Physica Scripta

PAPER

On elliptic trigonometric form of the Zernike system and polar limits

To cite this article: Natig M Atakishiyev *et al* 2019 *Phys. Scr.* **94** 045202

IOP Publishing Physica Scripta
Phys. Scr. **94** (2019) 045202 (13pp) https://doi.org/10.1088/1402-4896/aafecb

On elliptic trigonometric form of the Zernike system and polar limits

Natig M Atakishiyev[1], George S Pogosyan[2,3,4], Kurt Bernardo Wolf[5,7] and Alexander Yakhno[6]

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Cuernavaca, Mexico
[2] Departamento de Matemáticas, Centro Universitario de Ciencias Exactas e Ingenierías, Universidad de Guadalajara, México
[3] Yerevan State University, Yerevan, Armenia
[4] Joint Institute for Nuclear Research, Dubna, Russia
[5] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Cuernavaca, Mexico
[6] Departamento de Matemáticas, Centro Universitario de Ciencias Exactas e Ingenierías, Universidad de Guadalajara, México

E-mail: bwolf@fis.unam.mx

Received 5 October 2018, revised 10 January 2019
Accepted for publication 15 January 2019
Published 31 January 2019

Abstract
The Zernike system provides orthogonal polynomial solution bases on the unit disk that separate in coordinates that are generically elliptic. This is a superintegrable system whose optical realization is a scalar wavefield on the plane of a circular pupil. Here we describe the solution set in the trigonometric form of elliptic coordinates expressed in terms of special functions, and examine closely the two limits where the explicit form of the wavefunctions in elliptic coordinates reduce to wavefunctions in polar coordinates.

Keywords: Zernike system, separation of variables, elliptic trigonometric coordinates, limits to polar spherical coordinates

1. Introduction

The two-dimensional differential equation proposed by Frits Zernike in 1934 [1] and its solutions with boundary conditions to be seen below, has been of interest both for their relevance and applications in optics [2–4] for circular pupils, as well as for their mathematical properties [5–8]. We have dedicated recent research to examine the Zernike system from the points of view of its classical and quantum (or scalar-wave) realizations [9–12]. The model is defined by Zernike's differential equation, which is written as a Schrödinger equation with the 'Zernike' Hamiltonian $\widehat{Z}$ given by

$$\widehat{Z}\Psi(\mathbf{r}) := (\nabla^2 - (\mathbf{r}\cdot\nabla)^2 - 2\,\mathbf{r}\cdot\nabla)\Psi(\mathbf{r}) = -E\,\Psi(\mathbf{r}) \quad (1)$$

on the two-dimensional plane $\mathbf{r} = (x, y)$ restricted to the unit disk $\mathcal{D} := \{|\mathbf{r}| \leqslant 1\}$, whose square-integrable solutions $\Psi(\mathbf{r}) \in \mathcal{L}^2(\mathcal{D})$ have free but finite boundary conditions [1]

$$\Psi(\mathbf{r})|_{|\mathbf{r}|^2=1} < \infty. \quad (2)$$

Because the disk $\mathcal{D}$ has a *closed* boundary, the condition (2) determines only a *pre*-Hilbert space of solutions. The eigenfunctions of (1) have 'energy' eigenvalues E given by

$$E = n(n+2), \qquad \text{for} \quad n \in \{0,\ 1,\ 2,\ \dots\} =: \mathcal{Z}_0^+, \qquad \text{and} \quad (n+1) - \text{fold degenerate}, \quad (3)$$

as if it were a two-dimensional open oscillator system—which it is definitely *not*.

It will be noticed that (1) is a linear combination of the axially-symmetric quadratic operators ∇^2 and $\mathbf{r}\cdot\nabla$, plus a square of the latter. The commutator of the former two yields $\sim|\mathbf{r}|^2$, which completes the generators of the symplectic sp(2,R) Lie algebra. The addition of that square, $(\mathbf{r}\cdot\nabla)^2$, turns this [9, 10] into a superintegrable cubic Higgs algebra [13]. In that sense it can be seen to have a structure similar to the quartic oscillator, which adds a $(|\mathbf{r}|^2)^2 = |\mathbf{r}|^4$ term, and to the Kerr medium, which adds $(-\nabla^2 + |\mathbf{r}|^2)^2$ to the quantum oscillator Hamiltonian, although the ranges of $\mathbf{r}$ are different.

We recall that in [9–12] a key step to find solutions to the two-dimensional equation (1) that separate in a pair of simultaneous one-dimensional equations, was to project the

[7] Author to whom any correspondence should be addressed.

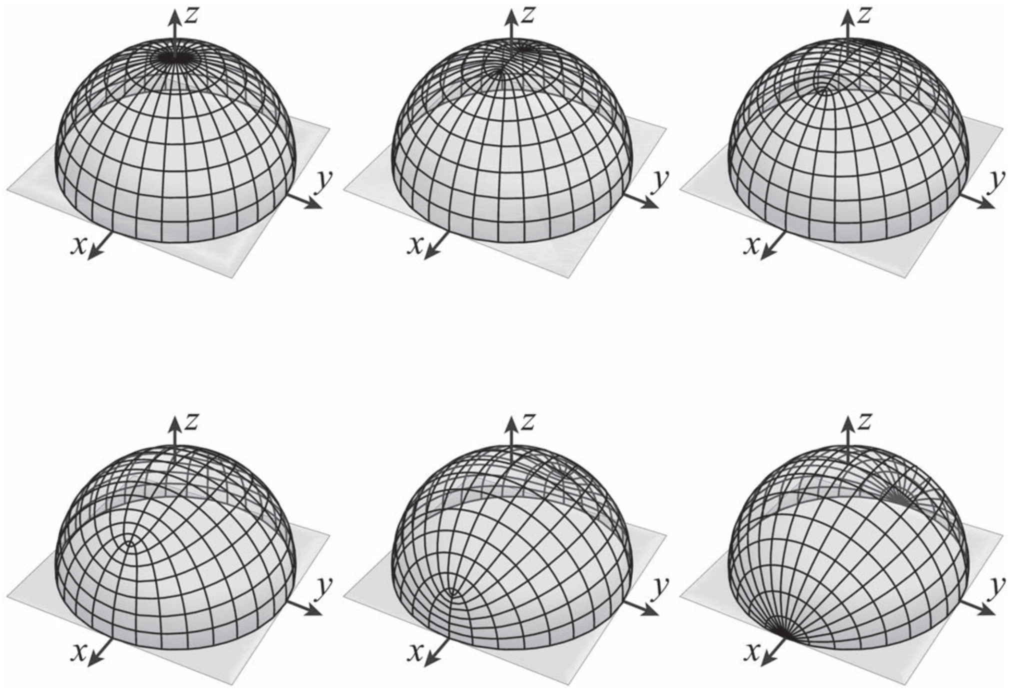

Figure 1. Elliptic coordinate systems on the upper half-sphere $\mathcal{S}^2_+$, with the angles $\frac{1}{2}\kappa$ between each focus and the $+z$ direction and the ellipticity parameter $k = \sin\frac{1}{2}\kappa$. The upper-left figure has $\kappa = 0$ and shows System I ($k = 0$); the lower-right figure shows $\frac{1}{2}\kappa = \frac{1}{2}\pi$ for System II ($k = 1$). The illustrated angles are $\kappa = 0, \frac{1}{8}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi$, and π.

disk vertically on the upper half of a two-dimensional sphere $\mathcal{S}^2$, that we indicate as

$$\mathcal{S}^2_+ := \{|\vec{\xi}\,| = 1,\ \xi_3 \geqslant 0\} \quad \text{with}\ \vec{\xi} = (\xi_1, \xi_2, \xi_3),$$
$$\xi_1 := x, \quad \xi_2 := y, \quad \xi_3 = \sqrt{1 - x^2 - y^2}. \qquad (4)$$

The orthogonal coordinate systems on the two-dimensional surface of the sphere $\mathcal{S}^2$ are generically elliptic [14–16] determined by two pairs of antipodal foci and, modulo rotations, the systems are characterized by the angle $0 \leqslant \kappa \leqslant \pi$ between a pair of foci on the same hemisphere. To qualify for the half-sphere we must add the proviso that the common boundary between $\mathcal{D}$ and $\mathcal{S}^2_+$ correspond to the constant value of $\xi_3 = 0$. When a system has solutions which separate in more than one system of coordinates, it is indicative that an associate higher symmetry or a superintegrable algebra exists [17–20].

In figure 1 we show a subset of the generic elliptic case, modulo rotations around the vertical axis, that is determined by the single parameter $k := \sin\frac{1}{2}\kappa \in [0, 1]$, with κ the angle between the foci. The value $k = 0$ determines System I, which served to separate Zernike's original solution [1] in polar coordinates, the only orthogonal ones on the disk, as shown in figure 2. On the other hand, when $k = 1$, the coordinates are again polar and orthogonal on the sphere but non-orthogonal on the disk; they were called System II and the corresponding separated solutions were given in [10, 11]. The generic solution in elliptic coordinates was broached in [21] using Jacobi elliptic coordinates and parameters [15].

The solutions separated in Systems I and II are products of hypergeometric polynomials: Legendre, Gegenbauer and Jacobi [22–24], while those separated in the generic elliptic case $0 < k < 1$are products of Heun polynomials [25, 26]. The purpose of this paper is to provide solutions to the Zernike system separated in a continuous one-parameter family of elliptic coordinates that interpolate between systems I and II, called *trigonometric* elliptic coordinates, which depend on the single parameter $0 \leqslant k \leqslant 1$. They appear to be better suited than Jacobi ones to establish the $k \to 0$ and $k \to 1$ limits, keeping track of the 'radial' and 'angular' parts of the separated solutions.

In section 2 we write (1) in elliptic trigonometric coordinates and in section 3 solve the separated solutions. Sections 4 and 5 derive the $k \to 0$ and $k \to 1$ limits respectively, for the Frobenius recurrence relations and the solution wavefunctions. Finally, in section 6 we present some conclusions regarding the new results that have been obtained, within the context of previous investigations [9, 10] into the algebraic properties of the Zernike system, stressing that some features, particularly those pertaining interbasis expansions may benefit from further research into the relation between symmetry and supersymmetry. This is relevant because the Zernike system is both used in optical applications and provides a physical realization of the Higgs cubic algebra.

2. Elliptic coordinate systems

The two-dimensional surface of the sphere $\mathcal{S}^2$ can be parametrized using elliptic coordinate systems (ϑ, φ), all of which are orthogonal. These systems can be best related to the three Cartesian coordinates (4), written as in [9, section 4.5], classified by the parameter $k := \sin\frac{1}{2}\kappa \in [0, 1]$ where, as

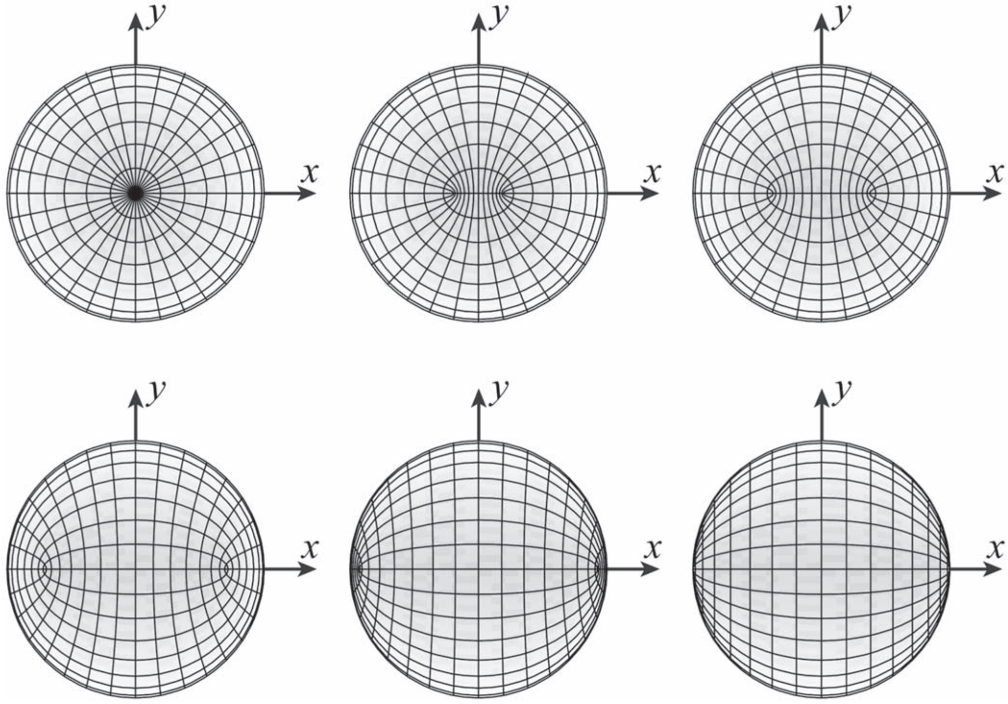

Figure 2. Elliptic coordinate systems on disk $\mathcal{D}$ for the same values of the angles κ and parameters $k = \sin\frac{1}{2}\kappa$ as in figure 1. The upper-left is the polar coordinate system that was used in the original work of Zernike [1] and was the only one considered before our work.

mentioned above, $\kappa \in [0, \pi]$. They are given by

$$\xi_1 = \sqrt{1 - k'^2\cos^2\vartheta}\,\cos\varphi,\quad \xi_2 = \sin\vartheta\sin\varphi,\quad \xi_3 = \cos\vartheta\sqrt{1 - k^2\cos^2\varphi}, \tag{5}$$

where we introduced for brevity $k' := \sqrt{1-k^2} = \cos\frac{1}{2}\kappa \in [0, 1]$. Any other elliptic coordinate system can be obtained from (5) through rotation of the sphere. The half-sphere where $\xi_3 \geqslant 0$ that we consider in this article, $\mathcal{S}^2_+$, is covered by the parameter ranges

$$\vartheta \in [0, \tfrac{1}{2}\pi], \qquad \varphi \in (-\pi, \pi]. \tag{6}$$

In figure 1, the line $(\vartheta = 0, \varphi)$ is twice the half-circle at the intersection of $\mathcal{S}^2_+$ with the $\xi_2 = 0$ plane, while $(\vartheta = \frac{1}{2}\pi, \varphi)$ is the ground circle on the $\xi_3 = 0$ plane; the $\xi_1 = 0$ plane contains the quarter-circles $(\vartheta,0)$ and (ϑ, π). When one coordinate is constant, the other defines lines whose points sum constant distances over the surface of the sphere to the two foci, $(\vartheta, \varphi) = (0, 0)$ and $(0, \pi)$ with the metric (11) given below. These foci, in Cartesian coordinates of $\mathcal{S}^2_+$, fall at $\vec{\xi} = (\pm k, 0, k')$, as can be seen in figure 1.

To bind the Zernike solutions given in terms of the elliptic trigonometric coordinates at the end of section 3 to functions on the disk, where $x = \xi_1$ and $y = \xi_2$, the relations inverse to (5) can be written as

$$\sin^2\vartheta = \frac{2k^2\xi_2^2}{k^2 - \xi_1^2 - k'^2\xi_2^2 + \sqrt{(k^2 - \xi_1^2 - k'^2\xi_2^2)^2 + 4k^2k'^2\xi_2^2}}, \tag{7}$$

$$\sin^2\varphi = \frac{\xi_2^2}{\sin^2\vartheta} = \frac{k^2 - \xi_1^2 - k'^2\xi_2^2 + \sqrt{(k^2 - \xi_1^2 - k'^2\xi_2^2)^2 + 4k^2k'^2\xi_2^2}}{2k^2}. \tag{8}$$

Two pairs of foci on $\mathcal{S}^2$ coincide in poles on the $\pm\xi_3$ axis when $k = 0$ and thus $k' = 1$; they define the usual polar coordinates of

$$\text{System I:}\quad \xi_1 = \sin\vartheta\cos\varphi,\quad \xi_2 = \sin\vartheta\sin\varphi,\quad \xi_3 = \cos\vartheta \geqslant 0, \tag{9}$$

in the range (6) for $\mathcal{S}^2_+$. On the other hand, when the pairs of foci coalesce into poles on the $\pm\xi_1$ axis, $k = 1$ and $k' = 0$, the system of coordinates is again polar, and defines

$$\text{System II:}\quad \xi_1 = \cos\varphi,\quad \xi_2 = \sin\vartheta\sin\varphi,\quad \xi_3 = \cos\vartheta\,|\sin\varphi| \geqslant 0, \tag{10}$$

with the same range (6).

The separability afforded by elliptic coordinates requires that three maximal circles on $\mathcal{S}^2$ do *not* depend on k, namely the ground circle $\vartheta = \frac{1}{2}\pi$, the half-circle through the two foci at $\vartheta = 0$, and the half-circle orthogonal to the other two, at $\vartheta = 0$ and $\frac{1}{2}\pi$. These circles lie in the planes $\xi_3 = 0$, $\xi_2 = 0$ and $\xi_1 = 0$ respectively, where parity under reflection across the later two will be present in our considerations. It is instructive to follow in figure 2 the lines drawn out by the ϑ and φ variables. In System I (upper left in figure 2) when ϑ is kept constant, φ draws out circles so we can call it the 'angular' coordinate for all following $0 < k \leqslant 1$ cases, up to System II (lower right in figure 2). Meanwhile, ϑ qualifies as the 'radial' coordinate, evident in System I, but degenerating into vertical parallel lines in System II. The solutions will be shown below to separate into functions $\Theta(\vartheta)$ and $\Phi(\varphi)$.

We now vertically *lift* the Zernike differential equation (1) from the disk to the half-sphere, and change its coordinates from $\mathbf{r} := (x, y)$, via $\vec{\xi}$, to elliptic coordinates $\omega := (\vartheta, \varphi)$. The distance element on $\mathcal{D}$ is $ds^2 = dx^2 + dy^2$, while on the the sphere it is

$$ds^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 = (k'^2 \sin^2\vartheta + k^2 \sin^2\varphi) \times \left(\frac{d\vartheta^2}{1 - k'^2\cos^2\vartheta} + \frac{d\varphi^2}{1 - k^2\cos^2\varphi}\right), \quad (11)$$

showing that their metric tensors are diagonal (recall that $k'^2 = 1 - k^2$). The surface element on the disk is $d^2\mathbf{r} = dx\, dy$, while on the half-sphere $\mathcal{S}_+^2$ it is

$$d^2S = \frac{d\xi_1\, d\xi_2}{\xi_3} = \frac{k'^2\sin^2\vartheta + k^2\sin^2\varphi}{\sqrt{(1 - k'^2\cos^2\vartheta)(1 - k^2\cos^2\varphi)}} d\vartheta\, d\varphi = \frac{dx\, dy}{\sqrt{1 - |\mathbf{r}|^2}} \quad (12)$$

Finally, the Laplace–Beltrami operator in these coordinates, with $\widehat{L}_i := \xi_j \partial_{\xi_k} - \xi_k \partial_{\xi_j}$, is

$$\Delta_{LB} = \widehat{L}_1^2 + \widehat{L}_2^2 + \widehat{L}_3^2 \quad (13)$$

$$= \frac{1}{k'^2\sin^2\vartheta + k^2\sin^2\varphi}(\widehat{D}(k'; \vartheta) + \widehat{D}(k; \varphi)), \quad (14)$$

where, writing $(\kappa; \psi)$ for $(k'; \vartheta)$ or $(k; \varphi)$,

$$\widehat{D}(\kappa; \psi) := \sqrt{1 - \kappa^2\cos^2\psi}\, \frac{\partial}{\partial\psi} \sqrt{1 - \kappa^2\cos^2\psi}\, \frac{\partial}{\partial\psi}. \quad (15)$$

When we set

$$\Upsilon_i(\omega) = \frac{1}{\sqrt{w(\omega)}} \Psi_i(\mathbf{r}(\omega)), \quad w(\omega) := \frac{1}{\cos\vartheta\sqrt{1 - k^2\cos^2\varphi}} = \frac{1}{\xi_3} = \frac{1}{\sqrt{1 - |\mathbf{r}|^2}}, \quad (16)$$

functions $\Psi_i(\mathbf{r})$ on the disk are unitarily related to functions $\Upsilon_i(\omega)$ on the half-sphere under their natural products

$$(\Psi_1, \Psi_2)_{\mathcal{D}} := \int_{\mathcal{D}} d^2\mathbf{r}\, \Psi_1^*(\mathbf{r})\, \Psi_2(\mathbf{r}) = \int_{\mathcal{S}_+^2} d^2S(\omega)\, \Upsilon_1^*(\omega)\, \Upsilon_2(\omega) =: (\Upsilon_1, \Upsilon_2)_{\mathcal{S}_+^2} = \int_0^{\frac{1}{2}\pi} d\vartheta \int_{-\pi}^{\pi} d\varphi\, \frac{k'^2\sin^2\vartheta + k^2\sin^2\varphi}{\sqrt{(1 - k^2\cos^2\varphi)(1 - k'^2\cos^2\vartheta)}} \times \Upsilon_1(\vartheta, \varphi)^*\, \Upsilon_2(\vartheta, \varphi). \quad (17)$$

The Zernike operator $\widehat{Z}$ in (1), which is Hermitian in $\mathcal{D}$ under (17), will be correspondingly mapped onto another operator $\widehat{W} := w(\omega)^{-1/2}\widehat{Z}\, w(\omega)^{1/2}$, which is Hermitian under the inner product in $\mathcal{S}_+^2$ and is given by

$$\widehat{W} = \Delta_{LB} + \frac{\xi_1^2 + \xi_2^2}{4\xi_3^2} + 1 = \Delta_{LB} + \frac{1}{4\cos^2\vartheta(1 - k^2\cos^2\varphi)} + \frac{3}{4}. \quad (18)$$

The Zernike differential equation (1) thus becomes

$$\widehat{W}\, \Upsilon(\vartheta, \varphi) = -E\, \Upsilon(\vartheta, \varphi), \quad (19)$$

where the value of the eccentricity k is present. It may be interesting to note that if, due to (18), this is understood as a Schrödinger two-dimensional Hamiltonian $-\frac{1}{2}\Delta_{LB} + V_W$, it allows the interpretation of the second summands as two-dimensional potentials on the disk and sphere

$$V_W := -\frac{\xi_1^2 + \xi_2^2}{8\xi_3^2} - \frac{1}{2} = -\frac{1}{8\cos^2\vartheta(1 - k^2\cos^2\varphi)} - \frac{3}{8} = -\frac{1}{8(1 - |\mathbf{r}|^2)} - \frac{3}{8}, \quad (20)$$

which is radially repulsive and drops inverse-quadratically to $-\infty$ at the $\mathcal{D}$-$\mathcal{S}_+^2$ boundary. The new two-dimensional differential equation to solve now is (19), and the boundary condition on the solutions, stemming from (2), is

$$\left.\frac{\Upsilon(\omega(\vec{\xi}\,))}{\sqrt{\xi_3}}\right|_{\xi_3=0} = \left.\frac{\Upsilon(\vartheta, \varphi)}{\sqrt{\cos\vartheta}}\right|_{\substack{\vartheta=\pi/2\\ k\neq 1}} < \infty. \quad (21)$$

3. Separation and solution of Zernike's equation

Now we propose that the solutions to (19) separate as the product of two functions, each depending on the eccentricity parameter and coordinate as

$$\Upsilon(\vartheta, \varphi) = \Theta(k'; \vartheta)\Phi(k; \varphi), \quad (22)$$

so we are led to two separate simultaneous equations with a separation constant $\lambda(k)$,

$$\left[\widehat{D}(k'; \vartheta) - \left(\left(E + \tfrac{3}{4}\right)k'^2\cos^2\vartheta - \frac{1}{4\cos^2\vartheta}\right)\right]\Theta(k'; \vartheta) = +\lambda\, \Theta(k'; \vartheta), \quad (23)$$

$$\left[\widehat{D}(k; \varphi) + \left(\left(E + \tfrac{3}{4}\right)(1 - k^2\cos^2\varphi) - \frac{k'^2}{4(1 - k^2\cos^2\varphi)}\right)\right] \times \Phi(k; \varphi) = -\lambda\, \Phi(k; \varphi). \quad (24)$$

If we ascribe to these two $\widehat{D}(\kappa; \psi)$'s the role of one-dimensional kinetic terms, clearly equations (23) and (24) contain distinct potential terms, so their solutions will be distinct functions. These differential equations belong to the class of those with periodic coefficients: from (5) it follows that the invariances under $\vartheta \to \vartheta + 2\pi$ and $\varphi \to \varphi + 2\pi$ imply the uniqueness of the solutions, $\Theta(k'; \vartheta + 2\pi) = \Theta(k'; \vartheta)$ and $\Phi(k, \varphi + 2\pi) = \Phi(k, \varphi)$. Also, Zernike's equation (1) is invariant under the two reflections, $x \leftrightarrow -x$ and $y \leftrightarrow -y$, which correspond to $\vartheta \leftrightarrow -\vartheta$ and $\varphi \leftrightarrow \varphi + \pi$; we thus expect that due to parity the solutions will split into *four classes*, even and odd under each reflection of $\mathcal{D}$ and $\mathcal{S}^2_+$. Moreover these inversions also entail the π-*periodicity* of the solutions under $\vartheta \to \vartheta + \pi$ and $\varphi \to \varphi + \pi$.

Equation (23) has a singularity at the $\mathcal{S}^2_+$ boundary $\vartheta = \frac{1}{2}\pi$, while (24) has two complex singularities at $\pm\cos\varphi = 1/k \geqslant 1$. To find square-integrable solutions we first take out these singular points with the substitution

$$\Theta(k'; \vartheta) = \sqrt{\cos\vartheta}\,\widetilde{\Theta}(k', \vartheta),$$
$$\Phi(k; \varphi) = (1 - k^2\cos^2\varphi)^{\frac{1}{4}}\,\widetilde{\Phi}(k, \varphi). \quad (25)$$

Then the differential equations (23) and (24) become

$$(1 - k'^2\cos^2\vartheta)\frac{\mathrm{d}^2\widetilde{\Theta}}{\mathrm{d}\vartheta^2} + (2k'^2\cos\vartheta\sin\vartheta - \tan\vartheta)\frac{\mathrm{d}\widetilde{\Theta}}{\mathrm{d}\vartheta} + (Ek'^2\sin^2\vartheta + \tfrac{1}{4}k'^2 - \Lambda - \tfrac{1}{4})\widetilde{\Theta} = 0, \quad (26)$$

$$(1 - k^2\cos^2\varphi)\frac{\mathrm{d}^2\widetilde{\Phi}}{\mathrm{d}\varphi^2} + 2k^2\cos\varphi\sin\varphi\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}\varphi} + (\tfrac{1}{4}k^2 + Ek^2\sin^2\varphi + \Lambda)\widetilde{\Phi} = 0, \quad (27)$$

where we introduced a new separation constant $\Lambda := \lambda + k'^2(E + \frac{1}{2})$.

To solve these differential equations we finally make the substitutions of variables

$$u := \sin^2\vartheta, \qquad v := \sin^2\varphi, \quad (28)$$

to rewrite (26) and (27) as

$$\frac{\mathrm{d}^2\widetilde{\Theta}}{\mathrm{d}u^2} + \left(\frac{1/2}{u} + \frac{1}{u-1} + \frac{1/2}{u + k^2/k'^2}\right)\frac{\mathrm{d}\widetilde{\Theta}}{\mathrm{d}u} + \frac{k'^2Eu - \frac{1}{4}k^2 - \Lambda}{4u(1-u)(k^2 + k'^2u)}\widetilde{\Theta} = 0, \quad (29)$$

$$\frac{\mathrm{d}^2\widetilde{\Phi}}{\mathrm{d}v^2} + \left(\frac{1/2}{v} + \frac{1/2}{v-1} + \frac{1}{v + k'^2/k^2}\right)\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}v} + \frac{\frac{1}{4}k^2 + Ek^2v + \Lambda}{4v(1-v)(k'^2 + k^2v)}\widetilde{\Phi} = 0. \quad (30)$$

For finite u, equation (29) has three real singular points at $u = 0$, 1, and $-k^2/k'^2$, while equation (30), has them real at $v = 0$, 1 and $-k'^2/k^2$. Now we take out the singularities at the points $u = 0$, 1 and at $v = 0$, $-k'^2/k^2$, because of (25), and consider the series expansions around the points $u = 0$ and $v = 0$. We can thus write the two series with the same coefficients $\{b_s\}_0^\infty$ as

$$\widetilde{\Theta}^{(\alpha_1,\alpha_2)}(k'; u) = (k^2 + k'^2u)^{\frac{1}{2}\alpha_1}\,u^{\frac{1}{2}\alpha_2}\sum_{s=0}^{\infty} b_s\,(k'^2\,u)^s, \quad (31)$$

$$\widetilde{\Phi}^{(\alpha_1,\alpha_2)}(k; v) = (1 - v)^{\frac{1}{2}\alpha_1}\,v^{\frac{1}{2}\alpha_2}\sum_{s=0}^{\infty} b_s\,(-k^2v)^s, \quad (32)$$

where the exponents α_i, $i = 1, 2$, must satisfy $\alpha_i(\alpha_i - 1) = 0$, i.e. $\alpha_i = 0$ or 1. The series coefficients b_s are those given by the three-term recurrence relations (depending on α_1, α_2),

$$k^2\,k'^2\,A_s\,b_{s+1} + (B_s - \tfrac{1}{4}\Lambda)b_s + C_s\,b_{s-1} = 0,$$
$$\text{with } b_{-1} = 0,\ b_0 = 1, \quad (33)$$

$$\text{and } A_s^{(\alpha_2)} = (s+1)\left(s + \alpha_2 + \tfrac{1}{2}\right), \quad (34)$$

$$B_s^{(\alpha_1,\alpha_2)} = k'^2\left(\tfrac{1}{2}(\alpha_1 + \alpha_2) + s\right)^2 - k^2\left(s + \tfrac{1}{2}\alpha_2 + \tfrac{1}{4}\right)^2, \quad (35)$$

$$C_s^{(\alpha_1,\alpha_2)} = \tfrac{1}{4}(E - (2s + \alpha_1 + \alpha_2)(2s + \alpha_1 + \alpha_2 - 2)). \quad (36)$$

The coefficients $\{b_s\}_{s=0}^\infty$ can be then obtained from an infinite system of homogeneous algebraic equations, asking for nontrivial solutions to the determinant equation

$$D^{(\alpha_1,\alpha_2)}(\Lambda) := \begin{vmatrix} B_0 - \frac{1}{4}\Lambda & k^2\,k'^2A_0 & 0 & 0 & \cdots \\ C_1 & B_1 - \frac{1}{4}\Lambda & k^2\,k'^2A_1 & 0 & \cdots \\ 0 & C_2 & B_2 - \frac{1}{4}\Lambda & k^2\,k'^2A_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} = 0. \quad (37)$$

Since it is evident that at the point $\vartheta = 0$ the wave function $\widetilde{\Theta}^{(\alpha_1,\alpha_2)}(k'; u)$ in (31) is a constant, let us now consider the asymptotic behavior of the solution (31) at the singular point $\vartheta = \frac{1}{2}\pi$; the convergence of the power series is determined by the behavior of quotient b_{s+1}/b_s for large s. Dividing the recurrence relations (33) by b_{s-1}, we have

$$\frac{b_{s+1}}{b_s}\frac{b_s}{b_{s-1}} = -\frac{1}{k^2\,k'^2}\frac{B_s - \frac{1}{4}\Lambda}{A_s}\frac{b_s}{b_{s-1}} - \frac{1}{k^2k'^2}\frac{C_s}{A_s}. \quad (38)$$

Now suppose that for the large s the behavior of their ratio is

$$\frac{b_{s+1}}{b_s} \approx c_0 + \frac{c_1}{s} + \frac{c_2}{s^2}, \qquad \frac{b_s}{b_{s-1}} \approx c_0 + \frac{c_1}{s-1} + \frac{c_2}{(s-1)^2} \approx c_0 + \frac{c_1}{s} + \frac{c_1 + c_2}{s^2}, \quad (39)$$

then putting this assumption into the three-term recurrence relation (33), and taking into account that the coefficients A_s and B_s behave asymptotically as

$$\frac{B_s - \frac{1}{4}\Lambda}{A_s} \approx (1 - 2k^2) + \frac{1}{s}\left(\left(\alpha_1 - \tfrac{3}{2}\right) - k^2\left(\alpha_1 - \tfrac{5}{2}\right)\right),$$
$$\frac{C_s}{A_s} \approx -1 + \frac{\frac{5}{2} - \alpha_1}{s}, \quad (40)$$

we can write the coefficients for s^2 and s, finding the equations for coefficients c_0 and c_1 as

$$k^2\,k'^2c_0^2 + (1-2k^2)c_0 - 1 = 0, \tag{41}$$

$$2k^2\,k'^2c_0\,c_1 + (1-2k^2)c_1 = c_0\left(k^2\left(\alpha_1-\tfrac{5}{2}\right)-\alpha_1+\tfrac{3}{2}\right) + \alpha_1 - \tfrac{5}{2}, \tag{42}$$

so we obtain two cases for coefficients c_0 and c_1:

$$\text{(a)}\qquad c_0^{(1)} = -\frac{1}{k^2},\qquad c_1^{(1)} = -\frac{1}{k^2}\left(\alpha_1-\tfrac{3}{2}\right) \tag{43}$$

$$\text{(b)}\qquad c_0^{(2)} = \frac{1}{k'^2},\qquad c_1^{(2)} = -\frac{1}{k'^2} \tag{44}$$

We now consider the expansion of $\widetilde{\Theta}^{(\alpha_1,\alpha_2)}(k';\,u)$ in (31) letting $k'^2 > k^2$, because then $|c_0^{(1)}| > |c_0^{(2)}|$. This is the case (b), which presents a so-called 'minimal solution'—while the case (a) is a 'maximal' solution. For the minimal solution (b) in (44) we have

$$\frac{b_{s+1}}{b_s} \approx \frac{1}{k'^2}\left(1-\frac{1}{s}\right)\quad\text{so that}$$
$$b_s \approx \frac{1}{(k'^2)^s}\prod_{\sigma=2}^{s}\left(1-\frac{1}{\sigma}\right) = \frac{1}{s}\,\frac{1}{(k'^2)^s}. \tag{45}$$

Therefore, at the point $\vartheta = \frac{1}{2}\pi$,

$$\widetilde{\Theta}^{(\alpha_1,\alpha_2)}(k';\,\vartheta)\Big|_{\vartheta\to\frac{1}{2}\pi} \approx \sum_{s=1}^{\infty}\frac{1}{s} \approx \ln s, \tag{46}$$

which diverges logarithmically, and thus by (25) also the functions $\Theta^{(\alpha_1,\alpha_2)}(k';\,\vartheta)$ diverge logarithmically. Analyzed in the same way, the 'maximal' case (a) gives an even more divergent solution.

Therefore, to obtain a regular solution of (23) for the any value of the parameter k, the series (31) has to be truncated to some member N, as $b_{N+1} = b_{N+2} = \cdots = 0$. So we let the coefficients of the three-term recurrence relation (33) start from $b_0 = 1$. Then, after the substitution $s = N + 1$, we have

$$k^2\,k'^2A_{N+1}b_{N+2} + (B_{N+1} - \tfrac{1}{4}\Lambda)b_{N+1} + C_{N+1}b_N = 0. \tag{47}$$

Taking into account that $b_N \neq 0$, we find that $C_{N+1} = 0$, or:

$$E = (2N+\alpha_1+\alpha_2)(2N+\alpha_1+\alpha_2+2) = n(n+2), \tag{48}$$

where $n := 2N + \alpha_1 + \alpha_2$ is the *principal* quantum number. Hence, instead of (33), the coefficients b_s will obey following three-term recurrence relations

$$k^2\,k'^2(s+1)\left(s+\alpha_2+\tfrac{1}{2}\right)b_{s+1} + \left(B_s - \tfrac{1}{4}\Lambda\right)b_s + (N-s+1)(N+s+\alpha_1+\alpha_2)b_{s-1} = 0. \tag{49}$$

Therefore the expansion of $\widetilde{\Theta}^{(\alpha_1,\alpha_2)}(k';\,u)$ in (31), and also the expansion of $\widetilde{\Phi}^{(\alpha_1,\alpha_2)}(k;\,u)$ in (32), which has the same coefficients b_s, will be truncated to a polynomial. Returning through (28) to the variables trigonometric elliptic coordinates ϑ and φ, we rewrite the wave functions $\Theta^{(\alpha_1,\alpha_2)}(k';\,\vartheta)$ and $\Phi^{(\alpha_1,\alpha_2)}(k;\,\varphi)$ in polynomial form as

$$\Theta^{(\alpha_1,\alpha_2)}(k';\,\vartheta) = \sqrt{\cos\vartheta}\,(1-k'^2\cos^2\vartheta)^{\frac{1}{2}\alpha_1} \times (\sin\vartheta)^{\alpha_2}\sum_{s=0}^{N} b_s(k'^2)^s(\sin\vartheta)^{2s}, \tag{50}$$

$$\Phi^{(\alpha_1,\alpha_2)}(k;\,\varphi) = (1-k^2\cos^2\varphi)^{\frac{1}{4}}(\cos\varphi)^{\alpha_1} \times (\sin\varphi)^{\alpha_2}\sum_{s=0}^{N} b_s(-k^2)^s(\sin\varphi)^{2s}. \tag{51}$$

Now we can solve the problem of the eigenvalues of the separation constant Λ: we rewrite the three-term recurrence relations (49) as a system of $N + 1$ homogeneous algebraic equations,

$$\begin{aligned}
&\left(B_0-\tfrac{1}{4}\Lambda\right)b_0 + k^2k'^2\left(\alpha_2+\tfrac{1}{2}\right)b_1 = 0,\\
&N(N+\alpha_1+\alpha_2+1)b_0 + \left(B_1-\tfrac{1}{4}\Lambda\right)b_1 + 2k^2k'^2\left(\alpha_2+\tfrac{3}{2}\right)b_2 = 0,\\
&\cdots \qquad\qquad \cdots\\
&(2N+\alpha_1+\alpha_2)b_{N-1} + \left(B_N-\tfrac{1}{4}\Lambda\right)b_N = 0.
\end{aligned} \tag{52}$$

This system has nontrivial solutions when the corresponding tridiagonal $(N + 1) \times (N + 1)$ determinant vanishes,

$$D_N^{(\alpha_1,\alpha_2)}(\Lambda) = \begin{vmatrix}
B_0-\frac{1}{4}\Lambda & k^2k'^2A_0 & 0 & 0 & \cdots & & 0\\
C_1 & B_1-\frac{1}{4}\Lambda & k^2k'^2A_1 & 0 & \cdots & & 0\\
0 & C_2 & B_2-\frac{1}{4}\Lambda & k^2k'^2A_2 & \ddots & & 0\\
\vdots & \vdots & \ddots & \ddots & C_{N-1} & B_{N-1}-\frac{1}{4}\Lambda & k^2k'^2A_{N-1}\\
0 & 0 & 0 & \cdots & 0 & C_N & B_N-\frac{1}{4}\Lambda
\end{vmatrix} = 0. \tag{53}$$

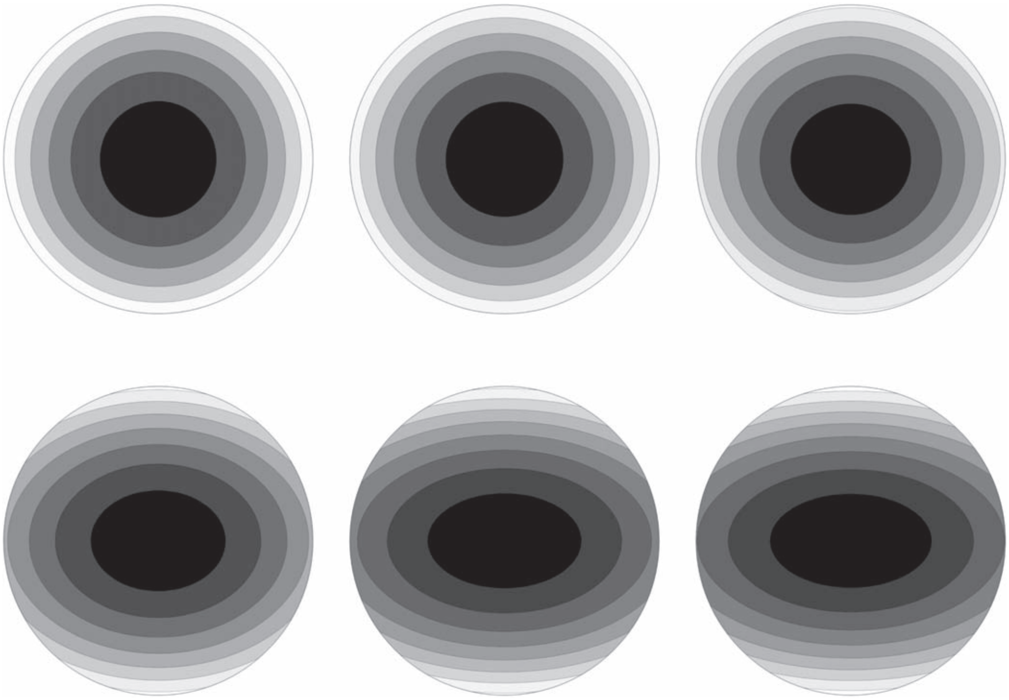

Figure 3. Homotopy of the Zernike solution $\Psi^{(0,0)}_{1,0}(x, y)$, in elliptic trigonometric coordinates characterized by the eccentricity parameter $0 \leqslant k \leqslant 1$. The angle between the foci is $0 \leqslant \kappa \leqslant \pi$ and $k = \sin\frac{1}{2}\kappa$. As in the previous figures 1 and 2, we show in the first row: System I $\kappa \approx 0$ ($k \approx 0$), $\kappa = \frac{1}{8}\pi$ ($k = 0.1951$), $\kappa = \frac{1}{4}\pi$ ($k = 0.3827$); in the second row: $\kappa = \frac{1}{2}\pi$ ($k = 0.7071$), $\kappa = \frac{3}{4}\pi$ ($k = 0.9239$), and System II $\kappa \approx \pi$ ($k \approx 1$). The numerical computation becomes unstable at the limits $k = 0, 1$ so we used $k = 0.01$ and 0.99 instead.

Such determinants are known to have real and distinct roots [27], which means that the eigenvalues Λ can be enumerated by an integer index q, as $\Lambda_{N,q}(k^2)$, with $q \in \{0, 1, 2, \ldots, N\}$, the degeneracy for a fixed N being equal to $N + 1$. Since the coefficients $A_s^{(\alpha_2)}$, $B_s^{(\alpha_1,\alpha_2)}$ and $C_s^{(\alpha_1,\alpha_2)}$ in (34)–(36) depend on (α_1, α_2), which can be (0, 0), (0, 1), (1, 0), or (1, 1), the separation constants $\Lambda^{(\alpha_1,\alpha_2)}_{N,q}$ are determined by *four* determinants (53). Each one of these will provide a 'radial' $\Theta^{(\alpha_1,\alpha_2)}_{N,q}(k'; \vartheta)$ and 'angular' $\Phi^{(\alpha_1,\alpha_2)}_{N,q}(k, \varphi)$ solution to (50) and (51). In the trigonometric form of elliptic coordinates the total number of zeros for the 'angular' function $\Phi^{(\alpha_1,\alpha_2)}_{N,q}(k; \varphi)$ in $\sin\varphi$ is $t \coloneqq 2q + \alpha_1 + \alpha_2$, while $n = 2N + \alpha_1 + \alpha_2$ and, in all cases, $0 \leqslant q \leqslant N$. In the appendix we write explicitly the 6 expressions for the one $n = 0$, two $n = 1$, and three $n = 2$ lowest-'energy' states given by (50), (51) for arbitrary $0 \leqslant k \leqslant 1$ with their normalizations constants. Integrals that are useful to find the normalization constants of the generic $\Upsilon^{(\alpha_1,\alpha_2)}_{N,q}(\vartheta, \varphi)$ in (22) are also provided.

The corresponding functions $\Psi^{(\alpha_1,\alpha_2)}_{N,q}(k; \mathbf{r})$ on the Zernike disk $\mathbf{r} \in \mathcal{D}$ can be recovered now by inverting (16) and multiplying as in (22) the radial and angular solutions

$$\Psi^{(\alpha_1,\alpha_2)}_{N,q}(k; \mathbf{r}(\vartheta, \varphi)) = \sqrt{w(\vartheta, \varphi)}\ \Theta^{(\alpha_1,\alpha_2)}_{N,q}(k'; \vartheta)\Phi^{(\alpha_1,\alpha_2)}_{N,q}(k; \varphi),$$
$$1/w(\vartheta, \varphi) \coloneqq \cos\vartheta\sqrt{1 - k^2\cos^2\varphi} = \xi_3 = \sqrt{1 - |\mathbf{r}|^2}, \tag{54}$$

and recalling the inverse coordinate transformations in (7) and (8) for the factor functions in (50) and (51). In figure 3 we show the state (54) of even–even parity $(\alpha_1, \alpha_2) = (0, 0)$ with indices $N = 1$ and $q = 0$ on the disk, for various values of the eccentricity parameter $0 < k < 1$. Clearly, this appears to be a smooth homotopic transformation depending on the parameter k. Yet we note that at the endpoints $k = 0$ and $k = 1$ of this interval, the upper diagonal of the tri-diagonal determinant (53) vanishes. This implies that the recurrence relations for the coefficients $\{b_n\}$ in (50), (51) for the solutions of Systems I and II in (9), (10) will change radically, although the functions themselves present a smooth limit to the previously known solutions of Systems I and II. Finally, in the generic elliptic case the sub-indices N, q follow their enumeration in the determinant equation (53). Their relation with the indices n, m of the System I solutions [10], and the indices n_1, n_2 of the System II solutions [12] will be made explicit in the following two sections.

4. Limit $k \to 0$ to the spherical basis of System I

In this section we examine the limit relations when the ellipticity parameter $k \to 0$, reproducing the formulas corresponding to System I of polar coordinates for the Zernike system.

4.1. Limit of the recurrence relations

In the limit $k \to 0$, all terms $k^2 k'^2 A_s$, $s = 0, 1, 2, \ldots, N - 1$ in determinant (53) on the diagonal above the main one can be neglected. The determinant thus reduces to a lower-triangular form, and is equal to the product of its diagonal elements, namely

$$\lim_{k\to 0} D_N(\Lambda) = \prod_{s=0}^{N}\left(B_s(0) - \tfrac{1}{4}\Lambda\right) = 0. \tag{55}$$

Let us assume that this product is zero due to one particular factor $s = q$, i.e. $B_q(0) - \frac{1}{4}\Lambda_{N,q}^{(\alpha_1,\alpha_2)} = 0$. This means that

$$\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(0) = (2q + \alpha_1 + \alpha_2)^2, \tag{56}$$

and consequently

$$\lim_{k\to 0}\left(B_s(k^2) - \tfrac{1}{4}\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2)\right) = (s-q)(s+q+\alpha_1+\alpha_2) =: \widetilde{B}_s. \tag{57}$$

Writing now the three-term recurrence relation (33) successively for $s = 0, 1, 2, \ldots$, and taking into account that $b_{-1} = 0$, we conclude that, as $k^2 \to 0$, this becomes

$$k^2 A_s b_{s+1} + \widetilde{B}_s b_s = 0 \quad \text{for} \quad 0 \leqslant s \leqslant q-1. \tag{58}$$

Repeating a similar procedure starting from $b_{N+1} = 0$ down, one arrives at the conclusion that the expression formula (33) reduces in the limit $k \to 0$ to

$$\widetilde{B}_s b_s + C_s b_{s-1} = 0 \quad \text{for} \quad q+1 \leqslant s \leqslant N. \tag{59}$$

In the case when $s = q$, we have $\widetilde{B}_q = 0$ and it becomes necessary to consider the next approximation term for small k^2,

$$\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2)|_{k^2=0} = \Lambda_{N,q}^{(\alpha_1,\alpha_2)}(0) + k^2 \left.\frac{\mathrm{d}\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2)}{\mathrm{d}\,k^2}\right|_{k^2=0} + O(k^4)$$
$$= (2q+\alpha_1+\alpha_2)^2 + k^2 \left.\frac{\mathrm{d}\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2)}{\mathrm{d}\,k^2}\right|_{k^2=0} + O(k^4). \tag{60}$$

Taking into account this relation and (35) for the coefficients B_s, one obtains

$$B_q(k^2) - \tfrac{1}{4}\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2) \sim \epsilon_q\, k^2, \tag{61}$$

where the smallness parameter is

$$\epsilon_q = -\frac{1}{4}\frac{\mathrm{d}\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2)}{\mathrm{d}\,k^2}|_{k^2=0} - \tfrac{1}{4}(2q+\alpha_1+\alpha_2)^2 - \tfrac{1}{4}\left(2q+\alpha_2+\tfrac{1}{2}\right)^2. \tag{62}$$

Hence for $s = q$ the three-term recurrence relation (33) takes the form

$$k^2 A_q b_{q+1} + \epsilon_q k^2 b_q + C_q b_{q-1} = 0. \tag{63}$$

Since in accordance with equations (58) and (59)

$$b_{q-1} = -b_q\, k^2 A_{q-1}/\widetilde{B}_{q-1}, \qquad b_{q+1} = -b_q\, C_{q+1}/\widetilde{B}_{q+1}, \tag{64}$$

substituting these relations into (63), one finds

$$\epsilon_q = \frac{A_q C_{q+1}}{\widetilde{B}_{q+1}} + \frac{A_{q-1}C_q}{\widetilde{B}_{q-1}}, \tag{65}$$

and since equation (62) defines the value of the derivative of $\Lambda_{N,q}^{(\alpha_1,\alpha_2)}(k^2)$ at $k \to 0$, (65) represents the restriction under which the cutoff conditions at $s = -1$ and $s = N+1$ are consistent with each other.

4.2. Limit of the wavefunctions

From the two-term recurrence relations (58) and (59) we obtain directly

$$b_s \underset{k\to 0}{\longrightarrow} \frac{\widetilde{B}_0\widetilde{B}_1\cdots\widetilde{B}_{s-1}}{A_0A_1\cdots A_{s-1}}\frac{(-1)^s}{(k^2)^s} = \frac{(-q)_s(\alpha_1+\alpha_2+q)_s}{\left(\alpha_2+\frac{1}{2}\right)_s s!}\frac{(-1)^s}{(k^2)^s}, \tag{66}$$

for $0 \leqslant s \leqslant q$, and

$$b_{q+s} \underset{k\to 0}{\longrightarrow} (-1)^s\frac{C_{q+1}C_{q+2}\cdots C_{q+s}}{\widetilde{B}_{q+1}\widetilde{B}_{q+2}\cdots\widetilde{B}_{q+s}} b_q = \frac{(-N+q)_s(N+q+\alpha_1+\alpha_2+1)_s}{(2q+\alpha_1+\alpha_2+1)_s s!} b_q, \tag{67}$$

for $1 \leqslant s \leqslant N-q$, where we use the Pochhammer symbol $(a)_m := a(a+1)\cdots(a+m-1) = \Gamma(a+m)/\Gamma(a)$. The coefficients b_q for the functions $\Theta(k'; u)$ and $\Phi(k; v)$ in (50) and (51) can be now calculated from (64), and they are

$$b_q \underset{k\to 0}{\longrightarrow} (k^2)^{-q}\frac{(q+\alpha_1+\alpha_2)_q}{\left(\alpha_2+\frac{1}{2}\right)_q}. \tag{68}$$

For the 'radial' functions $\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k'; \vartheta)$ (50), with $k' = \sqrt{1-k^2}$, and enumerated by the integer index $q \in \{0, 1, 2, \ldots, N\}$, we obtain

$$\lim_{k\to 0}\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k'; \vartheta) = \frac{1}{(k^2)^q}\frac{(q+\alpha_1+\alpha_2)_q(N-q)!}{\left(\alpha_2+\frac{1}{2}\right)_q(2q+\alpha_1+\alpha_2+1)_{N-q}} \times |\sin\vartheta|^{\alpha_1}(\sin\vartheta)^{2q+\alpha_2}\,|\cos\vartheta|^{\frac{1}{2}}\, P_{N-q}^{(2q+\alpha_1+\alpha_2,0)}(\cos 2\vartheta). \tag{69}$$

We eliminated the first sum in this formula because in the limit $k \to 0$ $(k' \to 1)$ the largest coefficient among the b_s's, according to (66), is b_q. We can thus list the four radial functions $\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k'; \vartheta)$, indicating their principal quantum number $n := 2N + \alpha_1 + \alpha_2$, and the index $m := 2q + \alpha_1 + \alpha_2$ that counts the number of zeros of the angular function that we introduced above, in terms of Legendre polynomials

for $\alpha_1 = \alpha_2 = 0$, $n = 2N$, $m = 2q$,

$$\Theta_{N,q}^{(0,0)}(k'; \vartheta) \underset{k\to 0}{\longrightarrow} \frac{2^{2q-1}}{k^{2q}}\frac{(2q)!(N-q)!}{(N+q)!} \times (\sin\vartheta)^{2q}\,|\cos\vartheta|^{\frac{1}{2}} P_{N-q}^{(2q,0)}(\cos 2\vartheta); \tag{70}$$

for $\alpha_1 = 0$, $\alpha_2 = 1$, $n = 2N+1$, $m = 2q+1$,

$$\Theta_{N,q}^{(0,1)}(k'; \vartheta) \underset{k\to 0}{\longrightarrow} \frac{2^{2q}}{k^{2q}}\frac{(2q)!(N-q)!}{(N+q+1)!} \times (\sin\vartheta)^{2q+1}\,|\cos\vartheta|^{\frac{1}{2}} P_{N-q}^{(2q+1,0)}(\cos 2\vartheta); \tag{71}$$

for $\alpha_1 = 1,\ \alpha_2 = 0,\ n = 2N + 1,\ m = 2q + 1,$

$$\Theta^{(1,0)}_{N,q}(k';\,\vartheta) \underset{k\to 0}{\longrightarrow} \frac{2^{2q}}{k^{2q}}\frac{(2q+1)!(N-q)!}{(N+q+1)!}\,|\sin\vartheta| \times (\sin\vartheta)^{2q}\,|\cos\vartheta|^{\frac{1}{2}}P^{(2q+1,0)}_{N-q}(\cos 2\vartheta); \tag{72}$$

for $\alpha_1 = \alpha_2 = 1,\ n = 2N + 2,\ m = 2q + 2,$

$$\Theta^{(1,1)}_{N,q}(k';\,\vartheta) \underset{k\to 0}{\longrightarrow} \frac{2^{2q+1}}{k^{2q}}\frac{(2q+1)!(N-q)!}{(N+q+2)!}\,|\sin\vartheta| \times (\sin\vartheta)^{2q+1}\,|\cos\vartheta|^{\frac{1}{2}}P^{(2q+2,0)}_{N-q}(\cos 2\vartheta). \tag{73}$$

Correspondingly, for the 'angular' functions $\Phi^{(\alpha_1,\alpha_2)}_{N,q}(k;\,\varphi)$ in (51) we obtain

$$\lim_{k\to 0}\Phi^{(\alpha_1,\alpha_2)}_{N,q}(k;\,\varphi) = (\cos\varphi)^{\alpha_1}\,(\sin\varphi)^{\alpha_2} \times {}_2F_1(-q,\,q+\alpha_1+\alpha_2;\,\alpha_2+\tfrac{1}{2};\,\sin^2\varphi). \tag{74}$$

Using the relations between trigonometric and hypergeometric functions given by

$$\begin{aligned}\cos az &= {}_2F_1\left(-\tfrac{1}{2}a,\,\tfrac{1}{2}a;\,\tfrac{1}{2};\,\sin^2 z\right)\\ &= \cos z\ {}_2F_1\left(\tfrac{1}{2}-\tfrac{1}{2}a,\,\tfrac{1}{2}+\tfrac{1}{2}a;\,\tfrac{1}{2};\,\sin^2 z\right),\\ \sin az &= a\sin z\ {}_2F_1\left(\tfrac{1}{2}-\tfrac{1}{2}a,\,\tfrac{1}{2}+\tfrac{1}{2}a;\,\tfrac{3}{2};\,\sin^2 z\right)\\ &= a\sin z\,\cos z\ {}_2F_1\left(1-\tfrac{1}{2}a,\,1+\tfrac{1}{2}a;\,\tfrac{3}{2};\,\sin^2 z\right),\end{aligned}$$

and recalling that $q \in \{0, 1, 2, \ldots, N\}$ and as above, $n = 2N + \alpha_1 + \alpha_2$ and $m = 2q + \alpha_1 + \alpha_2$, we obtain in all cases $(\alpha_1,\,\alpha_2)$:

$$\begin{aligned}&\text{for}(0,\,0),\ m = 2q, && \Phi^{(0,0)}_{N,q}(k;\,\varphi) \underset{k\to 0}{\longrightarrow} \cos m\varphi;\\ &\text{for}(0,\,1),\ m = 2q+1, && \Phi^{(0,1)}_{N,q}(k;\,\varphi) \underset{k\to 0}{\longrightarrow} \sin m\varphi;\\ &\text{for}(1,\,0),\ m = 2q+1, && \Phi^{(1,0)}_{N,q}(k;\,\varphi) \underset{k\to 0}{\longrightarrow} \cos m\varphi;\\ &\text{for}(1,\,1),\ m = 2q+2, && \Phi^{(1,1)}_{N,q}(k;\,\varphi) \underset{k\to 0}{\longrightarrow} \sin m\varphi.\end{aligned} \tag{75}$$

We thus have the complete wave functions on $\mathcal{S}^2_+$ in (22) given in the limit $k \to 0$, for $(0,\,0)$, $n = 2N$,

$$\Upsilon^{(0,0)}_{N,q}(\varphi,\,\vartheta) = C^{(0,0)}_{N,q}\,2^{2q-1}\frac{(2q)!(N-q)!}{(N+q)!} \times (\sin\vartheta)^{2q}\ |\cos\vartheta|^{\frac{1}{2}}\ P^{(2q,0)}_{N-q}(\cos 2\vartheta)\cos 2q\varphi; \tag{76}$$

for $(0,\,1)$, $n = 2N + 1$,

$$\Upsilon^{(0,1)}_{N,q}(\varphi,\,\vartheta) = C^{(0,1)}_{N,q}\,2^{2q}\frac{(2q)!(N-q)!}{(N+q+1)!} \times (\sin\vartheta)^{2q+1}\ |\cos\vartheta|^{\frac{1}{2}}\ P^{(2q+1,0)}_{N-q}(\cos 2\vartheta)\sin(2q+1)\varphi; \tag{77}$$

for $(1,\,0)$, $n = 2N + 1$,

$$\Upsilon^{(1,0)}_{N,q}(\varphi,\,\vartheta) = C^{(1,0)}_{N,q}\,2^{2q}\frac{(2q+1)!(N-q)!}{(N+q+1)!} \times (\sin\vartheta)^{2q+1}\ |\cos\vartheta|^{\frac{1}{2}}\ P^{(2q+1,0)}_{N-q}(\cos 2\vartheta)\cos(2q+1)\varphi; \tag{78}$$

for (1,1), $n = 2N + 2$,

$$\Upsilon^{(1,1)}_{N,q}(\varphi,\,\vartheta) = C^{(1,1)}_{N,q}\,2^{2q+1}\frac{(2q+1)!(N-q)!}{(N+q+2)!} \times (\sin\vartheta)^{2q+2}\ |\cos\vartheta|^{\frac{1}{2}}\ P^{(2q+2,0)}_{N-q}(\cos 2\vartheta)\sin(2q+2)\varphi; \tag{79}$$

where $C^{(\alpha_1,\alpha_2)}_{N,q}$ are the appropriate normalization constants multiplied by the factor k^{2q}. The four cases (76)–(79) can be written as a single expression using (75), recalling that $N = \frac{1}{2}(n - \alpha_1 - \alpha_2)$ and $q = \frac{1}{2}(m - \alpha_1 - \alpha_2)$, and introducing an integer 'radial' quantum number $n_r := N - q = (n - m)/2 \geqslant 0$. The four expressions (76)–(79) can be written as

$$\Upsilon^{(\alpha_1,\alpha_2)}_{N,q}(\varphi,\,\vartheta) = C^{(\alpha_1,\alpha_2)}_{N,q}(\sin\vartheta)^m\ |\cos\vartheta|^{\frac{1}{2}}\ P^{(|m|,0)}_{n_r}(\cos 2\vartheta) \times \begin{cases}\cos m\varphi & \text{for}(0,\,0)\text{and}(1,\,0),\\ \sin m\varphi & \text{for}(0,\,1)\text{and}(1,\,1),\end{cases} \tag{80}$$

for n, m both even or both odd. In the literature we find the Zernike solutions in System I classified by the radial quantum number n_r and trigonometric or complex exponential functions with the 'angular momentum' label $m \in \{-n,\,-n+2,\,\ldots,\,n\}$, so that $n_r = \frac{1}{2}(n - |m|)$ remains integer. With complex linear combinations we thus regain the familiar expression [1, 10] in polar coordinates $(r,\,\phi)$ with the normalization constant and standard phase given by

$$\Psi^{\mathrm{I}}_{n,m}(r,\,\phi) = (-1)^{n_r}\sqrt{\frac{n+1}{\pi}}\ r^{|m|}\ P^{(|m|,0)}_{n_r}(1-2r^2)\ \mathrm{e}^{\mathrm{i}m\phi}. \tag{81}$$

5. Limit $k'\to 0$ to the Cartesian basis of System II

In this section we follow the limit relations when the characteristic ellipticity parameter approaches $k = 1$, i.e. $k' \to 0$, that reproduce the corresponding expressions for the Zernike system in the 'Cartesian' coordinates that defined System II in [12].

5.1. Limit of recurrence relations

We follow the same path as in the previous section, eliminating the elements on the upper diagonal in the determinant $D^{(\alpha_1,\alpha_2)}_N(\Lambda)$ in (53), which are proportional to k'^2. Thus

$$\lim_{k'\to 0} D^{(\alpha_1,\alpha_2)}_N(\Lambda) = \prod_{s=0}^{N}\left(B_s(0) - \tfrac{1}{4}\Lambda^{(\alpha_1,\alpha_2)}_{N,s}\right) = 0. \tag{82}$$

Let us assume now that the vanishing factor here occurs for some particular term $s = p$, i.e. $B_p(0) - \frac{1}{4}\Lambda_{N,p}^{(\alpha_1,\alpha_2)} = 0$ and depending on k'. Then

$$\lim_{k'\to 0} \Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2) = -\left(2p + \alpha_2 + \tfrac{1}{2}\right)^2, \tag{83}$$

and consequently

$$\lim_{k'\to 0} \left(B_s(k'^2) - \tfrac{1}{4}\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2)\right)$$
$$= (p - s)(p + s + \alpha_2 + \tfrac{1}{2}) =: \widetilde{B}_s. \tag{84}$$

As in the previous section, in the limit $k' \to 0$ the three-term recurrence relation (33) splits into two two-term recurrence relations,

$$k'^2 A_s b_{s+1} + \widetilde{B}_s b_s = 0, \quad \text{for} \quad 0 \leqslant s \leqslant p - 1, \tag{85}$$

$$\widetilde{B}_s b_s + C_s b_{s-1} = 0, \quad \text{for} \quad p + 1 \leqslant s \leqslant N. \tag{86}$$

In the case $s = p$,

$$\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2)|_{k'^2=0} = \Lambda_{N,p}^{(\alpha_1,\alpha_2)}(0) + k'^2 \frac{\mathrm{d}\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2)}{\mathrm{d}\, k'^2}|_{k'^2=0} + O(k'^4)$$
$$= -\left(2p + \alpha_2 + \tfrac{1}{2}\right)^2 + k'^2 \frac{\mathrm{d}\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2)}{\mathrm{d}k'^2}|_{k'^2=0} + O(k'^4). \tag{87}$$

With this formula, one gets

$$B_p(k'^2) - \tfrac{1}{4}\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2) \sim \epsilon_p k'^2, \tag{88}$$

where now the smallness parameter is

$$\epsilon_p = -\frac{1}{4}\frac{\mathrm{d}\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2)}{\mathrm{d}k'^2}|_{k'^2=0} + \tfrac{1}{4}(\alpha_1 + \alpha_2 + 2p)^2 + \tfrac{1}{4}\left(2p + \alpha_2 + \tfrac{1}{2}\right)^2. \tag{89}$$

The three-term recurrence relation (33) for $s = p$ takes the form

$$k'^2 A_p b_{p+1} + \epsilon_p k'^2 b_p + C_p b_{p-1} = 0. \tag{90}$$

But according to the formulas (85) and (86), we obtain, corresponding to (64)

$$b_{p+1} = -b_p\, C_{p+1}/\widetilde{B}_{p+1},$$
$$b_{p-1} = -b_p\, k'^2 A_{p-1}/\widetilde{B}_{p-1}. \tag{91}$$

Putting now equations (91) into (90), one arrives at

$$\epsilon_p = \frac{A_p C_{p+1}}{\widetilde{B}_{p+1}} + \frac{A_{p-1}C_p}{\widetilde{B}_{p-1}}, \tag{92}$$

and from (89) one can then evaluate $\mathrm{d}\Lambda_{N,p}^{(\alpha_1,\alpha_2)}(k'^2)/\mathrm{d}\, k'^2|_{k'^2=0}$.

5.2. Limit of the wavefunctions

As previously in (66), from two-term recurrence relations (85) and (86) we obtain for $0 \leqslant s \leqslant p - 1$,

$$b_s \underset{k'\to 0}{\longrightarrow} \frac{\widetilde{B}_0\widetilde{B}_1 \cdots \widetilde{B}_{s-1}}{A_0 A_1 \cdots A_{s-1}} \frac{(-1)^s}{(k'^2)^s} = \frac{(-p)_s\left(p + \alpha_2 + \frac{1}{2}\right)_s}{\left(\alpha_2 + \frac{1}{2}\right)_s s!} \frac{1}{(k'^2)^s}, \tag{93}$$

while for $1 \leqslant s \leqslant N - p$,

$$b_{p+s} \underset{k'\to 0}{\longrightarrow} (-1)^s \frac{C_{p+1}C_{p+2} \cdots C_{s+p}}{\widetilde{B}_{p+1}\widetilde{B}_{p+2} \cdots \widetilde{B}_{s+p}} b_p = \frac{(-N + p)_s (N + p + \alpha_1 + \alpha_2 + 1)_s}{(-1)^s \left(2p + \alpha_2 + \frac{3}{2}\right)_s s!} b_p, \tag{94}$$

For b_p, the calculation yields

$$b_p = \frac{\left(p + \alpha_2 + \frac{1}{2}\right)_p}{(-k'^2)^p \left(\alpha_2 + \frac{1}{2}\right)_p}. \tag{95}$$

Proceeding as before, from (31) we obtain the limit $k' \to 0$ for the 'radial' functions

$$\lim_{k'\to 0} \Theta_{N,p}^{(\alpha_2)}(k';\, \vartheta) \sim (\sin\vartheta)^{\alpha_2}\, |\cos\vartheta|^{\frac{1}{2}}\; {}_2F_1(-p, p + \alpha_2 + \tfrac{1}{2};\, \alpha_2 + \tfrac{1}{2};\, \sin^2\vartheta), \tag{96}$$

which for $k' \to 0$ do not depend on the parity α_1. Let us introduce the quantum number $n_1 := 2p + \alpha_2$ having the same parity as α_2, so that $p = \frac{1}{2}(n_1 - \alpha_2)$ is integer; then we only need to list the two cases, which involve Legendre polynomials: for $\alpha_2 = 0$, $n_1 = 2p$,

$$\Theta_{N,p}^{(0)}(k';\, \vartheta) \underset{k'\to 0}{\longrightarrow} |\cos\vartheta|^{\frac{1}{2}}\; {}_2F_1\left(-p, p + \tfrac{1}{2};\, \tfrac{1}{2};\, \sin^2\vartheta\right) = (-1)^p \frac{2^{2p}(p!)^2}{(2p)!} |\cos\vartheta|^{\frac{1}{2}} P_{2p}(\sin\vartheta); \tag{97}$$

for $\alpha_2 = 1$, $n_1 = 2p + 1$,

$$\Theta_{N,p}^{(1)}(k';\, \vartheta) \underset{k'\to 0}{\longrightarrow} \sin\vartheta\, |\cos\vartheta|^{\frac{1}{2}}\; {}_2F_1\left(-p, p + \tfrac{3}{2};\, \tfrac{3}{2};\, \sin^2\vartheta\right) = (-1)^p \frac{2^{2p}(p!)^2}{(2p + 1)!} |\cos\vartheta|^{\frac{1}{2}} P_{2p+1}(\sin\vartheta). \tag{98}$$

These two expressions can be subsumed in a single form for the radial function

$$\Theta_{N,p}^{(\alpha_2)}(k';\, \vartheta) \underset{k'\to 0}{\longrightarrow} (-1)^p \frac{2^{2p}(p!)^2}{n_1!} |\cos\vartheta|^{\frac{1}{2}} P_{n_1}(\sin\vartheta)$$
$$n_1 = 2p + \alpha_2. \tag{99}$$

Next, for the angular function $\Phi_{N,p}^{(\alpha_1,\alpha_2)}(k;\, \varphi)$, from (32) and taking into account (93) and (94) we have that in the $k' \to 0$

limit

$$\Phi_{N,p}^{(\alpha_1,\alpha_2)}(k;\,\varphi) \sim (\sin\varphi)^{2p+\alpha_2}\,|\sin\varphi|^{\frac{1}{2}}\,(\cos\varphi)^{\alpha_1} \times \frac{\left(p+\alpha_2+\frac{1}{2}\right)_p}{(k'^2)^p\left(\alpha_2+\frac{1}{2}\right)_p} \times {}_2F_1(-N+p,\,N+p+\alpha_1+\alpha_2+1;\, 2p+\alpha_2+\tfrac{3}{2};\,\sin^2\varphi), \tag{100}$$

where the first summation is eliminated because it is one order more in k'^2. Because of the parities α_1, α_2, this contains *four* cases, and now the hypergeometric functions are Gegenbauer polynomials $C_n^{\alpha}(x)$. For the case of (0, 0) parities, we have

$$\begin{aligned}\Phi_{N,p}^{(0,0)}(k;\,\varphi) &\underset{\kappa'\longrightarrow 0}{\longrightarrow} \frac{\left(p+\frac{1}{2}\right)_p}{(\kappa')^{2p}\left(\frac{1}{2}\right)_p}(\sin\varphi)^{2p}\,|\sin\varphi|^{\frac{1}{2}} \\ &\times {}_2F_1(-N+p,\,N+p+1;\,2p+\tfrac{3}{2};\,\sin^2\varphi) \\ &= \frac{\left(p+\frac{1}{2}\right)_p}{(\kappa')^{2p}\left(\frac{1}{2}\right)_p}\frac{(2N-2p)!\,\Gamma(2(2p+1))}{\Gamma(2(2p+1)+2(N-p))} \\ &\times (\sin\varphi)^{2p+\frac{1}{2}}\,C_{2N-2p}^{2p+1}(\cos\varphi),\end{aligned} \tag{101}$$

and the three other cases follow similarly. As for the radial function, the angular functions can be subsumed by a single expression with appropriate indices

$$\Phi_{N,p}^{(\alpha_1,\alpha_2)}(k;\,\varphi) \underset{\kappa'\longrightarrow 0}{\longrightarrow} \frac{n_2!\,\Gamma(\alpha_2+\frac{1}{2})\Gamma(n_1+\frac{1}{2})\Gamma(2n_1+2)}{(\kappa')^{2p}\Gamma\left(p+\alpha_2+\frac{1}{2}\right)^2\Gamma(2n_1+n_2+2)} \times (\sin\varphi)^{n_1}\,|\sin\varphi|^{\frac{1}{2}}\,C_{n_2}^{n_1+1}(\cos\varphi). \tag{102}$$

With $n = 2N+\alpha_1+\alpha_2$, $\quad n_1 = 2p+\alpha_1+\alpha_2$, $n_2 := n - n_1 = 2N - 2p$.

Finally, the complete wave functions (22) built from both the radial and angular parts are in the limit $k' \to 0$, the solutions reported for System II in [12], with the exchange of coordinates $\vartheta \leftrightarrow \varphi + \frac{1}{2}\pi$ and given by

$$\Upsilon_{N,p}^{(\alpha_1,\alpha_2)}(\vartheta,\,\varphi) = \Upsilon_{n_1,n_2}^{\mathrm{II}}(\varphi,\,\vartheta) = \widetilde{C}_{n_1,n_2}^{(\alpha_1,\alpha_2)}\,|\cos\vartheta|^{\frac{1}{2}}\,P_{n_1}(\sin\vartheta)\,|\sin\varphi|^{n_1+\frac{1}{2}}\,C_{n_2}^{n_1+1}(\cos\varphi), \tag{103}$$

where $\widetilde{C}_{n_1,n_2}^{(\alpha_1,\alpha_2)}$ is an appropriate normalization constant multiplied by $(k')^{2p}$. This exchange of radial and angular coordinates and the corresponding rotation $(x,\,y) \leftrightarrow (-y,\,x)$ on the unit disk [10] $\mathbf{r}(\vartheta,\,\varphi)$ in (54) yields, for $k = 1$ the Zernike solutions on the disk in System II

$$\begin{aligned}\Psi_{n_1,n_2}^{\mathrm{II}}(\mathbf{r}) &= \Upsilon_{n_1,n_2}^{\mathrm{II}}(\vartheta,\,\varphi)/\sqrt{\cos\vartheta\sqrt{1-k^2\cos^2\varphi}} \\ &= 2^{n_1}n_1!\sqrt{\frac{(2n_1+1)(n_1+n_2+1)n_2!}{\pi(2n_1+n_2+1)!}}(1-y^2)^{\frac{1}{2}n_1} \\ &\times P_{n_1}\left(\frac{x}{\sqrt{1-y^2}}\right)C_{n_2}^{n_1+1}(y).\end{aligned} \tag{104}$$

6. Conclusions

We have constructed the explicitly separated solutions to the Zernike system in elliptic trigonometric coordinates. We thus verified the consistency of our results with those previously obtained in [12] by addressing their limits $k \to 0$ and $k \to 1$ to the polar coordinates of Systems I and II.

The importance of integrable and superintegrable systems in two or more dimensions, is that their differential equations and solutions separate in more than one system of coordinates. In the particular case of the Zernike system, the generic coordinate system is elliptic and its separation is ruled by the Heun differential equation, which has four regular singular points, while in the two limits examined here these reduce to hypergeometric differential equations with three such points. As figure 3 shows, there is a continuous homotopy between the two extremes $k = 0$ and 1 as we vary the eccentricity parameter.

A defining characteristic of superintegrable dynamical systems is that their governing Hamiltonians—in this case equation (1)—can be written as a nonlinear combination of the operators that correspond to the extra constants of the motion. This connection was provided explicitly for the polar coordinates of System I in (10, equation (78)), but we consider that repeating this analysis for the generic elliptic coordinates would take us beyond the stated purpose of the present paper.

The existence of more than one system of separating coordinates and thus of separated solutions, also raises the question of interbasis expansion coefficients [28–30]. In [11, 12] we found the overlap coefficients between systems I, II, and its $\frac{1}{2}\pi$-rotated version called III, to be given by special Hahn and Racah polynomials—the former also given as special Clebsch–Gordan coefficients. Having here a continuum of elliptic coordinate systems raises the question of their interbasis expansions between generic rotated elliptic coordinates. These considerations suggest that orthogonal sets of other special functions of higher order can be expected in further research on the Zernike system.

Acknowledgments

We thank Guillermo Krötsch for his invaluable help with the figures. GSP and AY thank the support of project PRO-SNI-2018 (Universidad de Guadalajara); NMA and KBW thank project IG-100119 *Óptica Matemática* awarded by the Dirección General de Asuntos del Personal Académico, Universidad Nacional Autónoma de México.

Appendix

Here we write out the Zernike eigenfunctions separated in elliptic coordinates, in their form

$$\Upsilon_{N,q}^{(\alpha_1,\alpha_2)}(\vartheta,\varphi)=C_{N,q}^{(\alpha_1,\alpha_2)}\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k';\vartheta)\Phi_{N,q}^{(\alpha_1,\alpha_2)}(k;\varphi),$$

where $C_{N,q}^{(\alpha_1,\alpha_2)}$ is a normalization constant, for the six lowest-lying values of the 'energy' that correspond to $n=0$, 1, and 2 (see (53) *et seq.*) which (recalling that $n=2N+\alpha_1+\alpha_2$, and choosing $b_0=1$) involves the first (uppermost) three eigenvalues $\Lambda_{N,q}^{(\alpha_1,\alpha_2)}$. We also give below two integrals that are useful to compute the normalization constants $C_{N,q}^{(\alpha_1,\alpha_2)}$

$\underline{n=0,\quad q=0,\quad \alpha_1=\alpha_2=0,}$ so $N=0$, and $\Lambda_{0,0}^{(0,0)}=-\frac{1}{4}k^2$:

$$\Upsilon_{0,0}^{(0,0)}(\varphi,\vartheta)=\frac{1}{\sqrt{\pi}}(\cos\vartheta)^{1/2}(1-k^2\cos^2\varphi)^{1/4}=\frac{1}{\sqrt{\pi}}\xi_3^{1/2}.$$

In this case, and in the following two $n=1$ cases, the normalization constants can be found directly by integration over the unit disk

$\underline{n=1,\quad q=0,\quad \alpha_1=0,\quad \alpha_2=1,}$ so $N=0$ and $\Lambda_{0,0}^{(0,1)}=1-\frac{13}{4}k^2$:

$$\Upsilon_{0,0}^{(0,1)}(\varphi,\vartheta)=\frac{2}{\sqrt{\pi}}(\cos\vartheta)^{1/2}\sin\vartheta\sin\varphi(1-k^2\cos^2\varphi)^{1/4}=\frac{2}{\sqrt{\pi}}\xi_2\,\xi_3^{1/2}.$$

$\underline{n=1,\quad q=0,\quad \alpha_1=1,\quad \alpha_2=0,}$ so $N=0$ and $\Lambda_{0,0}^{(1,0)}=1-\frac{5}{4}k^2$:

$$\Upsilon_{0,0}^{(1,0)}(\varphi,\vartheta)=\frac{2}{\sqrt{\pi}}(\cos\vartheta)^{1/2}(1-k'^2\cos^2\vartheta)^{1/2}\times\cos\varphi(1-k^2\cos^2\varphi)^{1/4}=\frac{2}{\sqrt{\pi}}\xi_1\,\xi_3^{1/2}.$$

$\underline{n=2,\quad q=0,\quad \alpha_1=\alpha_2=0,}$ so $N=1$ and $\Lambda_{1,0}^{(0,0)}=2k'^2-\frac{13}{4}k^2-\sqrt{4k'^2+9k^4}$:

$\underline{n=2,\quad q=0,\quad \alpha_1=\alpha_2=1,}$ so $N=0$ and $\Lambda_{0,0}^{(1,1)}=4k'^2-\frac{9}{4}k^2$,

$$\Upsilon_{0,0}^{(1,1)}(\varphi,\vartheta)=2\sqrt{\frac{6}{\pi}}(\cos\vartheta)^{1/2}\sin\vartheta\,(1-k'^2\cos^2\vartheta)^{1/2}(1-k^2\cos^2\varphi)^{1/4}\sin\varphi\cos\varphi.$$

Regarding the normalization constants $C_{N,q}^{(\alpha_1,\alpha_2)}$, although we cannot give their general expression, we can fragment their computation into separate one-dimension integrals. This involves factorizing

$$\begin{aligned}I_{N,q}^{(\alpha_1,\alpha_2)}&=\int_0^{\pi/2}\mathrm{d}\vartheta\int_0^{2\pi}\mathrm{d}\varphi(\Upsilon_{N,q}^{(\alpha_1,\alpha_2)}(\varphi,\vartheta))^2\times\frac{k'^2\sin^2\vartheta+k^2\sin^2\varphi}{\sqrt{(1-k'^2\cos^2\vartheta)(1-k^2\cos^2\varphi)}}\\&=|C_{N,q}^{(\alpha_1,\alpha_2)}|^2\left[\int_0^{2\pi}(\Phi_{N,q}^{(\alpha_1,\alpha_2)}(k;\varphi))^2\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\cos^2\varphi}}\times\int_0^{\pi/2}(\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k';\vartheta))^2\frac{k'^2\sin^2\vartheta\,\mathrm{d}\vartheta}{\sqrt{1-k'^2\cos^2\vartheta}}\right.\\&\left.+\int_0^{2\pi}(\Phi_{N,q}^{(\alpha_1,\alpha_2)}(k;\varphi))^2\frac{k^2\sin^2\varphi\,\mathrm{d}\varphi}{\sqrt{1-k^2\cos^2\varphi}}\times\int_0^{\pi/2}(\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k';\vartheta))^2\frac{\mathrm{d}\vartheta}{\sqrt{1-k'^2\cos^2\vartheta}}\right],\end{aligned}$$

and setting $I_{N,q}^{(\alpha_1,\alpha_2)}=1$, thereby obtaining $C_{N,q}^{(\alpha_1,\alpha_2)}$ up to a phase.

Now, observing the forms of $\Theta_{N,q}^{(\alpha_1,\alpha_2)}(k';\vartheta)$ in (50) and of $\Phi_{N,q}^{(\alpha_1,\alpha_2)}(k;\varphi)$ in (51) and, having found the coefficients $\{b_s\}_{s=1}^N$ from (53), we have polynomials in trigonometric functions of ϑ whose integrals range in $\vartheta\in[0,\frac{1}{2}\pi]$, and functions of $\varphi\in[0,2\pi]$, thus yielding four distinct integrands for the four parity cases (α_1,α_2).

$$\begin{aligned}\Upsilon_{1,0}^{(0,0)}(\varphi,\vartheta)&=\frac{C_{1,0}^{(0,0)}}{k'^2k^2}(\cos\vartheta)^{1/2}(1-k^2\cos^2\varphi)^{1/4}\left(k'^2-\left[k'^2-\tfrac{3}{2}k^2-\sqrt{k'^2+\tfrac{9}{4}k^4}\right]\sin^2\varphi\right)\\&\times\left(k^2+\left[k'^2-\tfrac{3}{2}k^2-\sqrt{k'^2+\tfrac{9}{4}k^4}\right]\sin^2\vartheta\right),\\(C_{1,0}^{(0,0)})^2&=\frac{12k'^4k^4/\pi}{54k'^8-156k'^6+193k'^4-114k'^2+27-(18k'^6-38k'^4+31k'^2-9)\sqrt{9k'^4-14k'^2+9}}.\end{aligned}$$

$\underline{n=2,\quad q=1,\quad \alpha_1=\alpha_2=0,}$ so $N=1$ and $\Lambda_{1,1}^{(0,0)}=2k'^2-\frac{13}{4}k^2+\sqrt{4k'^2+9k^4}$,

Let us consider here only the case $(\alpha_1,\alpha_2)=(0,0)$, where the integrals over φ are solved using [31, equation

$$\begin{aligned}\Upsilon_{1,1}^{(0,0)}(\varphi,\vartheta)&=\frac{C_{1,1}^{(0,0)}}{k'^2k^2}(\cos\vartheta)^{1/2}(1-k^2\cos^2\varphi)^{1/4}\left(k'^2-\left[k'^2-\tfrac{3}{2}k^2+\sqrt{k'^2+\tfrac{9}{4}k^4}\right]\sin^2\varphi\right)\\&\times\left(k^2+\left[k'^2-\tfrac{3}{2}k^2+\sqrt{k'^2+\tfrac{9}{4}k^4}\right]\sin^2\vartheta\right),\\(C_{1,1}^{(0,0)})^2&=\frac{12k'^4k^4/\pi}{54k'^8-156k'^6+193k'^4-114k'^2+27+(18k'^6-38k'^4+31k'^2-9)\sqrt{9k'^4-14k'^2+9}}.\end{aligned}$$

2.5.12(32)]

$$\int_0^{2\pi} \sin^m\varphi \cos^n\varphi \, d\varphi = (1 + [-1]^n) \times \Gamma(\tfrac{1}{2}[m+1])\Gamma\left(\tfrac{1}{2}[n+1]\right)/\Gamma\left(\tfrac{1}{2}[m+n]+1\right),$$

for $m + n$ even and zero otherwise. For the integral in $x = \cos\vartheta \in [0, 1]$ and $z = k/k'$, we can use the expression obtained from [31, equation (1.2.43(6)],

$$\int_0^1 \frac{x^{2p}\, dx}{\sqrt{z^2 + x^2}} = \frac{\sqrt{1+z^2}}{2p} \times \left[1 + \sum_{\ell=1}^{p-1} \frac{(2p-1)(2p-3)\cdots(2p-2\ell-1)}{2^\ell (p-1)(p-2)\cdots(p-\ell)} (-z^2)^\ell\right] + (-z^2)^p \frac{(2p-1)!!}{(2p)!!} \operatorname{arcsinh}\frac{1}{z},$$

which is valid for $p = 2, 3, \ldots$; for $p = 0$ the first summand is absent, while for $p = 1$ the summation over ℓ is excluded.

ORCID iDs

Kurt Bernardo Wolf https://orcid.org/0000-0002-1103-0470
Alexander Yakhno https://orcid.org/0000-0002-8531-8936

References

[1] Zernike F 1934 Beugungstheorie des Schneidenverfahrens und Seiner Verbesserten Form der Phasenkontrastmethode *Physica* **1** 689–704
[2] Bhatia A B and Wolf E 1954 On the circle polynomials of Zernike and related orthogonal sets *Math. Proc. Camb. Phil. Soc.* **50** 40–8
[3] Tango W J 1977 The circle polynomials of Zernike and their application in optics *Appl. Phys.* **13** 327–32
[4] Shakibaei B H and Paramesran R 2013 Recursive formula to compute Zernike radial polynomials *Opt. Lett.* **38** 2487–9
[5] Myrick D R 1966 A Generalization of the radial polynomials of F. Zernike *SIAM J. Appl. Math.* **14** 476–89
[6] Kintner E C 1976 On the mathematical properties of the Zernike polynomials *Opt. Acta* **23** 679–80
[7] Wünsche A 2005 Generalized Zernike or disc polynomials *J. Comput. Appl. Math.* **174** 135–63
[8] Ismail M E H and Zhang R 2016 Classes of bivariate orthogonal polynomials, special issue on symmetry, integrability and geometry: methods and applications *SIGMA* **12** 021
[9] Pogosyan G S, Wolf K B and Yakhno A 2017 Superintegrable classical Zernike system *J. Math. Phys.* **58** 072901
[10] Pogosyan G S, Salto-Alegre C, Wolf K B and Yakhno A 2017 Quantum superintegrable Zernike system *J. Math. Phys.* **58** 072101
[11] Pogosyan G S, Wolf K B and Yakhno A 2017 New separated polynomial solutions to the Zernike system on the unit disk and interbasis expansion *J. Opt. Soc. Am.* A **34** 1844–8
[12] Atakishiyev N M, Pogosyan G S, Wolf K B and Yakhno A 2017 Interbasis expansions in the Zernike system *J. Math. Phys.* **58** 103505
[13] Higgs P W 1979 Dynamical symmetries in a spherical geometry *J. Phys. A: Math. Nucl. Gen.* **12** 309–23
[14] Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1985 Elliptic basis of circular oscillator *Nuovo Cimento* B **88** 43–56
[15] Kalnins E G and Miller W Jr 2005 Jacobi elliptic coordinates, functions of Heun and Lamé type and the Niven transform *Regular Chaotic Dyn.* **10** 487–508
[16] Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 2006 *Quantum Systems with Hidden Symmetry. Interbasis Expansions* (Moscow: Nauka, Fizmatlit) (in Russian)
[17] Lukac I and Smorodinskiĭ Y A 1970 Wave functions for the asymmetric top *Sov. Phys.—JETP* **30** 728–30
[18] Pogosyan G S, Sissakian A N and Winternitz P 2002 Separation of variables and Lie algebra contractions. Applications to special functions *Phys. Part. Nuclei* **33** S123–44
[19] Kalnins E G, Kress J M, Miller W Jr and Pogosyan G S 2001 Completeness of superintegrability in two dimensional constant curvature spaces *J. Phys. A: Math. Gen.* **34** 4705–20
[20] Miller W Jr, Kalnins E G and Pogosyan G S 2006 Exact and quasi-exact solvability of second-order superintegrable systems: I. Euclidean space preliminaries *J. Math. Phys.* **47** 033502
[21] Atakishiyev N M, Pogosyan G S, Wolf K B and Yakhno A 2018 Elliptic basis for the Zernike system: Heun function solutions *J. Math. Phys.* **59** 073503
[22] Erdélyi A (ed) 1955 *Bateman Manuscript Project: Higher Transcendental Functions* vol 3 (New York: McGraw-Hill)
[23] Gradshteyn I S and Ryzhik I M 2000 *Table of Integrals, Series, and Products* 6th edn (New York: Academic)
[24] Koekoek R, Lesky P A and Swarttouw R F 2010 *Hypergeometric Orthogonal Polynomials and Their q-Analogues* (Berlin: Springer) p 204
[25] Patera J and Winternitz P 1973 A new basis for the representations of the rotation group. lamé and heun polynomials *J. Math. Phys.* **14** 1130–9
[26] Arscott F M 1995 Heun's equation *Heun's Differential equations* ed A Ronveaux (Oxford: Oxford University Press) pp 3–44
[27] Coulson C A and Robertson P D 1958 Wave functions for the Hydrogen atom in spheroidal coordinates: I. The derivation and properties of these functions *Proc. Phys. Soc.* **71** 815–27
[28] Dobrev V K and Hilgert J 1998 *Lie Theory and Its Applications in Physics Proc. of II Int. Workshop* ed H D Doebner, V K Dobrev and J Hilgert (Singapore: World Scientific) pp 243–51
[29] Hakobyan Y M, Pogosyan G S, Sissakian A N and Vinitsky S I 1999 Isotropic oscillator in the space of constant positive curvature. Interbasis expansions *Phys. At. Nuclei* **62** 623–37
[30] Cariñena J F, Rañada M F and Santander M 2008 The harmonic oscillator on Riemannian and Lorentzian configuration spaces of constant curvature *J. Math. Phys.* **49** 032703
[31] Prudnikov A P, Brychkov Y A and Marichev O I 1986 *Integrals and Series* (*Elementary Functions*) vol 1 (London: Gordon and Breach)