>1$, while $\theta^{(p)}_{12}$ runs over $[0,2\pi)$. The volume of the group is thus $\mathrm{vol}SO_n=\mathrm{vol}SO_{n-1}|S_{n-1}|$, where $|S_{n-1}|=2\pi^{n/2}/\Gamma(n/2)$ is the surface of the $(n-1)$-sphere, and $\mathrm{vol}SO_2=|S_1|=2\pi$.

The Wigner D-functions (UIR matrix elements) for SO_n are then constructed and decomposed as

$$D^J_{\overline{L}'\overline{L}}(R_n)\equiv\langle J\overline{L}'|R_n|J\overline{L}\rangle=\sum_{\overline{M}''}D^{L'}_{\overline{M}'\overline{M}''}(R_{n-1})\,E^J_{L'\overline{M}'',\overline{L}}(S_{n-1}),\tag{2.4}$$

where we have defined the E functions

$$E^J_{\overline{L}'\overline{L}}(S_{n-1})\equiv\langle J\overline{L}'|S_{n-1}|J\overline{L}\rangle=d^J_{L'M'L}(\theta_{n-1,n})E^L_{\overline{M}'\overline{M}}(S_{n-2}),\tag{2.5}$$

using the Wigner d-functions

$$d^J_{L'ML}(\theta_{n-1,n})\equiv\langle JL'M\overline{N}|r(\theta_{n-1,n})|JLM\overline{N}\rangle,\tag{2.6}$$

which are diagonal in M (UIR label of SO_{n-2}) and independent of $\overline{N}$ (row label of SO_{n-2}). The E functions are generalizations of the spherical harmonics: for SO_3, $E^l_{m'm}(\theta,\varphi)=d^l_{m'm}(\theta)e^{im\varphi}$ and for SO_4, $E^{J_1J_2}_{l'm',lm}(\zeta\theta\varphi)=d^{J_1J_2}_{l'm'l}(\zeta)E^l_{m'm}(\theta,\varphi)$, etc. Orthogonality and completeness relations for these functions will be analyzed below.

Consider the *right* action from the group on functions *on* the coset manifold S_{n-1} as

$$\mathcal{T}(R'_n)f(S_{n-1})=f(S_{n-1}R'_n);\tag{2.7}$$

the $E^J_{\overline{L}'\overline{L}}(S_{n-1})$ functions then transform as the ket $|J\overline{L}\rangle$. The indices $\overline{L}'$ do not enter into the transformation properties of the ket, and will be called *redundant* labels. They are only asked to respect the branching relations. They *do* distinguish, however, between different E functions transforming in the same way but are *not* eigenlabels of any operator in the enveloping algebra of SO_n from the *right*. (They are the eigenvalues, however, of operators acting on the group from the left.)

We know from the Peter-Weyl theorem[7] that the D functions are orthogonal and complete over the space $\mathcal{L}^2(SO_n)$ of square integrable functions over the SO_n group manifold. The orthogonality relation is

$$\int dR_n D^{J_1}_{\overline{L}'_1\overline{L}_1}(R_n)^*D^{J_2}_{\overline{L}'_2\overline{L}_2}(R_n)=\frac{\mathrm{vol}SO_n}{\mathrm{dim}J_1}\delta_{J_1,J_2}\delta_{\overline{L}_1,\overline{L}_2}\delta_{\overline{L}'_1,\overline{L}'_2},\tag{2.8}$$

where $\mathrm{dim}J$ is the dimension of the UIR labelled by J, and where the δ's in the collective indices $J,\overline{L}$, etc. are products of δ's in the individual indices. From (2.4) and (2.8) we can write the generalized orthogonality relations for the E functions as

$$\int dS_{n-1}\sum_{\overline{M}}E^{J_1}_{L\overline{M},\overline{L}_1}(S_{n-1})^*E^{J_2}_{L\overline{M},\overline{L}_2}(S_{n-1})=|S_{n-1}|\frac{\mathrm{dim}L}{\mathrm{dim}J_2}\delta_{J_1,J_2}\delta_{\overline{L}_1,\overline{L}_2},\tag{2.9}$$

where $\int dS_{n-1}$ stands for the integration, with the correct measure, over the coset space S_{n-1}. The completeness relation of the D's is given by

$$\sum_J\frac{\mathrm{dim}J}{\mathrm{vol}SO_n}\mathrm{Tr}[D^J(R_n)^\dagger D^J(R'_n)]=\frac{1}{\omega(R_n)}\delta(R_n,R'_n)\tag{2.10}$$

where the trace sums over all row and column indices and the δ in R_n and R'_n is the product of δ's in the individual angles. The right-hand side of (2.10) is the reproducing kernel under the Haar integral in (2.8) and conversely, the right-hand side of (2.8) is the reproducing kernel in the Plancherel sum in (2.10). We can find a generalized completeness relation for the E's if we write the right-hand side of (2.10) as

$$\frac{1}{\omega(R_n)}\delta(R_n,R'_n)=\frac{1}{\omega(S_{n-1})}\delta(S_{n-1},S'_{n-1})\frac{1}{\omega(R_{n-1})}\delta(R_{n-1},R'_{n-1}),\tag{2.11}$$

with the weight on S_{n-1} given by $\omega(S_{n-1})=\omega(R_n)/\omega(R_{n-1})$. Writing the last two factors of (2.11) as (2.10) with $n\to n-1$ multiplying by $D^{L_0}_{\overline{M}'_0\overline{M}_0}(R'_{n-1})^*$ and integrating over $R'_{n-1}\in SO_{n-1}$ with the appropriate Haar measure, we obtain

$$\sum_J\frac{1}{|S_{n-1}|}\frac{\mathrm{dim}J}{\mathrm{dim}L_0}\sum_{\overline{L}}E^J_{L_0\overline{M}'_0,\overline{L}}(S_{n-1})^*E^J_{L_0\overline{M}_0,\overline{L}}(S'_{n-1})=\frac{1}{\omega(S_{n-1})}\delta(S_{n-1},S'_{n-1})\delta_{\overline{M}'_0,\overline{M}_0}.\tag{2.12}$$

3. MULTIPLIER REPRESENTATIONS AND THE CANONICAL CHAIN

It has been shown in Refs. 9-12 that one can deform the SO_n algebra of elements M_{ij} $(i,j=1,\ldots,n)$ into the $SO_{n,1}$ algebra through the addition of the "noncompact" generators

$$M^{(\sigma)}_{i,n+1}=\tfrac{1}{2}[x_i,\Phi]+\sigma x_i\tag{3.1}$$

where Φ is the second-order SO_n Casimir operator and x_i is a point on S_{n-1}. Since one can decompose an arbitrary element g of $SO_{n,1}$ into double cosets as $g=hbh'$ where $h,h'\in SO_n$ and $b\in SO_n\backslash SO_{n,1}/SO_n$, it suffices to consider[11] the "last boost" generated by

$$M^{(\sigma)}_{n,n+1}=\sin\theta\frac{\partial}{\partial\theta}+\lambda\cos\theta=\frac{\partial}{\partial\omega}+\lambda\tanh\omega,\qquad \lambda=\tfrac{1}{2}(n-1)-\sigma,\tag{3.2}$$

where $\theta\equiv\theta^{(n)}_{n-1,n}$ and $\omega=\ln\tan\frac{1}{2}\theta$. The range $[0,\pi]$ for θ implies $-\infty<\omega<\infty$. We construct a multiplier representation of $SO_{n,1}$ on $R_n=(R_{n-1},\theta,S_{n-2})$ such that, for the "last boost" generated by (3.2), we have

$$\mathcal{T}(\exp[\zeta M^{(\sigma)}_{n,n+1}])f(R_{n-1},\theta,S_{n-2})=\left(\frac{\sin\theta}{\sin\theta'}\right)^\lambda f(R_{n-1},\theta',S_{n-2}),\tag{3.3a}$$

where

$$\tan\tfrac{1}{2}\theta'=e^\zeta\tan\tfrac{1}{2}\theta,\tag{3.3b}$$

and the action (2.7) holds for the elements in $SO_n\subset SO_{n,1}$. When $\lambda=-\frac{1}{2}(n-1)-i\tau$ (τ real), the multiplier just offsets the change in SO_n measure and provides the principal series of unitary representations of $SO_{n,1}$ on the space $\mathcal{L}^2(SO_n)$. This is, essentially, Mackey's method of induced representations.

Now, we *do* have a complete and orthogonal set of functions over SO_n, namely, the $D^J_{\overline{L}'\overline{L}}(R_n)$. These were shown[11] to transform under $SO_{n,1}$ in this realization as the (λ,L') bases for UIRs with *row* $\overline{J}=J,L,\overline{M}$, i.e.,

$$\left\langle R_n \middle| \begin{matrix}\lambda L' \\ J\end{matrix} \right\rangle_{(\overline{M}')} = \left[\frac{\dim J}{\mathrm{vol} SO_n}\right]^{1/2} D^J_{L'\overline{M}',\overline{L}}(R_n), \qquad (3.4)$$

where the factor has been added to make the kets orthonormal over SO_n. The important points to notice are (i) the *column* labels L' enter as UIR labels for $SO_{n,1}$ satisfying the correct branching relations with respect to J, (ii) the principal series of UIRs is obtained for $\lambda = -\frac{1}{2}(n-1) - i\tau$ (τ real), and (iii) the column labels $\overline{M}'$ do not affect the transformation properties of (3.4) and are thus *redundant* labels in the same sense as these seen in the former section. There are $\frac{1}{2}[\frac{1}{2}(n-2)^2]$ of these labels.[8] For fixed $SO_{n,1}$ and redundant labels, the functions (3.4) form an irreducible basis. The functions (3.4) were used in Ref. 11 in order to find the Bargmann d-functions for $SO_{n,1}$ in a recursive fashion. Here, we only wish to stress that the orthogonality relation (2.8) is written for the kets (3.4) as

$$_{(\overline{M}_1')}\left\langle \begin{matrix}\lambda L_1' \\ J_1\end{matrix} \middle| \begin{matrix}\lambda L_2' \\ J_2\end{matrix} \right\rangle_{(\overline{M}_2')} = \delta_{J_1 J_2}\delta_{L_1' L_2'}\delta_{\overline{M}_1' \overline{M}_2'}, \qquad (3.5)$$

and completeness (2.10) becomes

$$\sum_{JL'\overline{M}'} \left| \begin{matrix}\lambda L' \\ J\end{matrix} \right\rangle_{(\overline{M}')(\overline{M}')}\left\langle \begin{matrix}\lambda L' \\ J\end{matrix} \right| = \mathbf{1}, \qquad (3.6)$$

i.e., the redundant labels $\overline{M}'$ *do* enter into the orthogonality and completeness relations, only λ is never invoked and stems from the realization of the $SO_{n,1}$ algebra we are working with. Notice that the total number of labels in the ket (3.4), excepting λ which is fixed, is $\frac{1}{2}n(n-1)$, equal to the number of parameters of SO_n. It should be emphasized that although the representation (3.3) is reducible over $\mathcal{L}^2(SO_n)$, the irreducible components are given explicitly by the redundant labels. These appear in block-diagonal form in Eq. (3.5).

4. THE NONCANONICAL CHAIN

We will now *construct* orthogonal and complete sets of basis functions with definite transformation properties under the subgroup $SO_{1,1} \otimes SO_{n-1}$ of $SO_{n,1}$. The number of ket labels specifying the $SO_{n,1}$ rows provided by the canonical chain is $\frac{1}{2}[\frac{1}{2}n^2]$ (see Ref. 8), while that provided by the noncanonical chain $SO_{1,1} \otimes SO_{n-1} \supset SO_{n-2} \supset \cdots \supset SO_2$ is $1 + \frac{1}{2}[\frac{1}{2}(n-1)^2]$. The number of *missing* labels in the noncanonical chain is thus $[\frac{1}{2}n] - 1$. The functions we wish to construct must first contain an E function on S_{n-2} in order to have the necessary SO_{n-1} labels. Secondly, they must be eigenfunctions of the generator of $SO_{1,1}$ so that they are classified by its label. This generator can be taken to be (3.2) in the variable $\omega = \ln \tan\frac{1}{2}\theta$ so that the eigenfunctions are obtained from the differential equation

$$M^{(\sigma)}_{n,n+1}\tilde{f}^\lambda_\nu(\omega) = \nu \tilde{f}^\lambda_\nu(\omega) \qquad (4.1a)$$

which are

$$\tilde{f}^\lambda_\nu(\omega) = (2\pi)^{-1/2}\cosh^{-\lambda}\omega e^{i\nu\omega}, \qquad (4.1b)$$

complete and orthogonal over the ν and ω real lines as

$$\int_{-\infty}^{\infty} d\nu \tilde{f}^\lambda_\nu(\omega)^* \tilde{f}^\lambda_\nu(\omega') = \cosh^{n-1}\omega\,\delta(\omega-\omega'), \qquad (4.2a)$$

$$\int_{-\infty}^{\infty} \cosh^{n-1}\omega d\omega \tilde{f}^\lambda_\nu(\omega)^* \tilde{f}^\lambda_{\nu'}(\omega) = \delta(\nu-\nu'), \qquad (4.2b)$$

when we recall that $\lambda = -\frac{1}{2}(n-1) - i\tau$ (τ real) for the principal series. Furthermore, the redefinition of the argument brings in the functions

$$f^\lambda_\nu(\theta) = \tilde{f}^\lambda_\nu(\ln \tan\tfrac{1}{2}\theta) = (2\pi)^{-1/2}2^\lambda \sin^{\lambda+i\nu}\tfrac{1}{2}\theta \cos^{\lambda-i\nu}\tfrac{1}{2}\theta, \qquad (4.3)$$

which are complete as in (4.2a) and orthogonal in θ under the measure $\sin^{n-2}\theta d\theta$.

Lastly, our noncanonical basis functions should be functions of the whole of the SO_n manifold, so that the whole of $SO_{n,1}$ can be applied to it as a multiplier representation. This can be achieved multiplying the functions by a D function on the remaining SO_{n-1} manifold as

$$\left\langle R_n \middle| \begin{matrix}\lambda L' \\ \nu(M'') \\ \overline{L}\end{matrix} \right\rangle_{(\overline{M}')} = \left(\frac{\mathrm{vol}SO_{n-2}}{\mathrm{vol}^2 SO_{n-1}}\,\frac{\dim L \dim L'}{\dim M''}\right)^{1/2} \times \sum_{\overline{N}''} D^{L'}_{\overline{M}',M''\overline{N}''}(R_{n-1}) f^\lambda_\nu(\theta) E^L_{M''\overline{N}'',\overline{M}}(S_{n-2}), \qquad (4.4)$$

where M'' are the "missing" labels found by our scheme and $\overline{M}'$ the "redundant" ones. In all there are again $\frac{1}{2}n(n-1)$ labels (excluding λ).

Using (2.8), (2.9), and (4.2b) we can verify that these functions are indeed orthogonal over SO_n, i.e.,

$$_{(\overline{M}_1')}\left\langle \begin{matrix}\lambda L_1' \\ \nu_1(M_1'') \\ \overline{L}_1\end{matrix} \middle| \begin{matrix}\lambda L_2' \\ \nu_2(M_2'') \\ \overline{L}_2\end{matrix} \right\rangle_{(\overline{M}_2')} = \delta(\nu_1-\nu_2)\delta_{L_1'L_2'}\delta_{M_1''M_2''}\delta_{\overline{L}_1\overline{L}_2}\delta_{\overline{M}_1'\overline{M}_2'}, \qquad (4.5)$$

and using (2.10), (2.12), and (4.2a), completeness over $\mathcal{L}^2(SO_n)$ holds:

$$\int_{-\infty}^{\infty} d\nu \sum_{L'M''\overline{L}\overline{M}'} \left| \begin{matrix}\lambda L' \\ \nu(M'') \\ \overline{L}\end{matrix} \right\rangle_{(\overline{M}')(\overline{M}')}\left\langle \begin{matrix}\lambda L' \\ \nu(M'') \\ \overline{L}\end{matrix} \right| = \mathbf{1}. \qquad (4.6)$$

This formula allows us to decompose any function in $\mathcal{L}^2(SO_n)$ in terms of the noncanonical basis functions (4.4) as a sum over discrete labels and an integral over the continuous label ν. We stress the fact that the "redundant" labels enter in the noncanonical chain, that is, relations (4.5) and (4.6) in the *same* way as they do for the canonical chain. This means that the $SO_{n,1}$ labels L' and the redundant labels $\overline{M}'$ not only appear in block diagonal form with the corresponding subgroup chain, but also appear in block diagonal form in the overlap functions computed explicitly in the next section. For fixed $SO_{n,1}$ and "redundant" labels, we obtain an irreducible subspace and a basis in this subspace is given by the remaining labels, including the "missing" labels, which are essential. All of the discrete labels, of course, are constrained by the branching rules. In particular, for the "missing" labels $M'' \equiv J''_{n-2}$ we have

$$\min(J_{n-1,k}, J'_{n-1,k}) \geq J''_{n-2,k} \geq \max(J_{n-1,k+1}, J'_{n-1,k+1}), \qquad (4.7a)$$

for $k = 1, \ldots, [n/2] - 2$, while

$$J''_{n-2,[n/2]-1} \geq 0 \quad \text{for } n \text{ odd} \qquad (4.7b)$$

and

$$\min(J_{n-1,[n/2]-1}, J'_{n-1,[n/2]-1}) \geq |J''_{n-2,[n/2]-1}| \quad \text{for } n \text{ even.} \qquad (4.7c)$$

The number of possible "missing" labels J''_{n-2} for a fixed $SO_{n,1}$ UIR and for a fixed $SO_{1,1} \otimes SO_{n-1}$ UIR gives the *multiplicity* of the decomposition.

5. THE OVERLAP FUNCTIONS

As both the canonical and noncanonical bases functions are orthogonal and complete, one can easily obtain an integral representation for the overlap function

$$
{}_{(\overline{M'_1})}\left\langle \begin{matrix} \lambda L'_1 \\ \nu(M'') \\ \overline{L}_1 \end{matrix} \middle| \begin{matrix} \lambda L'_2 \\ J \\ \overline{L}_2 \end{matrix} \right\rangle_{(\overline{M'_2})} = \delta_{L'_1 L'_2} \delta_{\overline{L}_1 \overline{L}_2} \delta_{\overline{M'_1} \overline{M'_2}} 2^{-\lambda-3/2}
$$
$$
\times \left(\pi^{-3/2} \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}[n-1])} \frac{\dim J \ \dim M''}{\dim L_1 \ \dim L'_1} \right)^{1/2}
$$
$$
\times \int_0^\pi d\theta \ \sin^{-\lambda-1-i\nu}\tfrac{1}{2}\theta \ \cos^{-\lambda-1+i\nu}\tfrac{1}{2}\theta \ d^J_{L'_1, M'' L'_1}(\theta), \qquad (5.1)
$$

where we have used the orthogonality relations for SO_{n-1}, (2.8) and (2.9). Equation (5.1) contains much information. The appearance of the "missing" labels M'' is explicitly in the d function inside the integral while the result does not depend on the "redundant" labels $\overline{M'}$ (which appear only in the δ's) nor on the UIR labels of SO_p, $p \leqslant n-2$. The singularity structure[13] and asymptotic behavior in ν can be examined noting that the d functions can be written as polynomials in $\sin^{k'}\frac{1}{2}\theta \ \cos^{k''}\frac{1}{2}\theta$ where $k', k'' \geqslant 0$, and run over a finite range. A typical term occurring in the integral (5.1) yields an integral representation for the Beta function[14]

$$
\int_0^\pi d\theta \ \sin^{-\lambda-1-i\nu+k'}\tfrac{1}{2}\theta \ \cos^{-\lambda-1+i\nu+k''}\tfrac{1}{2}\theta
$$
$$
= B(\tfrac{1}{2}[-\lambda - i\nu + k'], \tfrac{1}{2}[-\lambda + i\nu + k'']), \qquad (5.2)
$$

from which we can see that the overlap functions exhibit simple poles at the points $\nu = \pm i(\lambda - 2k)$, where $k = 0, 1, 2, \cdots$. In particular cases there may be zeroes cancelling some poles due to the influence of the d function. There are no other singularities, however. Moreover, from (5.2) the asymptotic behavior of (5.1) can be obtained from Stirling's formula[14] to be $\sim|\nu|^\gamma \exp(-\frac{1}{2}\pi|\nu|)$, for some fixed γ, which is typical of many such overlap functions and assures the convergence of the decomposition.

When changing bases, the integration contour over ν runs along the real axis and we see that none of the poles of either (5.1) nor its complex conjugate interfere with the integration. If we analytically continue the $SO_{n,1}$ UIRs to the supplementary series[15] by allowing λ to take values $0 > \lambda > -(n-1)$ [or equivalently τ to lie on the imaginary axis between $-i\frac{1}{2}(n-1)$ and $i\frac{1}{2}(n-1)$, not including the endpoints], we see that still none of the above poles interfere with the integration contour. Thus our decomposition remains valid for the supplementary series of $SO_{n,1}$ as well.

6. OUTLOOK

We have discussed the example $SO_{n,1} \supset SO_{1,1} \otimes SO_{n-1}$ for its relative simplicity. The corresponding unitary groups can be worked out using the results of Ref. 16 and for the linear groups we can point to Ref. 17. Future work[18] should provide the framework for the reduction $SO_{n,1} \supset SO_{n-k} \otimes SO_{k,1}$ and $SO_{n,k} \supset SO_n \otimes SO_k$ and their unitary and symplectic counterparts. This is due to the relative ease in constructing multiplier representations of noncompact groups. The compact groups should be treatable through analytic continuation and, indeed, the solution of the multiplicity problem does not seem to depend on the noncompact nature of the example presented.

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APPENDIX A: $SO_{3,1} \supset SO_{1,1} \otimes SO_2$

In order to find the overlap coefficients, we apply the formula (5.1) keeping in mind that, as $n = 3$, all SO_{n-2} labels disappear and there are no missing nor redundant labels. A straightforward calculation yields, for $\lambda = -1 - i\tau$,

$$
\left\langle \begin{matrix} \lambda M \\ \nu \\ m \end{matrix} \middle| \begin{matrix} \lambda M \\ l \\ m \end{matrix} \right\rangle = 2^{-1+i\tau}\pi^{-1/2}(2l+1)^{1/2}
$$
$$
\times \int_0^\pi d\theta \ \sin^{i(\tau-\nu)}\tfrac{1}{2}\theta \ \cos^{i(\tau+\nu)}\tfrac{1}{2}\theta \, d^l_{Mm}(\theta)
$$
$$
= 2^{-\lambda-1}(2\pi)^{-1/2} i^{\mu_s - \mu_i} \ (2l+1)^{1/2}
$$
$$
\times \left(\frac{(l+\mu_s)!\,(l-\mu_i)!}{(l-\mu_s)!\,(l+\mu_i)!} \right)^{1/2} \frac{\Gamma(\frac{1}{2}[\mu_s + \mu_i - \lambda + i\nu])}{(\mu_s - \mu_i)!}
$$
$$
\times \frac{\Gamma(\frac{1}{2}[\mu_s - \mu_i - \lambda - i\nu])}{\Gamma(\mu_s - \lambda)}
$$
$$
\times {}_3F_2 \left[\begin{matrix} \mu_s - l \ \ \mu_s + l + 1 \ \ \frac{1}{2}(\mu_s - \mu_i - \lambda - i\nu) \\ \mu_s - \mu_i \ \ \mu_s - \lambda \end{matrix} ; 1 \right], \qquad \text{(A1)}
$$

where $\mu_s \equiv \max(m, M)$ and $\mu_i = \min(m, M)$, and which can be compared with Kuznetzov et al.[3] We see that the poles occur at $\nu = \pm i[\lambda - (\mu_s \mp \mu_i) - 2k]$, $k = 0, 1, 2, \cdots$.

APPENDIX B: $SO_{4,1} \supset SO_{1,1} \otimes SO_3$

Using the SO_4 d-functions as given, e.g., in Ref. 11, one finds for $SO_{4,1}$ that

$$
{}_{(M')}\left\langle \begin{matrix} \lambda L' \\ \nu(M'') \\ L \\ M \end{matrix} \middle| \begin{matrix} \lambda L' \\ J_1 J_2 \\ L \\ M \end{matrix} \right\rangle_{(M')}
$$
$$
= 2^{-\lambda}(2\pi)^{-1} \left(\frac{\dim(J_1, J_2)}{\dim L \ \dim L'} \right)^{1/2} \int_0^\pi d\theta \ \sin^{-\lambda-1-i\nu}\tfrac{1}{2}\theta
$$
$$
\cos^{-\lambda-1+i\nu}\tfrac{1}{2}\theta \ d^{J_1 J_2}_{L' M'' L}(\theta)
$$
$$
= 2^{-\lambda}(2\pi)^{-1} \left(\frac{(J_1+1)^2 - J_2^2}{(2L+1)(2L'+1)} \right)^{1/2} \sum_m C(\tfrac{1}{2}[J_1 + J_2],
$$
$$
\tfrac{1}{2}[J_1 - J_2], L'; \ \tfrac{1}{2}[M'' + m], \tfrac{1}{2}[M'' - m], M'')
$$
$$
\times C(\tfrac{1}{2}[J_1 + J_2], \tfrac{1}{2}[J_1 - J_2], L;
$$
$$
\tfrac{1}{2}[M'' + m], \tfrac{1}{2}[M'' - m], M'') \sum_n i^n \binom{n}{2m}
$$
$$
\times B(\tfrac{1}{2}[n - \lambda - i\nu], \tfrac{1}{2}[2m - n - \lambda + i\nu]), \qquad \text{(B1)}
$$

which can be seen to be independent of the SO_2 and redundant labels M and M'. There are poles due to the B function at $\nu = \pm i(\lambda - 2k)$ $(k = 0, 1, 2, \cdots)$. Notice that the "missing" label M'' appears in a rather "geometric" fashion through the entries of the Clebsch-Gordan coefficient C. If a given UIR of $SO_{1,1} \otimes SO_3$ (in the noncanonical chain) given by (ν, L) appears in the decomposition of a $(\lambda L')$ UIR of $SO_{4,1}$, the multiplicity is given by the possible values of M''. This is constrained by the minimum of $2L + 1$ and $2L' + 1$, hence the multiplicity is $2 \min(L, L') + 1$.

The overlap coefficients for $SO_{5,1}$ can be obtained in a similar fashion from the SO_5 d's as given by Holman.[19]

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