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Interbasis expansions in the Zernike system

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The differential equation with free boundary conditions on the unit disk that was proposed by Frits Zernike in 1934 to find Jacobi polynomial solutions (indicated as I) serves to define a classical system and a quantum system which have been found to be superintegrable. We have determined two new orthogonal polynomial solutions (indicated as II and III) that are separable and involve Legendre and Gegenbauer polynomials. Here we report on their three interbasis expansion coefficients: between the I–II and I–III bases, they are given by ${}_3F_2(\cdots|1)$ polynomials that are also special su(2) Clebsch–Gordan coefficients and Hahn polynomials. Between the II–III bases, we find an expansion expressed by ${}_4F_3(\cdots|1)$'s and Racah polynomials that are related to the Wigner $6j$ coefficients. *Published by AIP Publishing.* https://doi.org/10.1063/1.5000915

I. INTRODUCTION

The two-dimensional differential equation proposed by Frits Zernike in 1934 to find a basis of orthogonal functions over the closed unit disk, $\mathcal{D} := \{\mathbf{r} = (x, y) \,|\, |\mathbf{r}| \le 1\}$, is[26]

$$\widehat{Z}\,\psi(\mathbf{r}) := \left(\nabla^2 - (\mathbf{r}\cdot\nabla)^2 - 2\mathbf{r}\cdot\nabla\right)\psi(\mathbf{r}) = -E\,\psi(\mathbf{r}). \quad (1)$$

The operator $\widehat{Z}$ is Hermitian under the natural inner product of functions over this region,

$$(\psi_1, \psi_2)_{\mathcal{D}} := \int_{\mathcal{D}} \mathrm{d}^2\mathbf{r}\,\psi_1(\mathbf{r})^*\,\psi_2(\mathbf{r}), \quad (2)$$

that defines the space $\mathcal{L}^2(\mathcal{D})$ of square-integrable functions over D. Therefore, $(\widehat{Z}\psi_1, \psi_2)_{\mathcal{D}} = (\psi_1, \widehat{Z}\psi_2)_{\mathcal{D}}$ and the eigenfunction solutions $\Psi_{n,m}(\mathbf{r})$ to (1) will be orthogonal when they belong to different eigenvalues $E_n = n(n+2)$ and/or different eigenvalues under the angular momentum operator $\widehat{M} := -\mathrm{i}(x\partial_y - y\partial_x)$, which corresponds to the evident rotational symmetry of Zernike's equation and the ensuing separation of variables in a polar coordinate system $\mathbf{r} = (r, \phi)$.

Equation (1) can be seen as a classical Hamiltonian (with momenta $\mathbf{p} = -\mathrm{i}\nabla$ as in Ref. 20) or a Schrödinger equation with a non-standard quantum Hamiltonian $\widehat{H} = -\frac{1}{2}\widehat{Z}$, as done in Ref. 19. In this paper, we shall address the expansions between the original Zernike eigenbasis (labeled I) and two of the new separated eigenbases reported in Ref. 19 (labeled II and III). All three eigenbases are solutions of (1) that *separate* into two polynomial factors: for I, in Jacobi polynomials of the radius r and trigonometric functions of the angle ϕ; for II and III, the factors are Legendre and Gegenbauer polynomials of coordinates that are not orthogonal over the disk.

We consider to be relevant that the Zernike system is one of the very few superintegrable systems that have been actually used in optics, concretely for phase-contrast microscopy.[26] Superintegrability means that the system has more constants of motion than degrees of freedom, that classically the system has closed orbits, and that Poisson brackets or commutators of the conserved quantities or operators will generally yield quadratic or higher elements of known Lie algebras; see Refs. 8–12, 7,

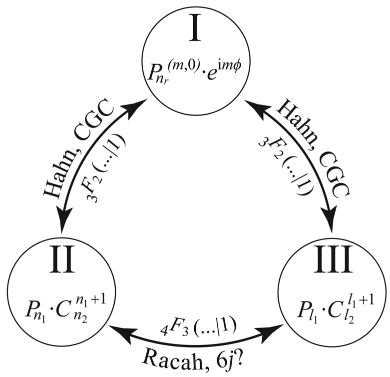

FIG. 1. Three bases and three interbasis expansions. The original Zernike disk solutions,[26] indicated as I, involve Jacobi polynomials and phases. These are related through interbasis expansions given by $_3F_2(\cdots \mid 1)$'s, which are Hahn polynomials and special Clebsch–Gordan coefficients, to the new solutions, II and III,[19] that involve Legendre and Gegenbauer polynomials. The relation between the II and III bases are given by $_4F_3(\cdots \mid 1)$'s that are Racah polynomials, which are suggested to be special $6j$ coupling coefficients.

and 13–16. As a quantum system, it is novel in the sense of not being restricted in space by potential barriers but by boundaries where the wave functions can have finite values and normal derivatives. Also, as a Hamiltonian operator, we note that $\widehat{Z}$ is built as a linear combination of generators of an $\mathsf{sp}(2,\mathsf{R})$ algebra ($\sim\nabla^2$, $\mathbf{r}\cdot\nabla$, and $|\mathbf{r}|^2$) plus one quadratic term, $(\mathbf{r}\cdot\nabla)^2$.

It has been pointed out that the generic superintegrable system leads by contraction to the Askey-Wilson polynomial scheme and that the results reported here can be understood to stem from its contraction to the Higgs oscillator system.[7,11,12]

In Sec. II, we succinctly recall the construction and expressions of the three said bases. Then we proceed to find the interbasis expansions I–II and I-III in Sec. III, which yield special Clebsch–Gordan coefficients that are Hahn polynomials, and II-III in Sec. IV, which are special $6j$ coefficients given by Racah polynomials, as illustrated in Fig. 1. We add conclusions in Sec. V, while necessary but extensive derivations are collected in the Appendixes A and B.

II. THREE ORTHONORMAL EIGENBASES

The key to find new coordinate systems where the solutions of Zernike's equation separate is to perform a vertical map from the disk D to a half-sphere $\mathcal{H}_+ := \{\vec{r} = (x, y, z) \mid |\vec{r}| = 1,\ z \geq 0\}$. Separation of the solutions occurs when the coordinates are such that one of them is constant on the boundary of the region, i.e., on the circle $|\mathbf{r}| = 1$ common to both D and $\mathcal{H}_+$. As shown in Fig. 2, we can use the spherical coordinate system (ϑ, φ) on the half-sphere, oriented in three distinct directions. When the line of poles coincides with the z-axis, one obtains the polar separation of coordinates of the Zernike basis I; when this line coincides with the x- or y-axis, we obtain the II or III bases, respectively.

To find the coordinate ranges for the three spherical coordinate systems, we introduce the unit 3-vector $\vec{\xi}$ of components,

$$\xi_1 := x, \quad \xi_2 := y, \quad \xi_3 := \sqrt{1 - x^2 - y^2} \geq 0. \tag{3}$$

The three coordinate systems and their ranges on $\mathcal{H}_+$ are defined as[19,20]

System I:

$$\xi_1 = \sin\vartheta\cos\varphi, \quad \xi_2 = \sin\vartheta\sin\varphi, \quad \xi_3 = \cos\vartheta, \qquad \vartheta|_0^{\pi/2},\ \varphi|_{-\pi}^{\pi}, \tag{4}$$

System II:

$$\xi_1 = \cos\vartheta', \quad \xi_2 = \sin\vartheta'\cos\varphi', \quad \xi_3 = \sin\vartheta'\sin\varphi', \qquad \vartheta'|_0^{\pi},\ \varphi'|_0^{\pi}, \tag{5}$$

System III:

$$\xi_1 = \sin\vartheta''\sin\varphi'', \quad \xi_2 = \cos\vartheta'', \quad \xi_3 = \sin\vartheta''\cos\varphi'', \qquad \vartheta''|_0^{\pi},\ \varphi''|_{-\pi/2}^{\pi/2}. \tag{6}$$

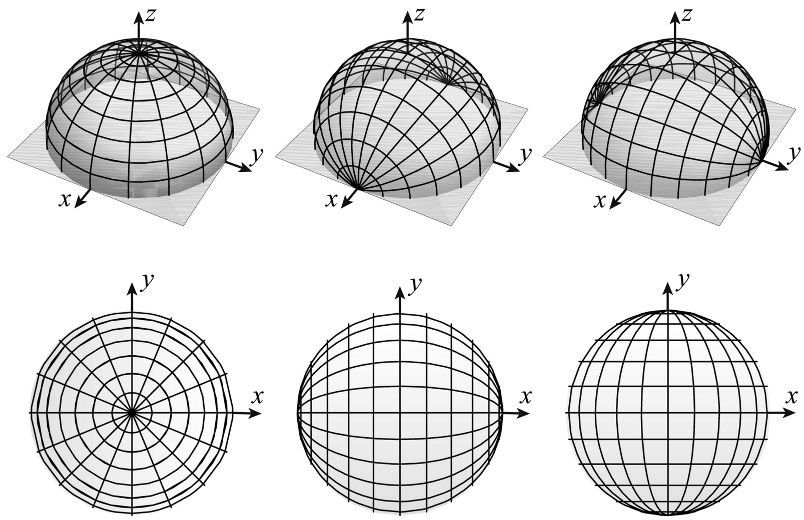

FIG. 2. *Top row*: the three coordinate systems (4)–(6) on the half-sphere $\mathcal{H}_+$. *Bottom row*: the same three coordinate systems under projection to the disk D.

The measure on $\mathcal{H}_+$ is related to that on D through

$$\mathrm{d}^2 S(\vec{\xi}) = \frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{\xi_3} = \frac{\mathrm{d}^2\mathbf{r}}{\sqrt{1-|\mathbf{r}|^2}} = \sin\vartheta\,\mathrm{d}\vartheta^\circ\,\mathrm{d}\varphi^\circ, \tag{7}$$

where $\vartheta^\circ, \varphi^\circ$ stands for any of the three spherical coordinates in (4)–(6), within their corresponding ranges for the inner product on $\mathcal{H}_+$, the upper half-sphere,

$$(\upsilon_1,\upsilon_2)_{\mathcal{H}_+} = \int_{\mathcal{H}_+} \mathrm{d}^2 S(\vec{\xi})\,\upsilon_1(\vec{\xi})^*\,\upsilon_2(\vec{\xi}) = (\psi_1,\psi_2)_{\mathcal{D}}. \tag{8}$$

In accordance with the change of measures, the functions and the Zernike operator on $\mathcal{H}_+$ and on D will then relate through

$$\upsilon(\vartheta^\circ,\varphi^\circ) \equiv \upsilon(\vec{\xi}) := (1-|\mathbf{r}|)^{1/4}\psi(\mathbf{r}), \qquad \widehat{W} := (1-|\mathbf{r}|)^{1/4}\widehat{Z}(1-|\mathbf{r}|)^{-1/4}. \tag{9}$$

On $\vec{\xi}\in\mathcal{H}_+$, the Zernike differential equation —now with $\widehat{W}$—has the simpler Schrödinger structure,

$$\widehat{W}\Upsilon(\vec{\xi}) = \Big(\Delta_{\mathrm{LB}} + \frac{\xi_1^2+\xi_2^2}{4\xi_3^2} + 1\Big)\Upsilon(\vec{\xi}) = -E\Upsilon(\vec{\xi}), \tag{10}$$

where $\Delta_{\mathrm{LB}} = L_1^2 + L_2^2 + L_3^2$ is the Laplace-Beltrami operator on the sphere, with the formal $\mathsf{so}(3)$ generators $L_i := \xi_j\partial_{\xi_k} - \xi_k\partial_{\xi_j}$ (i, j, k cyclic) and a repulsive oscillator-type of potential $\sim -r^2/(1-r^2)$ over the disk.

In Ref. 19, we wrote (10) in each of the three coordinate systems (4)–(6), separating each in successive or simultaneous differential equations in ϑ° and φ°, taking care that the solutions be square-integrable under the inner product (8) and allowing them to have finite values on the boundary circle.

The normalized eigen-solutions over $\mathcal{H}_+$ and $\mathcal{D}$ classified by "polar" (n, m) and "Cartesian" (n_1, n_2) eigenvalues are as follows:

The original Zernike system I in (4), shown in Fig. 3, is defined as

$$\Upsilon^{\mathrm{I}}_{n,m}(\vartheta,\varphi) = \sqrt{\frac{n+1}{\pi}}\,(\sin\vartheta)^{|m|}(\cos\vartheta)^{1/2}P^{(|m|,0)}_{\frac{1}{2}(n-|m|)}(\cos 2\vartheta)\,e^{\mathrm{i}m\varphi}, \tag{11}$$

$$\Psi^{\mathrm{I}}_{n,m}(r,\phi) = (-1)^{\frac{1}{2}(n-|m|)}\sqrt{\frac{n+1}{\pi}}\,r^{|m|}P^{(|m|,0)}_{\frac{1}{2}(n-|m|)}(1-r^2)\,e^{\mathrm{i}m\phi}, \tag{12}$$

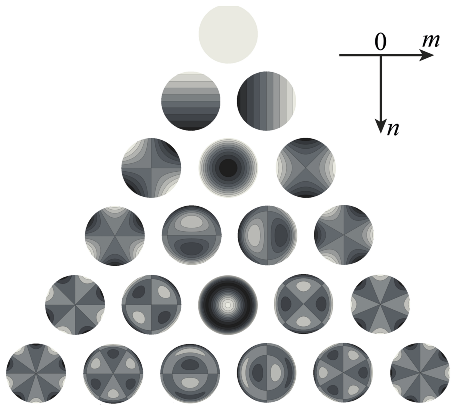

FIG. 3. The original Zernike solutions[26] over the disk, $\Psi^{\rm I}_{n,m}(r,\phi)$, in (12). The rows are counted from $n=0$ down and m crosswise. Since these functions are complex, $\Psi^{\rm I}_{n,m}=\Psi^{\rm I\,*}_{n,-m}$, we show their real part for $m\geq 0$ and their imaginary part for $m<0$.

the x-oriented system II in (5), shown in Fig. 4, is defined as

$$\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi')=C_{n_1,n_2}\,(\sin\vartheta')^{n_1+\frac{1}{2}}\,C^{n_1+1}_{n_2}(\cos\vartheta')\,\sqrt{\sin\varphi'}P_{n_1}(\cos\varphi'), \tag{13}$$

$$\Psi^{\rm II}_{n_1,n_2}(x,y)=C_{n_1,n_2}\,(1-x^2)^{\frac{1}{2}n_1}\,C^{n_1+1}_{n_2}(x)P_{n_1}\Big(\frac{y}{\sqrt{1-x^2}}\Big), \tag{14}$$

and the y-oriented system III in (6), shown in Fig. 5, is defined as

$$\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'')=C_{\ell_1,\ell_2}\,(\sin\vartheta'')^{\ell_1+\frac{1}{2}}\,C^{\ell_1+1}_{\ell_2}(\cos\vartheta'')\,\sqrt{\cos\varphi''}P_{\ell_1}(\sin\varphi''), \tag{15}$$

$$\Psi^{\rm III}_{\ell_1,\ell_2}(x,y)=C_{\ell_1,\ell_2}\,P_{\ell_1}\Big(\frac{x}{\sqrt{1-y^2}}\Big)\,(1-y^2)^{\frac{1}{2}\ell_1}\,C^{\ell_1+1}_{\ell_2}(y), \tag{16}$$

where, in (13)–(16), the multiplying constant is

$$C_{k_1,k_2}:=2^{k_1}\,k_1!\,\sqrt{\frac{(2k_1+1)(k_1+k_2+1)\,k_2!}{\pi(2k_1+k_2+1)!}}, \tag{17}$$

where $J^{(\alpha,\beta)}_\nu$, C^α_ν, and P_ν are the Jacobi, Gegenbauer, and Legendre polynomials of degree ν, respectively, and where $k_i\in\{0,1,2,\ldots\}=:\mathcal{Z}^+_0$ stands for n_i or ℓ_i. The range of these quantum numbers is

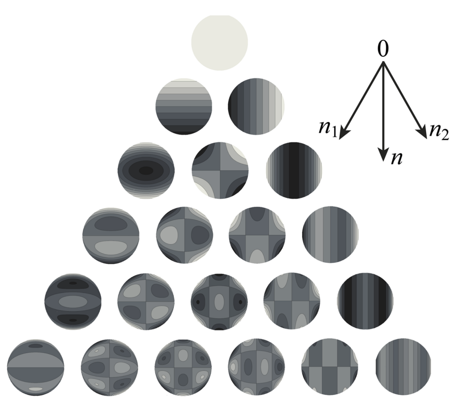

FIG. 4. The new solutions[19] of the Zernike equation over the disk, $\Psi^{\rm II}_{n_1,n_2}(x,y)$, in (14). The rows are counted from $n=0$ down, n_1 counted down left, and n_2 counted down right.

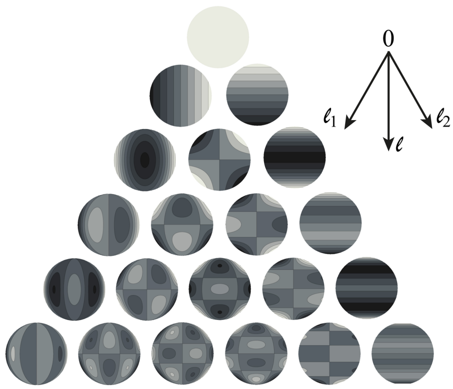

FIG. 5. The solutions $\Psi^{\rm III}_{\ell_1,\ell_2}(x,y)$ of the Zernike equation defined in (16) over the disk. The principal quantum number $\ell := \ell_1 + \ell_2$ here has the same interpretation as $n = n_1 + n_2$ in Fig. 4.

$$k_1 + k_2 = n = 2n_r + |m| \in \mathcal{Z}_0^+,\qquad n_r \in \mathcal{Z}_0^+,\quad m \in \{-n, -n+2, \ldots n\}. \tag{18}$$

Here $n = \ell$ is the *principal* quantum number that determines the "energy" eigenvalues $E_n = n(n + 2)$ in 1; in system I, n_r is the *radial* quantum number (that counts the radial nodes), m is the *angular* quantum number, while (k_1, k_2) in systems II and III qualify to be called the *Cartesian* numbers.

It is important to remark that although the multiplets formed with the quantum numbers (18) in Figs. 3–5 exhibit the same pattern as that of the two-dimensional quantum harmonic oscillator in polar and Cartesian coordinates, respectively, this analogy is misleading. There is no Lie algebra of raising and lowering operators to provide two-term transitions within the multiplet; only *three*-term differential and recursion relations have been found and are analyzed in several mathematical papers.[2,3,10,13,17,22,25] The constant of motion operators that one collects when separating the solutions forms instead a superintegrable cubic Higgs algebra[9] as shown in Ref. 19.

III. I-II AND I-III INTERBASIS EXPANSIONS

We first consider the interbasis expansion between the I and II orthonormal basis functions in (11) and (13) for fixed values of the principal quantum number, n,

$$\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi') = \sum_{m=-n\,(2)}^{n} W^{n,m}_{n_1,n_2}\,\Upsilon^{\rm I}_{n,m}(\vartheta,\varphi), \tag{19}$$

where $\sum_{m=-n\,(2)}^{n}$ indicates that the sum over m is in the range (18), so the radial quantum number $n_r = \frac{1}{2}(n - |m|) \in \mathcal{Z}_0^+$ is an integer. The relation between the primed and unprimed angles is found equating (4) and (5), as

$$\cos\vartheta' = \sin\vartheta\cos\varphi, \qquad \cos\varphi' = \frac{\sin\vartheta\sin\varphi}{\sqrt{1-\sin^2\vartheta\cos^2\varphi}}. \tag{20}$$

A. I–II interbasis with ${}_3F_2$'s

To calculate the explicit form of the expansion coefficients $W^{n,m}_{n_1,n_2}$, it is sufficient to fix the value of the coordinate that is constant over the boundary and then use the orthogonality of the wave functions for the other coordinate in the expansion (19). Thus consider $\vartheta = \frac{1}{2}\pi - \varepsilon$ for vanishing ε; then $\cos\vartheta = -\sin\varepsilon \approx -\varepsilon$ and $\sin\vartheta = \cos\varepsilon \approx 1$, and also $P_{n_1}(1) = 1$ and $P^{(|m|,0)}_{n_r}(-1) = (-1)^{n_r}$. Dividing by a common vanishing factor, the expansion (19) remains in terms only of functions of φ and reads

$$C_{n_1,n_2}\sqrt{\frac{\pi}{n+1}}(\sin\varphi)^{n_1}\, C_{n_2}^{n_1+1}(\cos\varphi) = \sum_{m=-n\,(2)}^{n} (-1)^{\frac{1}{2}(n-|m|)} W_{n_1,n_2}^{n,m}\, e^{\mathrm{i}m\varphi}. \tag{21}$$

Thus, using the orthogonality of the functions $e^{\mathrm{i}m\varphi}$ in m, we obtain the following integral representation for the interbasis expansion coefficients:

$$W_{n_1,n_2}^{n,m} = \frac{(-1)^{\frac{1}{2}(n-|m|)} C_{n_1,n_2}}{2\sqrt{\pi(n+1)}} \int_{-\pi}^{\pi} \mathrm{d}\varphi\,(\sin\varphi)^{n_1}\, C_{n_2}^{n_1+1}(\cos\varphi)\, e^{-\mathrm{i}m\varphi}. \tag{22}$$

This integral does not appear in the standard tables, so we perform its computation by writing the functions in the integrand as finite series in $e^{\mathrm{i}\varphi}$,

$$\sin^k\varphi = \frac{e^{\mathrm{i}k\varphi}}{(2\mathrm{i})^k}(1-e^{-2\mathrm{i}\varphi})^k = \frac{1}{(2\mathrm{i})^k}\sum_{l=0}^{k}\frac{(-1)^l\, k!}{l!\,(k-l)!}\, e^{\mathrm{i}(k-2l)\varphi}, \tag{23}$$

$$C_\mu^\lambda(\cos\varphi) = \sum_{j=0}^{\mu}\frac{\Gamma(\lambda+j)}{j!\,(\mu-j)!}\,\frac{\Gamma(\lambda+\mu-j)}{\Gamma^2(\lambda)}\, e^{-\mathrm{i}(\mu-2j)\varphi}. \tag{24}$$

Substituting these expansions in (22), replacing the constant C_{n_1,n_2} from (17) and using the orthogonality of the $e^{\mathrm{i}m\varphi}$ functions, we obtain a hypergeometric ${}_3F_2$ terminating series of unit argument,

$$\begin{aligned} W_{n_1,n_2}^{n,m} &= \frac{\mathrm{i}^{n_1}(-1)^{\frac{1}{2}(m+|m|)}\, n!}{\left(\frac{1}{2}(n_1-n_2-m)\right)!\left(\frac{1}{2}(n+m)\right)!}\sqrt{\frac{2n_1+1}{n_2!\,(n+n_1+1)!}} \\ &\quad\times {}_3F_2\left(\begin{matrix}-n_2, & n_1+1, & -\frac{1}{2}(n+m)\\ -n, & \frac{1}{2}(n_1-n_2-m)+1 & \end{matrix}\middle|\, 1\right), \end{aligned} \tag{25}$$

where we note that of the five parameters in ${}_3F_2(\cdots|1)$, only three, e.g., (n_1, n_2, m), are effectively present in the interbasis coefficients.

B. I–II interbasis with Clebsch–Gordan coefficients

Perhaps surprisingly, the interbasis coefficients $W_{n_1,n_2}^{n,m}$ in (25) can be compactly expressed in terms of $\mathsf{su}(2)$ Clebsch–Gordan coefficients $C_{a,\alpha;b,\beta}^{c,\gamma}$ of a special type. As given by Varshalovich *et al.*,[23] the generic coefficients that couple angular momentum states $|a,\alpha\rangle$ and $|b,\beta\rangle$ to form $|c,\gamma\rangle$, after a transformation between two ${}_3F_2$ forms, are

$$\begin{aligned} C_{a,\alpha;b,\beta}^{c,\gamma} &= \sqrt{\frac{(2c+1)(b+c-a)!(b-\beta)!(c+\gamma)!(c-\gamma)!}{(a+b-c)!(a-b+c)!(a+b+c+1)!(a+\alpha)!(a-\alpha)!(b+\beta)!}} \\ &\quad\times\frac{\delta_{\gamma,\alpha+\beta}(2a)!\,(c-b+\alpha)!}{(c-b+\alpha)!\,(c-a-\beta)!}\,{}_3F_2\left(\begin{matrix}-a-b+c, & -a+\alpha, & b-a+c+1\\ -2a, & c-a-\beta+1 & \end{matrix}\middle|\, 1\right). \end{aligned} \tag{26}$$

Now, comparing this with (25), we can write the $W_{n_1,n_2}^{n,m}$ coefficients, with $a=b=\frac{1}{2}n$, $\alpha=-\beta=-\frac{1}{2}m$, and $\gamma=0$, in terms of a special type of Clebsch–Gordan coefficients and a phase, as

$$W_{n_1,n_2}^{n,m} = \mathrm{i}^{n_1}(-1)^{\frac{1}{2}(m+|m|)}\, C^{n_1,0}_{\frac{1}{2}n,-\frac{1}{2}m;\,\frac{1}{2}n,\,\frac{1}{2}m}. \tag{27}$$

Let us note that values of $C^{n_1,0}_{\frac{1}{2}n,-\frac{1}{2}m;\,\frac{1}{2}n,\,\frac{1}{2}m}$ satisfy all necessary conditions [Ref. 23, Sec. 8.1.1] for valid $\mathsf{su}(2)$ Clebsch–Gordan coefficients, namely, the triangle condition because $0\le n_1\le n=n_1+n_2$, with integer or half-integer non-negative numbers $|m|\le n$ and $0\le n$. Finally, we note that while the original Zernike solutions are complex, $\Upsilon^{\mathrm{I}}_{n,m}=\Upsilon^{\mathrm{I}\,*}_{n,-m}$, the new ones, $\Upsilon^{\mathrm{II}}_{n_1,n_2}$ and $\Upsilon^{\mathrm{III}}_{\ell_1,\ell_2}$, are real. This property is assured by relation between $\pm m$ coefficients,

$$C^{n_1,0}_{\frac{1}{2}n,\,\frac{1}{2}m;\,\frac{1}{2}n,-\frac{1}{2}m} = (-1)^{n_2}\, C^{n_1,0}_{\frac{1}{2}n,-\frac{1}{2}m;\,\frac{1}{2}n,\,\frac{1}{2}m}. \tag{28}$$

The expansion inverse to (19), namely,

$$\Upsilon^{\mathrm{I}}_{n,m}(\vartheta,\varphi) = \sum_{n_1=0}^{n} \widetilde{W}^{n_1,n_2}_{n,m}\,\Upsilon^{\mathrm{II}}_{n_1,n_2}(\vartheta',\varphi'), \tag{29}$$

with $n_1 + n_2 = n$, follows from the orthogonality property of the $\mathsf{su}(2)$ Clebsch–Gordan coefficients. These II-I interbasis coefficients are thus given by

$$\widetilde{W}^{n_1,n_2}_{n,m} = (-\mathrm{i})^{n_1}(-1)^{\frac{1}{2}(m+|m|)}\, C^{n_1,\,0}_{\frac{1}{2}n,\,-\frac{1}{2}m;\,\frac{1}{2}n,\,\frac{1}{2}m} \tag{30}$$

and may be written in terms of ${}_3F_2$ hypergeometric functions through (26).

C. I–II interbasis with Hahn polynomials

The interbasis coefficients $W^{n,m}_{n_1,n_2}$ can be also expressed in terms of the N Hahn polynomials of degree p of a discrete variable x,[14]

$$Q_p(x;\,\alpha,\beta,N) := {}_3F_2\left(\begin{matrix}-p, & -x, & p+\alpha+\beta+1\\ & -N, & \alpha+1\end{matrix}\,\middle|\,1\right), \tag{31}$$

for $x, p \in \{0, 1, \ldots, N\}$. Applying the transformation,

$${}_3F_2\left(\begin{matrix}a, & b, & c\\ & d, & e\end{matrix}\,\middle|\,1\right) = \frac{(d+1)!\,(d-a-b+1)!}{(d-a+1)!\,(d-b+1)!}\,{}_3F_2\left(\begin{matrix}a, & b, & e-c\\ & a+b-d+1, & e\end{matrix}\,\middle|\,1\right) \tag{32}$$

to the ${}_3F_2$ hypergeometric polynomial in (25), we can write the interbasis coefficients $W^{n,m}_{n_1,n_2}$, with its three effective parameters, as

$$\begin{aligned} W^{n,m}_{n_1,n_2} = \frac{\mathrm{i}^{n_1}\,(-1)^{\frac{1}{2}(m+|m|)}\,(n!)^2}{\left(\frac{1}{2}(n-m)\right)!\,\left(\frac{1}{2}(n+m)\right)!}\sqrt{\frac{2n_1+1}{n_2!\,(n+n_1+1)!}}\\ \times Q_{n_2}\left(\tfrac{1}{2}(n+m);\,-n-1,\,-n-1,\,n\right). \end{aligned} \tag{33}$$

The discrete orthogonality relation for the Hahn polynomials is of the form [Ref. 14, Eq. (9.5.2)],

$$\sum_{j=0}^{N} \rho(j)\,Q_m(j;\alpha,\beta,N)\,Q_n(j;\alpha,\beta,N) = \delta_{m,n}\,d_n^2, \tag{34}$$

with the weight function $\rho(j)$ and the norm d_n,

$$\rho(j) = \left(\frac{N!}{j!\,(N-j)!}\right)^2, \qquad d_n = \frac{1}{N!}\sqrt{\frac{n!\,(2N+1-n)!}{2N-2n+1}}. \tag{35}$$

Thus (33) can be written in a more compact form as

$$W^{n,m}_{n_1,n_2} = \mathrm{i}^{n_1}\,(-1)^{\frac{1}{2}(m+|m|)}\,\frac{\sqrt{\rho(\frac{1}{2}(n+m))}}{d_{n_2}}\,Q_{n_2}\left(\tfrac{1}{2}(n+m);\,-(n+1),-(n+1),n\right). \tag{36}$$

In fact, the Hahn polynomials present here are particular cases of (31), with $\alpha = \beta = -(n+1) < -N$ and $N = n$ and symmetric under $m \leftrightarrow -m$, which coincide with the *dual* Hahn polynomials [Ref. 14, Eq. (9.5.2)],

$$R_{\frac{1}{2}(n+m)}\left(\lambda(n_2);\,-(n+1),-(n+1),n\right) = Q_{n_2}\left(\tfrac{1}{2}(n+m);\,-(n+1),-(n+1),n\right), \tag{37}$$

on the quadratic lattice $\lambda(n_2) := n_2(n_2-2n-1)$ (see remark in Ref. 14, p. 208). The expansion inverse to (19), namely (29), follows from the orthogonality of Hahn and dual Hahn polynomials given in Ref. 14, Eq. (9.6.2).

D. The I–III interbasis expansion

The coefficients of the interbasis expansion between the I and III bases can be found with the same method as for the I-II interbasis coefficients (25), (36), or (27) given above, by realizing that the spherical coordinates (ϑ',φ') in (5) and (6) are related through $\vartheta' \mapsto \vartheta''$ and $\varphi' \mapsto \varphi'' + \frac{1}{2}\pi$ and up to a phase $(-1)^{\ell_1}$. The expansion between the solutions defined in (11) and (15) is

$$\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'') = \sum_{m=-n\,(2)}^{n} \widehat{W}^{n,m}_{\ell_1,\ell_2}\, \Upsilon^{\rm I}_{n,m}(\vartheta,\varphi), \tag{38}$$

with the relation between the angles being now

$$\cos\vartheta'' = \sin\vartheta\sin\varphi \qquad \cos\varphi'' = \frac{\cos\vartheta}{\sqrt{1-\sin^2\vartheta\sin^2\varphi}}. \tag{39}$$

To find the coefficients $\widehat{W}^{n,m}_{\ell_1,\ell_2}$, one comes to an integral similar to (22) except for the trigonometric functions of φ, namely,

$$\widehat{W}^{n,m}_{\ell_1,\ell_2} = \frac{(-1)^{\frac{1}{2}(n-|m|)}C_{\ell_1,\ell_2}}{2\sqrt{\pi(n+1)}} \int_{-\pi}^{\pi} {\rm d}\varphi\,(\cos\varphi)^{\ell_1}\, C^{\ell_1+1}_{\ell_2}(\sin\varphi)\, e^{-{\rm i}m\varphi}, \tag{40}$$

so with the change of variables $\varphi \to \varphi + \frac{1}{2}\pi$ and the same procedure used in (22), we obtain

$$\widehat{W}^{n,m}_{\ell_1,\ell_2} = (-1)^{\ell_1}\exp(-{\rm i}\tfrac{1}{2}\pi m)\, W^{n,m}_{\ell_1,\ell_2}, \tag{41}$$

where the coefficients $W^{n,m}_{\ell_1,\ell_2}$ are those in (19) and (25), with (ℓ_1,ℓ_2) replacing (n_1,n_2).

IV. II-III INTERBASIS EXPANSIONS

We consider now the interbasis expansion between the two new spherical wave functions, $\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi')$ defined in (13) and $\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'')$ in (15), within the same multiplet characterized by the principal quantum number n that contains $n+1$ functions,

$$\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'') = \sum_{n_2=0}^{n} U^{n_1,n_2}_{\ell_1,\ell_2}\, \Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi'), \tag{42}$$

where $\ell_1+\ell_2 = n = n_1+n_2$, $\ell_i, n_i \in \mathcal{Z}^+_0$, and with the relation between the two spherical coordinate systems (ϑ'',φ'') and (ϑ',φ') being now

$$\cos\vartheta'' = \sin\vartheta'\cos\varphi',\ \sin\vartheta'' = \sqrt{1-\sin^2\vartheta'\cos^2\varphi'},$$
$$\cos\varphi'' = \frac{\sin\vartheta'\sin\varphi'}{\sqrt{1-\sin^2\vartheta'\cos^2\varphi'}},\ \sin\varphi'' = \frac{\cos\vartheta'}{\sqrt{1-\sin^2\vartheta'\cos^2\varphi'}}. \tag{43}$$

The interbasis expansion coefficients in (42) can be succinctly expressed by passing through $\Upsilon^{\rm I}_{n,m}(\vartheta,\varphi)$, using the coefficients for the inverse expansion in (38) and the direct one in (29), as

$$\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'') = \sum_{m=-n\,(2)}^{n} \widehat{W}^{n,m}_{\ell_1,\ell_2} \sum_{n_1=0}^{n} \widetilde{W}^{n_1,n_2}_{n,m}\, \Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi'). \tag{44}$$

Replacing now the Clebsch–Gordan coefficients from (27) with care of the phases, we find

$$U^{n_1,n_2}_{\ell_1,\ell_2} = (-1)^{\ell_1} \sum_{m=-n\,(2)}^{n} {\rm i}^{\ell_1-n_1-m} C^{\ell_1,0}_{\frac{1}{2}n,-\frac{1}{2}m;\,\frac{1}{2}n,\frac{1}{2}m}\; C^{n_1,0}_{\frac{1}{2}n,-\frac{1}{2}m;\,\frac{1}{2}n,\frac{1}{2}m}, \tag{45}$$

and changing the summation index m to $k = \frac{1}{2}(m+n) \in \{0,1,\ldots,n\}$, this expression is rewritten as

$$U^{n_1,n_2}_{\ell_1,\ell_2} = {\rm i}^{\ell_1+n_2} \sum_{k=0}^{n} (-1)^{\ell_1+k} C^{\ell_1,0}_{\frac{1}{2}n,\frac{1}{2}n-k;\,\frac{1}{2}n,-\frac{1}{2}n+k}\; C^{n_1,0}_{\frac{1}{2}n,\frac{1}{2}n-k;\,\frac{1}{2}n,-\frac{1}{2}n+k}. \tag{46}$$

As will be seen in Subsection IV A, when $\ell_1 + n_2$ or $\ell_2 + n_1$ are odd numbers, the coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$ are zero; the non-zero coefficients are real, as are the two sets of basis functions $\Upsilon^{\rm II}_{n_1,n_2}$ and $\Upsilon^{\rm III}_{\ell_1,\ell_2}$.

The expansion in (42) holds both for the $\Upsilon^{\rm II,III}_{k_1,k_2}(\vartheta^\circ,\varphi^\circ)$ functions and for the $\Psi^{\rm II,III}_{k_1,k_2}(x,y)$ functions on the disk. Although (45) is an explicit formula for the II–III interbasis expansion coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$, we consider worthwhile to pursue alternative closed expressions that will turn out to involve the Racah discrete polynomials, which occupy the highest rung in the Askey-Wilson classification.[14] Also, we will suggest that Wigner 6j recoupling coefficients may express this interbasis expansion.

A. Considerations on parities

Since both the Legendre and Gegenbauer polynomials have definite parities, so do the new Zernike solutions $\Psi^{\rm II}_{n_1,n_2}(x,y)$ and $\Psi^{\rm III}_{\ell_1,\ell_2}(x,y)$ in (14) and (16). These are

$$\Psi^{\rm II}_{n_1,n_2}(x,y) = (-1)^{n_2}\Psi^{\rm II}_{n_1,n_2}(-x,y) = (-1)^{n_1}\Psi^{\rm II}_{n_1,n_2}(x,-y), \tag{47}$$

$$\Psi^{\rm III}_{\ell_1,\ell_2}(x,y) = (-1)^{\ell_1}\Psi^{\rm III}_{\ell_1,\ell_2}(-x,y) = (-1)^{\ell_2}\Psi^{\rm III}_{\ell_1,\ell_2}(x,-y), \tag{48}$$

where we notice that the changes of sign in the 1 and 2 quantum numbers are intertwined.

The parity must be the same on both sides of the expansion (42), so with the aid of (43), we separate the sum into even and odd parts, writing $\sum_{n_2=0}^{n} = \sum_{n_2\ \text{even}} + \sum_{n_2\ \text{odd}}$. Under the transformation $x \mapsto -x$, (47) and (48) turn (42) into

$$(-1)^{\ell_1}\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'') = \sum_{n_2\ \text{even}} U^{n_1,n_2}_{\ell_1,\ell_2}\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi') - \sum_{n_2\ \text{odd}} U^{n_1,n_2}_{\ell_1,\ell_2}\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi'), \tag{49}$$

which compared with the original (42) imply that, when ℓ_1 is odd or even, the coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$ of either the even or the odd part of the sum are zero,

$$U^{n_1,2p_2}_{2q_1+1,\ell_2} = 0, \qquad U^{n_1,2p_2+1}_{2q_1,\ell_2} = 0, \tag{50}$$

where we have written odd $\ell_1 = 2q_1 + 1$ and even $\ell_1 = 2q_1$, as well as even $n_2 = 2p_2$ and odd $n_2 = 2p_2 + 1$, for integer q_1, p_2.

The summation over n_2 in (42) can be turned into a summation over $n_1 = n - n_2$ with the same division into even and odd terms and considered under the transformation $y \mapsto -y$ yielding

$$(-1)^{\ell_2}\Upsilon^{\rm III}_{\ell_1,\ell_2}(\vartheta'',\varphi'') = \sum_{n_1\ \text{even}} U^{n_1,n_2}_{\ell_1,\ell_2}\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi') - \sum_{n_1\ \text{odd}} U^{n_1,n_2}_{\ell_1,\ell_2}\Upsilon^{\rm II}_{n_1,n_2}(\vartheta',\varphi'). \tag{51}$$

Again comparing with the original (42), we conclude that when ℓ_1 is odd or even, then

$$U^{2p_1,n_2}_{\ell_1,2q_2+1} = 0, \qquad U^{2p_1+1,n_2}_{\ell_1,2q_2} = 0 \tag{52}$$

for integer p_1, q_2.

From (50) and (52), we reach the result that the coefficients $U^{a,b}_{\alpha,\beta}$ are *non-zero* only when (a, β) have the same parity and also (b, α) have the same parity. This leaves (42) broken up into *four* separate cases.

For even n states:

$$\Upsilon^{\rm III}_{2q_1,2q_2}(\vartheta'',\varphi'') = \sum_{p_1,p_2} U^{2p_1,2p_2}_{2q_1,2q_2}\,\Upsilon^{\rm II}_{2p_1,2p_2}(\vartheta',\varphi'), \tag{53}$$

$$\Upsilon^{\rm III}_{2q_1+1,2q_2+1}(\vartheta'',\varphi'') = \sum_{p_1,p_2} U^{2p_1+1,2p_2+1}_{2q_1+1,2q_2+1}\,\Upsilon^{\rm II}_{2p_1+1,2p_2+1}(\vartheta',\varphi'). \tag{54}$$

For odd n states:

$$\Upsilon^{\rm III}_{2q_1+1,2q_2}(\vartheta'',\varphi'') = \sum_{p_1,p_2} U^{2p_1,2p_2+1}_{2q_1+1,2q_2}\,\Upsilon^{\rm II}_{2p_1,2p_2+1}(\vartheta',\varphi'), \tag{55}$$

$$\Upsilon^{\rm III}_{2q_1,2q_2+1}(\vartheta'',\varphi'') = \sum_{p_1,p_2} U^{2p_1+1,2p_2}_{2q_1,2q_2+1}\,\Upsilon^{\rm II}_{2p_1+1,2p_2}(\vartheta',\varphi'). \tag{56}$$

In every case, the sum of each pair of indices will add to the principal quantum number n, and the relation (43) between the angles will hold. In Subsections IV B and IV C, we analyze separately the two cases presented by even n_2 in (53) and (56) and by odd n_2 in (54) and (55), following routes parallel to that used in Sec. III for the I–II interbasis coefficients.

B. Interbasis coefficients $U_{\ell_1,\ell_2}^{n_1,n_2}$ for even n_2

We consider first the interbasis expansion (42) for even n_2. The coefficients $U_{\ell_1,\ell_2}^{n_1,n_2}$ can be calculated again as we did following (20) for $\vartheta' = \pi/2 - \varepsilon$ on the common rim of the disk and half-sphere. As before, in the limit $\varepsilon \to 0$ on both sides of (51), with $\cos\vartheta'' = \cos\varphi'$, $\cos\varphi'' \approx 1$, and $\sin\varphi'' \approx \cos\vartheta'/\sin\varphi'$ in (5) and (6), where $x = \xi_1 = 0$. Using the expressions for Legendre and Gegenbauer polynomials in (14) and (16) for even $\ell_1 = 2q_1$ and $n_2 = 2p_2$, with q_1, p_2 non-negative integers,

$$P_{2q_1}(0) = \frac{(-1)^{q_1}}{2^{2q_1}}\frac{(2q_1)!}{(q_1!)^2}, \qquad C_{2p_2}^{n_1+1}(0) = (-1)^{p_2}\frac{(n_1+p_2)!}{n_1!\,p_2!}, \tag{57}$$

then multiplying (42) by $P_{n_1'}(x)\,dx$, integrating over the region $\cos\varphi' = x \in [-1,1]$, and taking into account the orthogonality and square norm of Legendre polynomials, $\int_{-1}^{1} dx\,[P_\ell(x)]^2 = 1/(\ell+\frac{1}{2})$, we obtain after integration

$$U_{2q_1,\ell_2}^{n_1,2p_2} = A_{q_1,\ell_2}^{n_1,p_2}\int_{-1}^{1} dx\,(1-x^2)^{q_1}\,C_{\ell_2}^{2q_1+1}(x)\,P_{n_1}(x), \tag{58}$$

with the coefficient

$$A_{q_1,\ell_2}^{n_1,p_2} = \frac{(-1)^{q_1-p_2}}{2^{n_1}}\frac{[(2q_1)!]^2\,p_2!}{(q_1!)^2\,(n_1+p_2)!} \times\sqrt{(2q_1+\tfrac{1}{2})(n_1+\tfrac{1}{2})\frac{\ell_2!\,(2n_1+2p_2+1)!}{(2p_2)!\,(4q_1+\ell_2+1)!}}. \tag{59}$$

To solve the integral (58), we use (A6) and (A7) from Appendix A, for

$$\begin{aligned} \ell_1 &= 2q_1, & n_1 &= 2p_1, \\ \ell_2 &= 2q_2, & n_2 &= 2p_2, \end{aligned} \quad \text{and} \quad 2q_1+2q_2 = n = 2p_1+2p_2,$$

to obtain the even–even coefficients (53) written in terms of ${}_4F_3$ hypergeometric polynomials as

$$\begin{aligned} U_{2q_1,2q_2}^{2p_1,2p_2} = {}& \frac{(-1)^{q_2}}{2^{2p_1}}\frac{p_2!}{p_1!}\frac{[(2q_1)!]^2}{(4q_1+1)!}\frac{\sqrt{(2p_1+\frac{1}{2})(2q_1+\frac{1}{2})}}{(2p_1+p_2)!} \\ & \times\frac{\Gamma(p_1+\frac{1}{2})}{(q_1-p_1)!\,\Gamma(q_1+p_1+\frac{3}{2})}\sqrt{\frac{(4p_1+2p_2+1)!\,(4q_1+2q_2+1)!}{(2p_2)!\,(2q_2)!}} \\ & \times{}_4F_3\left(\begin{matrix} -q_2, & 2q_1+q_2+1, & q_1+1, & q_1+1 \\ 2q_1+\frac{3}{2}, & q_1+p_1+\frac{3}{2}, & q_1-p_1+1 \end{matrix}\,\middle|\,1\right). \end{aligned} \tag{60}$$

On the other hand, for

$$\begin{aligned} \ell_1 &= 2q_1, & n_1 &= 2p_1+1, \\ \ell_2 &= 2q_2+1, & n_2 &= 2p_2, \end{aligned} \quad \text{and} \quad 2q_1+2q_2+1 = n = 2p_1+2p_2+1,$$

we use the same formulas from Appendix A to write (56) as

$$U_{2q_1,2q_2+1}^{2p_1+1,2p_2} = \frac{(-1)^{q_2}}{2^{2p_1+1}} \frac{p_2!}{p_1!} \frac{[(2q_1)!]^2}{(2p_1+p_2+1)!\,(4q_1+1)!} \sqrt{\frac{(2p_1+\frac{3}{2})(2q_1+\frac{1}{2})}{(2p_2)!\,(2q_2+1)!}}$$
$$\times \frac{\Gamma(p_1+\frac{3}{2})\sqrt{(4p_1+2p_2+3)!\,(4q_1+2q_2+2)!}}{\Gamma(q_1-p_1+1)\,\Gamma(q_1+p_1+\frac{5}{2})}$$
$$\times {}_4F_3\left(\begin{matrix} -q_2, & 2q_1+q_2+2, & q_1+1, & q_1+1 \\ 2q_1+\frac{3}{2}, & q_1+p_1+\frac{5}{2}, & q_1-p_1+1 \end{matrix}\middle|\, 1\right). \tag{61}$$

C. Interbasis coefficients $U_{\ell_1,\ell_2}^{n_1,n_2}$ for odd n_2

We now consider the interbasis expansion coefficients $U_{\ell_1,\ell_2}^{n_1,n_2}$ in (42) for odd $n_2 = 2p_2 + 1$. We divide both sides of this expansion by $\cos\vartheta'$ and again take the limit $\vartheta' \to \frac{1}{2}\pi - \varepsilon$ for $\varepsilon \to 0$. We require the following limit expressions for Legendre and Gegenbauer polynomials when the quantum numbers $\ell_1 = 2q_1 + 1$ and $n_2 = 2p_2 + 1$ are odd,

$$\left.\frac{P_{2q_1+1}(\sin\varphi'')}{\cos\vartheta'}\right|_{\vartheta'\to\frac{1}{2}\pi} = \frac{(-1)^{q_1}(\frac{3}{2})_{q_1}}{q_1!\,\sin\varphi'}, \tag{62}$$

$$\left.\frac{C_{2p_2+1}^{n_1+1}(\cos\vartheta')}{\cos\vartheta'}\right|_{\vartheta'\to\frac{1}{2}\pi} = 2(-1)^{p_2}\frac{(n_1+p_2+1)!}{n_1!\,p_2!}. \tag{63}$$

Using the orthogonality relation for Legendre polynomials and by analogy with the previous case of even indices, we obtain

$$U_{2q_1+1,\ell_2}^{n_1,2p_2+1} = B_{q_1,\ell_2}^{n_1,p_2} \int_{-1}^{1} \mathrm{d}x\,(1-x^2)^{q_1}\, C_{\ell_2}^{2q_1+2}(x)\, P_{n_1}(x), \tag{64}$$

with

$$B_{q_1,\ell_2}^{n_1,p_2} = \frac{(-1)^{q_1-p_2}}{2^{n_1}} \frac{p_2!}{(n_1+p_2+1)!} \sqrt{\frac{(2p_2+2n_1+2)!\,\ell_2!}{(\ell_2+4q_1+3)!\,(2p_2+1)!}}$$
$$\times \sqrt{(n_1+\tfrac{1}{2})(2q_1+\tfrac{3}{2})}\left(\frac{(2q_1+1)!}{q_1!}\right)^2. \tag{65}$$

As before, we must consider separately two cases: when ℓ_2, n_1 are both even or both odd. Using again formulas (A6) and (A7) from Appendix A, we obtain, for $\ell_2 = 2q_2$ and $n_1 = 2p_1$, thus $n = 2p_1 + 2p_2 + 1 = 2q_1 + 2q_2 + 1$,

$$U_{2q_1+1,2q_2}^{2p_1,2p_2+1} = \frac{(-1)^{q_2}}{2^{2p_1}} \frac{p_2!\,[(2q_1+1)!]^2}{p_1!\,(2p_1+p_2+1)!} \sqrt{\frac{(2p_1+\frac{1}{2})(2q_1+\frac{3}{2})}{(2p_2+1)!\,(2q_2)!}}$$
$$\times \frac{\Gamma(p_1+\frac{1}{2})}{\Gamma(4q_1+4)} \frac{\sqrt{(2p_1+n+1)!\,(2q_1+n+2)!}}{\Gamma(q_1+p_1+\frac{3}{2})\,\Gamma(q_1-p_1+1)}$$
$$\times {}_4F_3\left(\begin{matrix} -q_2, & q_2+2q_1+2, & q_1+1, & q_1+1 \\ 2q_1+\frac{5}{2}, & q_1+p_1+\frac{3}{2}, & q_1-p_1+1 \end{matrix}\middle|\, 1\right), \tag{66}$$

while for $\ell_2 = 2q_2 + 1$, $n_1 = 2p_1 + 1$, thus $n = 2p_1 + 2p_2 + 2 = 2q_1 + 2q_2 + 2$,

$$U^{2p_1+1,2p_2+1}_{2q_1+1,2q_2+1} = \frac{(-1)^{q_2}}{2^{2p_1+1}} \frac{p_2!\,[(2q_1+1)!]^2}{p_1!\,(2p_1+p_2+2)!} \sqrt{\frac{(2p_1+\frac{3}{2})(2q_1+\frac{3}{2})}{(2p_2+1)!\,(2q_2+1)!}}$$
$$\times \frac{\Gamma(p_1+\frac{3}{2})}{\Gamma(4q_1+4)} \frac{\sqrt{(2p_1+n+2)!\,(2q_1+n+2)!}}{\Gamma(q_1+p_1+\frac{5}{2})\,\Gamma(q_1-p_1+1)}$$
$$\times {}_4F_3\left(\begin{matrix} -q_2, & q_2+2q_1+3, & q_1+1, & q_1+1, \\ 2q_1+\frac{5}{2}, & q_1+p_1+\frac{5}{2}, & q_1-p_1+1 & \end{matrix}\middle| 1\right). \tag{67}$$

The four cases of nonzero II-III interbasis coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$ that appear in Eqs. (60) and (61) for even n_2, and in Eqs. (66) and (67) for odd n_2, have thus been given in terms of ${}_4F_3(\cdots|1)$ hypergeometric polynomials. We now proceed to express them in terms of Racah polynomials.

D. Coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$ as Racah polynomials

Recall that in the case of the I-II interbasis expansion in Sec. III, the coefficients $W^{n,m}_{n_1,n_2}$, depending on three effective parameters as is evident in their Clebsch–Gordan form (27), were written in terms of ${}_3F_2(\cdots|1)$'s and as Hahn polynomials, the latter two having five available parameters. Now we have written the four distinct nonzero sets of II-III interbasis coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$, which also depend on three effective parameters, in terms of ${}_4F_3(\cdots|1)$'s that in principle can provide *seven* available parameters. These have a special form though: first let us recall that when the parameters in a hypergeometric series

$${}_{k+1}F_k(a_1,a_2,\ldots,a_{k+1};b_1,\ldots,b_k;z)$$

are such that

$$a_1+a_2+\cdots+a_{k+1}+1 = b_1+b_2+\cdots+b_k, \tag{68}$$

the series is called *balanced* or *Saalschützian* (see Ref. 1, p. 188). It is not difficult to verify that ${}_4F_3$'s in (60), (61), (66), and (67) satisfy this condition and could enjoy six free parameters.

Saalschützian hypergeometric polynomials can be expressed in terms of Racah polynomials of degree n in the variable x, also with six effective parameters α, β, γ, δ, plus n and x, as

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) := {}_4F_3\left(\begin{matrix} -n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1 \\ \alpha+1,\ \beta+\delta+1,\ \gamma+1 \end{matrix}\middle| 1\right) \tag{69}$$

on the quadratic lattice $\lambda(x) := x(x+\gamma+\delta+1)$ of $x\in\{0,1,\ldots,N\}$. The range of degrees of the polynomials is $n\in\{0,1,2,\ldots,N\}$, where N is a nonnegative integer which can have one of the three values according to whether [Ref. 14, Eq. (9.2.1)]

$$\alpha+1=-N, \quad \text{or} \quad \beta+\delta=-N, \quad \text{or} \quad \gamma+1=-N. \tag{70}$$

In *each* of the ranges (70), a set of Racah polynomials is orthogonal over the points in the quadratic lattice $\lambda(x)$, with weight functions $\rho(x)$ and norms d_n, as was the case of the Hahn polynomials in (34). Their use will provide more compact formulas below. But first we must transform the ${}_4F_3(\cdots|1)$ hypergeometrics to their canonical form (69); this is done in Appendix B and results in the following forms for ${}_4F_3(\cdots|1)$'s expressible with Racah polynomials.

1. The coefficients $U^{2p_1,2p_2}_{2q_1,2q_2}$

Substituting now (B9) into (60), we obtain the interbasis coefficients for the even–even coefficients $U^{2p_1,2p_2}_{2q_1,2q_2}$ in (53), where the ${}_4F_3$ parameters can be readily compared with their "Racah form" in (69). Defining here the number $N := p_1 + p_2 = q_1 + q_2$ for all subsequent expressions, we write

$$U^{2p_1,2p_2}_{2q_1,2q_2} = (-1)^{p_1+q_2} 4^{q_1+p_1} \frac{(q_1+N)!\,(p_1+N)!}{q_2!\,p_2!} \times \sqrt{\frac{(4q_1+1)(4p_1+1)\,(2q_2)!\,(2p_2)!}{(2q_1+2N+1)!\,(2p_1+2N+1)!}} \times {}_4F_3\left(\begin{matrix}-p_1, & p_1+\frac{1}{2}, & q_1+\frac{1}{2}, & -q_1\\ & N+1, & 1, & -N\end{matrix}\middle| 1\right) \tag{71}$$

$$= (-1)^{p_1+q_2} \frac{\sqrt{\rho(p_1)}}{d_{q_1}} R_{q_1}\big(\lambda(p_1);\alpha,\beta,\gamma,\delta\big) \tag{72}$$

$$= (-1)^{p_1+q_2} \frac{\sqrt{\rho(q_1)}}{d_{p_1}} R_{p_1}\big(\lambda(q_1);\alpha,\beta,\gamma,\delta\big). \tag{73}$$

The identification with the parameters in (69) has $N = \frac{1}{2}n$ (because here $n = n_1 + n_2 = 2p_1 + 2p_2 = \ell_1 + \ell_2 = 2q_1 + 2q_2$), the quadratic lattice $\lambda(x) = x(x+\frac{1}{2})$, the parameters $\alpha = N$, $\beta = -\delta = -(N+\frac{1}{2})$, $\gamma = -(N+1)$, and the weight and norm factors

$$\rho(p_1) := 4^{2p_1} \frac{(2N+1)(4p_1+1)\,(2p_2)!\,[(p_1+N)!]^2}{(2p_1+2N+1)!\,(p_2!)^2},$$

$$d_{q_1} := \frac{q_2!}{4^{q_1}(q_1+N)!} \sqrt{\frac{(2N+1)(2k_1+2N+1)!}{(4q_1+1)(2q_2)!}}.$$

The ${}_4F_3$ hypergeometric in (71) represents a particular family of self-dual Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ because its parameters are interconnected by $\alpha + \beta = \gamma + \delta$, which means that this ${}_4F_3$ can be expressed by two equivalent Racah polynomials in the discrete variable,

$$R_{q_1}\big(\lambda(p_1);\alpha,\beta,\gamma,\delta\big) = R_{p_1}\big(\lambda(q_1);\alpha,\beta,\gamma,\delta\big), \tag{74}$$

over the same quadratic lattice $\lambda(x) = x(x+\frac{1}{2})$ and the same parameters α, β, γ, and δ.

2. *The coefficients* $U^{2p_1+1,2p_2}_{2q_1,2q_2+1}$

For the coefficients $U^{2p_1+1,2p_2}_{2q_1,2q_2+1}$ in (55), the ${}_4F_3$ hypergeometric in (61) can be again transformed as in Appendix B to the canonical form (69), which simplifies to

$$U^{2p_1+1,2p_2}_{2q_1,2q_2+1} = (-1)^{p_1+q_2} 2^{2q_1+2p_1+1} \frac{(q_1+N+1)!\,(p_1+N+1)!}{(N+1)\,q_2!\,p_2!} \times \sqrt{\frac{(4q_1+1)\,(4p_1+3)\,(2q_2+1)!\,(2p_2)!}{(2q_1+2N+2)!\,(2p_1+2N+3)!}} \times {}_4F_3\left(\begin{matrix}-p_1, & p_1+\frac{3}{2}, & q_1+\frac{1}{2}, & -q_1\\ & N+2, & 1, & -N\end{matrix}\middle| 1\right) \tag{75}$$

$$= (-1)^{p_1+q_2} \frac{\sqrt{\rho_1(q_1)}}{d^{(1)}_{p_1}} R_{p_1}\big(\lambda(q_1);\alpha,\beta,\gamma,\delta\big) \tag{76}$$

$$= (-1)^{p_1+q_2} \frac{\sqrt{\rho_2(p_1)}}{d^{(2)}_{q_1}} R_{q_1}\big(\mu(p_1);\alpha,\beta-1,\gamma,\delta+1\big). \tag{77}$$

Here $N := p_1 + p_2 = q_1 + q_2 = \frac{1}{2}(n-1)$, the parameters are $\alpha = -\gamma = N+1$ and $\beta = -\delta = -(N+\frac{1}{2})$; in the two last expressions, the quadratic lattices are $\lambda(x) = x(x+\frac{1}{2})$ and $\mu(x) = x(x+\frac{3}{2})$, and the weight and norm factors are

$$\rho_1(q_1) = 2^{4q_1+1}\frac{(4q_1+1)(2q_2+1)![(q_1+N+1)!]^2}{(N+1)(2q_1+2N+2)!(q_2!)^2},$$

$$d^{(1)}_{p_1} = \frac{p_2!}{4^{p_2}(p_1+N+1)!}\sqrt{\frac{(N+1)(2p_1+2N+3)!}{2(4p_1+3)(2p_2)!}},$$

$$\rho_2(p_1) = 2^{4p_1+1}\frac{(2N+1)(2N+3)(4p_1+3)(2p_2)![(p_1+N+1)!]^2}{3(N+1)(2p_1+2N+3)!(p_2!)^2},$$

$$d^{(2)}_{q_1} = \frac{q_2!}{4^{q_1}(q_1+N+1)!}\sqrt{\frac{(N+1)(2N+1)(2N+3)(2k_1+2N+2)!}{6(4q_1+1)(2q_2+1)!}}.$$

3. The coefficients $U^{2p_1,2p_2+1}_{2q_1+1,2q_2}$

Regarding the interbasis coefficients for odd n_2 in (66) and performing the ${}_4F_3$ transformations of Appendix B, we obtain

$$U^{2p_1,2p_2+1}_{2q_1+1,2q_2} = (-1)^{q_2+p_1}2^{2q_1+2p_1+1}\frac{(N+q_1+1)!\,(N+p_1+1)!}{(N+1)\,q_2!\,p_2!}$$
$$\times\sqrt{\frac{(4p_1+1)(4q_1+3)\,(2p_2+1)!\,(2q_2)!}{(2N+2p_1+2)!\,(2N+2q_1+3)!}}\qquad(78)$$
$$\times\,{}_4F_3\left(\begin{matrix}-p_1, & p_1+\frac{1}{2}, & q_1+\frac{3}{2}, & -q_1\\ & N+2, & 1, & -N\end{matrix}\,\middle|\,1\right)$$
$$= (-1)^{p_1+k_2}\frac{\sqrt{\rho_1(p_1)}}{d^{(1)}_{q_1}}R_{q_1}\left(\lambda(p_1);\alpha,\beta,\gamma,\delta\right)\qquad(79)$$
$$= (-1)^{p_1+q_2}\frac{\sqrt{\rho_2(q_1)}}{d^{(2)}_{p_1}}R_{p_1}\left(\mu(q_1);\alpha,\beta-1,\gamma,\delta+1\right).\qquad(80)$$

Here again $N := p_1 + p_2 = q_1 + q_2 = \frac{1}{2}(n-1)$, but the parameters are now $\alpha = -\gamma = N+1$ and $\beta = -\delta = -(N+\frac{1}{2})$, the quadratic lattices are $\lambda(x) = x(x+\frac{1}{2})$ and $\mu(x) = x(x+\frac{3}{2})$, and the weight and norm factors are

$$\rho_1(p_1) = 2^{4p_1+1}\frac{(4p_1+1)\,(2p_2+1)!\,[(p_1+N+1)!]^2}{(N+1)\,(2p_1+2N+2)!\,(p_2!)^2},$$

$$d^{(1)}_{q_1} = \frac{q_2!}{4^{q_1}\,(q_1+N+1)!}\sqrt{\frac{(N+1)\,(2q_1+2N+3)!}{2(4q_1+3)\,(2q_2)!}},$$

$$\rho_2(q_1) = 2^{4q_1+1}\frac{(2N+1)(2N+3)(4q_1+3)\,(2q_2)!\,[(q_1+N+1)!]^2}{3(N+1)\,(2q_1+2N+3)!\,(q_2!)^2},$$

$$d^{(2)}_{p_1} = \frac{p_2!}{4^{p_1}\,(p_1+N+1)!}\sqrt{\frac{(N+1)(2N+1)(2N+3)\,(2p_1+2N+2)!}{6(4p_1+1)\,(2p_2+1)!}}.$$

4. The coefficients $U^{2p_1+1,2p_2+1}_{2q_1+1,2q_2+1}$

Finally, the odd–odd interbasis coefficients in (67) can be brought in terms of Racah polynomials as

$$U^{2p_1+1,2p_2+1}_{2q_1+1,2q_2+1} = (-1)^{q_2+p_1} 2^{2q_1+2p_1+2} \frac{(N+q_1+2)!\,(N+p_1+2)!}{(N+1)(N+2)\,q_2!\,p_2!} \times \sqrt{\frac{(4p_1+3)(4q_1+3)\,(2p_2+1)!\,(2q_2+1)!}{(2N+2p_1+4)!\,(2N+2q_1+4)!}} \times {}_4F_3\left(\begin{matrix}-p_1, & p_1+\frac{3}{2}, & q_1+\frac{3}{2}, & -q_1\\ & N+3, & 1, & -N\end{matrix}\,\middle|\,1\right) \tag{81}$$

$$= (-1)^{p_1+q_2} \frac{\sqrt{\rho(p_1)}}{d_{q_1}} R_{q_1}\left(\mu(p_1);\alpha,\beta,\gamma,\delta\right) \tag{82}$$

$$= (-1)^{p_1+q_2} \frac{\sqrt{\rho(q_1)}}{d_{p_1}} R_{p_1}\left(\mu(q_1);\alpha,\beta,\gamma,\delta\right). \tag{83}$$

Here $N := p_1+p_2 = q_1+q_2 = \frac{1}{2}n-1$, the parameters are $\alpha = 1-\gamma = N+2$ and $\beta = -\delta = -(N+\frac{3}{2})$, the lattice is $\mu(x) = x(x+\frac{3}{2})$, and the weight and norm factors are

$$\rho(q_1) = 4^{2q_1+1} \frac{(2N+3)(4q_1+3)\,(2q_2+1)!\,[(q_1+N+2)!]^2}{3(N+1)(N+2)\,(2q_1+2N+4)!\,(q_2!)^2},$$

$$d_{p_1} = \frac{p_2!}{2^{2p_1+1}\,(p_1+N+2)!} \sqrt{\frac{(N+1)(N+2)(2N+3)\,(2p_1+2N+4)!}{3(4p_1+3)\,(2p_2+1)!}}.$$

Our final remark is to point out that the expressions for all II-III interbasis coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$ are equivalent to the summation formula (46) of the product of two special Clebsch–Gordan coefficients.

E. Relation with the Wigner 6*j* symbols

The coefficients that bridge two distinct coupling orders between three spins to the same total spin are known as Wigner 6*j* symbols. They contain six spin parameters: ℓ_1, ℓ_2, ℓ_3, their couplings to ℓ_{12}, ℓ_{23}, and the total ℓ, and they have a host of symmetry relations that can be seen in the literature.[24] These 6*j* coefficients can be expressed in terms of balanced ${}_4F_3(\cdots\,|\,1)$ functions (see for example Refs. 23 and 24), so they can be also written in terms of Racah polynomials through (69), having the same number of parameters, times a lengthy factor containing factorials and Kronecker triangle functions (1 when they couple properly and 0 if not).

In Subsection III B, we wrote the I-II interbasis coefficients $W^{n,m}_{n_1,n_2}$, containing three effective labels, in terms of *proper* Clebsch–Gordan coefficients, i.e., whose three spin indices and their projections are an integer or half-integer, form a triangle and vanish when not. On the other hand, their expression as Hahn polynomial functions in (33) allows for analytic continuation in all parameters.

Here we find that it is *not* always the case that the II-III interbasis coefficients $U^{n_1,n_2}_{\ell_1,\ell_2}$ correspond to *proper* 6*j* coefficients. Yet their relation is sufficiently close to merit attention. We thus proceed to examine the known equivalence between Wigner 6*j* symbols and balanced ${}_4F_3(\cdots\,|\,1)$ hypergeometric functions, which is (Ref. 24, Sec. 8.4.4)

$$\begin{Bmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_3 & \ell & \ell_{23}\end{Bmatrix} = c_0\, {}_4F_3\left(\begin{matrix}\ell_1-\ell_2-\ell_{12}, & \ell_3-\ell_2-\ell_{23}, & -\ell_1-\ell_2-\ell_{12}-1, & -\ell_2-\ell_3-\ell_{23}-1\\ & -2\ell_2, & \ell-\ell_2-\ell_{12}-\ell_{23}, & -\ell_2-\ell_{12}-\ell-\ell_{23}-1\end{matrix}\,\middle|\,1\right). \tag{84}$$

We shall consider only the case of even–even coefficients $U^{2p_1,2p_2}_{2q_1,2q_2}$ in (60), which is sufficiently illustrative for our purpose, concentrate on the ${}_4F_3$ functions, and avoid long and distracting prefactors with the notation c_i for those that are not essential to our present endeavour. Comparing the components of ${}_4F_3$ in (60) with those in (84),

$$\begin{aligned} \ell_1-\ell_2-\ell_{12} &= -q_2, & \ell_3-\ell_2-\ell_{23} &= 2q_1+q_2+1, \\ -\ell_1-\ell_2-\ell_{12}-1 &= q_1+1, & -\ell_2-\ell_3-\ell_{23}-1 &= q_1+1, \\ -2\ell_2 &= q_1-p_1+1, & \ell-\ell_2-\ell_{12}-\ell_{23} &= 2q_1+\tfrac{3}{2}, \\ & & -\ell_2-\ell_{12}-\ell-\ell_{23}-1 &= q_1+p_1+\tfrac{3}{2}, \end{aligned} \tag{85}$$

and solving this system of 6 simultaneous equations, we obtain

$$U^{2p_1,2p_2}_{2q_1,2q_2} = c_1 \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}, \tag{86}$$

with

$$\begin{aligned} &\ell_1 = -\tfrac{1}{2}N-1,\ \ell_2 = \tfrac{1}{2}(p_1-q_1-1),\ \ell_{12} = \tfrac{1}{2}(q_2-p_1-1), \\ &\ell_3 = \tfrac{1}{2}(N-1),\ \ \ell = \tfrac{1}{2}(q_1-p_1-1),\ \ \ell_{23} = -1-q_1-\tfrac{1}{2}(p_1+q_2), \end{aligned}$$

where $N := q_1+q_2 = p_1+p_2 = \frac{1}{2}n$ is a nonnegative integer in every case, so all parameters of the $6j$ symbol in (86) are integers or half-integers.

The fact that some of these parameters appear with negative signs is not a problem because with the help of "mirror" transformations (Ref. 23, Sec. 9.4) one can invert some $\ell_i \to -\ell_i-1$, with a sign on the $6j$ coefficient, and use it, for example, in $\ell_1 = -\frac{1}{2}N-1 \to \frac{1}{2}N$. What cannot be ascertained is that all triangle relations (e.g., $|\ell_1-\ell_2| \le \ell_{12} \le \ell_1+\ell_2$, etc.) are fulfilled. On the other hand, the II-III interbasis coefficients are well defined with Racah polynomials, regardless of these relations.

V. CONCLUSION

The Zernike system (1) in its classical and quantum realizations[19,20] is superintegrable and harbours several remarkable geometric and spectral properties. In this paper, we highlighted its relevance to special function theory.

In three sets of coordinates on the unit disk, the solutions to Eq. (1) involve Legendre, Gegenbauer, and Jacobi polynomials (and phases), as illustrated in Fig. 1, each characterized by two quantum numbers. Between them, the I-II and I-III relations are given by Hahn polynomials and II-III by Racah polynomials in three discrete parameters. All relations can be expressed also with Clebsch–Gordan coefficients, whose geometric interpretation still eludes us, while the role of $6j$ coefficients has only been suggested.

Finally, we underline the fact that the original Zernike polynomials have a great practical importance in phase-contrast microscopy and in the correction of wavefronts in circular pupils. Recent work[4–6,18] has extended this technique to pupils of essentially arbitrary shape through diffeomorphisms that conserve their basic properties. This has been applied to describe wavefronts in sectorial, annular, and polygonal-shaped pupils, the latter specifically tailored to the hexagonal components of large astronomical mirrors. As remarked in Ref. 21, the fact that among the members of each horizontal-n "multiplet" $\Psi^{\rm II}_{0,n}$ in (14) and $\Psi^{\rm III}_{0,n}$ in (16) are plane wave-like solutions can be of some relevance for applications in correcting cylindrical aberrations.

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APPENDIX A: SOLUTIONS TO INTEGRALS (58) AND (64)

To find the expressions (66) and (67), consider the integral

$$J_{n,m}^{\mu,\lambda} := \int_{-1}^{1} \mathrm{d}x\,(1-x^2)^{\mu}\, C_n^{\lambda}(x) P_m(x). \tag{A1}$$

Because of the oddness of the integrand, this integral is nonzero only when n and m are both even or both odd. We use the following expressions for the Gegenbauer and Legendre polynomials:

$$C_n^{\lambda}(x) = \begin{cases} \dfrac{(2\lambda)_{2q}}{(2q)!}\, {}_2F_1\!\left(\begin{matrix} -q, & q+\lambda \\ & \lambda+\frac{1}{2} \end{matrix}\middle|\, 1-x^2\right), & n=2q, \\ \dfrac{(2\lambda)_{2q+1}}{(2q+1)!}\, x\, {}_2F_1\!\left(\begin{matrix} -q, & q+\lambda+1 \\ & \lambda+\frac{1}{2} \end{matrix}\middle|\, 1-x^2\right), & n=2q+1, \end{cases} \tag{A2}$$

$$P_m(x) = \begin{cases} (-1)^p \dfrac{(\frac{1}{2})_p}{p!}\, {}_2F_1\!\left(\begin{matrix} -p, & p+\frac{1}{2} \\ & \frac{1}{2} \end{matrix}\middle|\, x^2\right), & m=2p, \\ (-1)^p \dfrac{(\frac{3}{2})_p}{p!}\, x\, {}_2F_1\!\left(\begin{matrix} -p, & p+\frac{3}{2} \\ & \frac{3}{2} \end{matrix}\middle|\, x^2\right), & m=2p+1, \end{cases} \tag{A3}$$

where $(x)_n := \Gamma(x+n)/\Gamma(x)$. Then, using

$$\int_0^1 \mathrm{d}y\, y^{\gamma-1}(1-y)^{\rho-1}\, {}_2F_1\!\left(\begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix}\middle|\, y\right) = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}, \tag{A4}$$

with $\mathrm{Re}\,\gamma > 0$, $\mathrm{Re}\,\rho > 0$, and $\mathrm{Re}\,(\gamma+\rho-\alpha-\beta) > 0$, we must consider separately the two parity cases:
A.1. When $n = 2q$ and $m = 2p$ are even, we rewrite the integral (A1) in the form

$$\begin{aligned} J_{2q,2p}^{\mu,\lambda} &= \int_{-1}^{1} \mathrm{d}x\,(1-x^2)^{\mu}\, C_{2q}^{\lambda}(x) P_{2p}(x) = (-1)^p \frac{\Gamma(2\lambda+2q)}{(2q)!\,\Gamma(2\lambda)} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\,p!} \\ &\times \sum_{s=0}^{q} \frac{(-q)_s\,(q+\lambda)_s}{(\lambda+\frac{1}{2})_s\, s!} \int_{-1}^{1} \mathrm{d}x\,(1-x^2)^{\mu+s}\, {}_2F_1\!\left(\begin{matrix} -p, & p+\frac{1}{2} \\ & \frac{1}{2} \end{matrix}\middle|\, x^2\right). \end{aligned} \tag{A5}$$

Substituting here $x^2 = y$ and using (A4) with $\alpha = -p$, $\beta = p + \frac{1}{2}$, $\gamma = \frac{1}{2}$, and $\rho = s + 1 + \mu$, we obtain

$$\begin{aligned} J_{2q,2p}^{\mu,\lambda} &= (-1)^p \frac{\Gamma(2\lambda+2q)\,\Gamma(p+\frac{1}{2})}{\Gamma(2\lambda)\,(2q)!\,p!} \frac{\Gamma(\mu+1)^2}{\Gamma(\mu+p+\frac{3}{2})\,\Gamma(\mu-p+1)} \\ &\times {}_4F_3\!\left(\begin{matrix} -q, & q+\lambda, & \mu+1, & \mu+1 \\ \lambda+\frac{1}{2}, & \mu+p+\frac{3}{2}, & \mu-p+1 & \end{matrix}\middle|\, 1\right). \end{aligned} \tag{A6}$$

A.2. When $n = 2q + 1$ and $m = 2p + 1$ are odd, we have the integral (A1) with $\alpha = -p$, $\beta = p + 3/2$, $\gamma = \frac{3}{2}$, and $\rho = \mu + s + 1$,

$$\begin{aligned} J_{2q+1,2p+1}^{\mu,\lambda} &= (-1)^p \frac{\Gamma(2\lambda+2q+1)\,\Gamma(p+\frac{3}{2})}{\Gamma(2\lambda)\,(2q+1)!\,p!} \frac{\Gamma(\mu+1)^2}{\Gamma(\mu+p+\frac{5}{2})\,\Gamma(\mu-p+1)} \\ &\times {}_4F_3\!\left(\begin{matrix} -q, & q+\lambda+1, & \mu+1, & \mu+1 \\ \lambda+\frac{1}{2}, & \mu+p+\frac{5}{2}, & \mu-p+1 & \end{matrix}\middle|\, 1\right). \end{aligned} \tag{A7}$$

APPENDIX B: SAALSCHÜTZIAN HYPERGEOMETRIC AND RACAH POLYNOMIALS

To transform the ${}_4F_3(\cdots|1)$ Saalschützian hypergeometric polynomials (60), (61), (66), and (67) into the canonical form for the Racah polynomials in (69), we use the symmetry properties of terminating hypergeometric series of the general form ${}_4F_3(-n, x, y, z; u,v,w; 1)$ that preserve their Saalschützian character.

Two such transformation formulas come to hand: the first is known in the literature as Whipple's formula for terminating balanced ${}_4F_3$ series [see Ref. 14, Eq. (1.7.6)], namely,

$$\begin{aligned} {}_4F_3\left(\begin{matrix} -n, & x, & y, & z \\ u, & v, & w & \end{matrix}\middle| 1\right) &= \frac{(v-z)_n\,(u-z)_n}{(v)_n\,(u)_n} \\ &\times {}_4F_3\left(\begin{matrix} -n, & w-x, & w-y, & z \\ 1-u+z-n, & 1-v+z-n, & w & \end{matrix}\middle| 1\right), \end{aligned} \tag{B1}$$

where $(x)_n := \Gamma(x+n)/\Gamma(x)$. The second transformation formula that we use is

$$\begin{aligned} {}_4F_3\left(\begin{matrix} -n, & x, & y, & z \\ u, & v, & w & \end{matrix}\middle| 1\right) &= (-1)^n \frac{(x)_n\,(y)_n\,(z)_n}{(u)_n\,(v)_n\,(w)_n} \\ &\times {}_4F_3\left(\begin{matrix} -n, & 1-u-n, & 1-v-n, & 1-w-n \\ & 1-x-n, & 1-y-n, & 1-z-n \end{matrix}\middle| 1\right), \end{aligned} \tag{B2}$$

which can be readily derived by reversing the order of summation in the definition of the series.

We can now write the expressions for the interbasis coefficients $U^{2p_1,2p_2}_{2q_1,2q_2}$ in (60) and $U^{2p_1+1,2p_2}_{2q_1,2q_2+1}$ in (61) in terms of Racah polynomials (69) by using three successive transformations, where the first two are (B1) and (B2). We start with the ${}_4F_3$ function in (60) and use (B1) with the parameters

$$\begin{aligned} &n = q_2, \quad x = q_1 + 1, \quad y = q_1 + N + 1, \quad z = q_1 + 1, \\ &u = 2q_1 + \tfrac{3}{2}, \quad v = q_1 + p_1 + \tfrac{3}{2}, \quad w = q_1 - p_1 + 1, \end{aligned} \tag{B3}$$

where $N = \frac{1}{2}n = q_1 + q_2 = p_1 + p_2$. This yields the relation

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} -q_2, & q_1+N+1, & q_1+1, & q_1+1 \\ 2q_1+\frac{3}{2}, & q_1+p_1+\frac{3}{2}, & q_1-p_1+1 & \end{matrix}\middle| 1\right) \\ &= \frac{(p_1+\frac{1}{2})_{q_2}\,(q_1+\frac{1}{2})_{q_2}}{(q_1+p_1+\frac{3}{2})_{q_2}\,(2q_1+\frac{3}{2})_{q_2}} \\ &\quad\times {}_4F_3\left(\begin{matrix} -q_2, & -p_1, & -(p_1+N), & q_1+1 \\ \frac{1}{2}-N, & \frac{1}{2}-p_1-q_2, & q_1-p_1+1 & \end{matrix}\middle| 1\right). \end{aligned} \tag{B4}$$

The second step is to apply the transformation (B2) with the parameters

$$\begin{aligned} &n = p_1, \quad x = -(p_1+N), \quad y = q_1 + 1, \quad z = -q_2, \\ &u = \tfrac{1}{2} - p_1 - q_2, \quad v = \tfrac{1}{2} - N, \quad w = q_1 - p_1 + 1, \end{aligned} \tag{B5}$$

to find

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} -q_2, & -p_1, & -(p_1+N), & q_1+1 \\ \frac{1}{2}-N, & \frac{1}{2}-p_1-q_2, & q_1-p_1+1 & \end{matrix}\middle| 1\right) \\ &= (-1)^{p_1} \frac{(-p_1-N)_{p_1}\,(q_1+1)_{p_1}\,(-q_2)_{p_1}}{(\frac{1}{2}-p_1-q_2)_{p_1}\,(\frac{1}{2}-N)_{p_1}\,(q_1-p_1+1)_{p_1}} \\ &\quad\times {}_4F_3\left(\begin{matrix} -p_1, & k_2+\frac{1}{2}, & p_2+\frac{1}{2}, & -k_1 \\ N+1, & -(p_1+q_1), & q_2-p_1+1 & \end{matrix}\middle| 1\right). \end{aligned} \tag{B6}$$

The third step applies to ${}_4F_3$ in (B6) the same transformation (B1) but with the parameters

$$\begin{aligned} &n = p_1, \quad x = q_2 + \tfrac{1}{2}, \quad y = p_2 + \tfrac{1}{2}, \quad z = -q_1, \\ &u = -(q_1+p_1), \quad v = q_2 - p_1 + 1, \quad w = N + 1. \end{aligned} \tag{B7}$$

This leads us to the form

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix} -p_1, & q_2+\frac{1}{2}, & p_2+\frac{1}{2}, & -q_1 \\ N+1, & -(p_1+q_1), & q_2-p_1+1 \end{matrix}\middle| 1\right)
&= \frac{(p_2+1)_{p_1}\,(-p_1)_{p_1}}{(q_2-p_1+1)_{p_1}\,(-q_1-p_1)_{p_1}} \\
&\quad\times {}_4F_3\left(\begin{matrix} -p_1, & p_1+\frac{1}{2}, & q_1+\frac{1}{2}, & -q_1 \\ & N+1, & 1, & -N \end{matrix}\middle| 1\right).
\end{aligned}
\tag{B8}
$$

As a result of these three successive transformations, we finally arrive at the canonical form of the Racah polynomials (69) in terms of the hypergeometric series ${}_4F_3$ in (60),

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix} -q_2, & q_1+N+1, & q_1+1, & q_1+1 \\ 2q_1+\frac{3}{2}, & q_1+p_1+\frac{3}{2}, & q_1-p_1+1 \end{matrix}\middle| 1\right)
&= (-1)^{p_1}\,2^{2q_1+4p_1+1}\,\frac{\Gamma(q_1+p_1+\frac{3}{2})\,\Gamma(q_1-p_1+1)}{\Gamma(p_1+\frac{1}{2})} \\
&\quad\times \frac{p_1!\,(2q_2)!\,(2p_2)!\,(q_1+N)!\,[(p_1+N)!]^2(4q_1+1)!}{q_2!\,[(p_2)!]^2\,[(2q_1)!]^2\,(2q_1+2N+1)!\,(2p_1+2N+1)!} \\
&\quad\times {}_4F_3\left(\begin{matrix} -p_1, & p_1+\frac{1}{2}, & q_1+\frac{1}{2}, & -q_1 \\ & N+1, & 1, & -N \end{matrix}\middle| 1\right).
\end{aligned}
\tag{B9}
$$

The parameters of ${}_4F_3$ hypergeometric functions for the interbasis coefficients $U^{2p_1,2p_2+1}_{2q_1+1,2q_2}$ in (66) and $U^{2p_1+1,2p_2+1}_{2q_1+1,2q_2+1}$ in (67) also enjoy the property (68) and can be transformed into the canonical form for Racah polynomials (69) by using the three steps (B4)–(B8) *mutatis mutandis*. This results in

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix} -q_2, & q_1+N+2, & q_1+1, & q_1+1 \\ 2q_1+\frac{5}{2}, & q_1+p_1+\frac{3}{2}, & q_1-p_1+1 \end{matrix}\middle| 1\right)
&= (-1)^{p_1}\,\frac{4^{q_1+2p_1+1}\,p_1!\,(2q_2)!\,(2p_2+1)!\,(q_1+N+1)!}{(N+1)\,q_2!\,[(p_2)!]^2\,[(2q_1+1)!]^2} \\
&\quad\times \frac{[(p_1+N+1)!]^2(4q_1+3)!\,\Gamma(q_1+p_1+\frac{3}{2})\,\Gamma(q_1-p_1+1)}{(2q_1+2N+3)!\,(2p_1+2N+2)!\,\Gamma(p_1+\frac{1}{2})} \\
&\quad\times {}_4F_3\left(\begin{matrix} -p_1, & p_1+\frac{1}{2}, & q_1+\frac{3}{2}, & -q_1 \\ & N+2, & 1, & -N \end{matrix}\middle| 1\right),
\end{aligned}
\tag{B10}
$$

where $N=q_1+q_2=p_1+p_2=\frac{1}{2}(n-1)$, and

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix} -q_2, & q_1+N+3, & q_1+1, & q_1+1 \\ 2q_1+\frac{5}{2}, & q_1+p_1+\frac{5}{2}, & q_1-p_1+1 \end{matrix}\middle| 1\right)
&= (-1)^{p_1}\,\frac{p_1!\,(2q_2+1)!\,(2p_2+1)!\,(q_1+N+1)!\,(p_1+N+1)!}{(N+1)(N+2)\,q_2!\,[(p_2)!]^2\,[(2q_1+1)!]^2} \\
&\quad\times \frac{4^{q_1+2p_1+1}\,(p_1+N+2)!\,(4q_1+3)!\Gamma(q_1+p_1+\frac{5}{2})\,\Gamma(q_1-p_1+1)}{(2q_1+2N+3)!\,(2p_1+2N+3)!\,\Gamma(p_1+\frac{3}{2})} \\
&\quad\times {}_4F_3\left(\begin{matrix} -p_1, & p_1+\frac{3}{2}, & q_1+\frac{3}{2}, & -q_1 \\ & N+3, & 1, & -N \end{matrix}\middle| 1\right),
\end{aligned}
\tag{B11}
$$

where $N=q_1+q_2=p_1+p_2=\frac{1}{2}n-1$.

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