Unitary rotation and gyration of pixelated images on rectangular screens

ALEJANDRO R. URZÚA¹ AND KURT BERNARDO WOLF²,*

¹Posgrado en Ciencias Físicas, Universidad Nacional Autónoma de México, Av. Universidad s/n, Cuernavaca, Morelos 62210, Mexico
²Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Av. Universidad s/n, Cuernavaca, Morelos 62251, Mexico
*Corresponding author: bwolf@fis.unam.mx

Received 7 December 2015; revised 10 February 2016; accepted 19 February 2016; posted 19 February 2016 (Doc. ID 255335); published 21 March 2016

In two space dimensions of screens in optical systems, rotations, gyrations, and fractional Fourier transformations form the Fourier subgroup of the symplectic group of linear canonical transformations: $U(2)_F \subset \text{Sp}(4, \mathbb{R})$. Here, we study the action of this Fourier group on pixelated images within generic rectangular $N_x \times N_y$ screens; its elements here compose properly and act unitarily, i.e., without loss of information.

© 2016 Optical Society of America

OCIS codes: (070.2025) Discrete optical signal processing; (080.2720) Mathematical methods (general); (350.6980) Transforms.

http://dx.doi.org/10.1364/JOSAA.33.000642

1. INTRODUCTION

Paraxial geometric, wave, and finite optical models with two-dimensional plane screens are covariant with the Fourier group $U(2)_F$. This group consists of joint $\text{SO}(2)$ phase space rotations between the coordinates $q = (q_x, q_y)$ and between their canonically conjugate momenta $p = (p_x, p_y)$; also, it contains joint $\text{SO}(2)$ gyrations in the $(q_x, p_y)$ and $(q_y, p_x)$ planes; and finally, of $\text{SO}(2)_x \otimes \text{SO}(2)_y$, fractional Fourier transformations that independently rotate the $(q_x, p_y)$ and $(q_y, p_x)$ planes, and all their compositions. In the geometric model, the group $U(2)_F$ is represented by $4 \times 4$ matrices that are both orthogonal and symplectic [1]; in the wave model of images on the screen, $f(q_x, q_y), \; q \in \mathbb{R}^2$, these are subject to integral linear canonical transforms [2,3] that represent this same group. In particular, gyrations rotate smoothly between Hermite–Gauss and Laguerre–Gauss fundamental beams [1]. In the finite model of optics, where images are matrices of values $f(q_x, q_y)$, the coordinates $q_x, q_y$ are integers that count the $N_x \times N_y$ pixels in a rectangular screen, so the Fourier group will be represented by square $N_x \times N_y$ matrices that are unitary. Of course, being elements of a group, these $U(2)_F$ transformations can be concatenated and inverted using their simplest $4 \times 4$ representation.

Previously, we have considered the action of the $U(2)_F$ Fourier group on finite systems, where the screens were $N_x \times N_y$ squares [4–6]. The extension to rectangular screens, where $N_x \neq N_y$, is not trivial because rotations and gyrations to “angular momentum” Laguerre-type modes require an extended form of symmetry importation [7,8].

In Sections 2 and 3, we recall the foundations of the finite model of pixelated optics and the definition of the Fourier group in the paraxial geometric model. The fractional Fourier transforms have their corresponding matrix Fourier–Kravchuk transform [9] within the finite model, i.e., they are domestic to it. Rotations and gyrations, however, require importation from the geometric model; this is done in Section 4, where we provide computed examples of these transformations and show the finite rectangular analogs of the Laguerre–Gauss modes of wave optics [10]. In Section 5, we offer some concluding remarks on applications to image processing.

2. CONTINUOUS AND FINITE OSCILLATOR SYSTEMS

The linear finite oscillator system arises as the algebra and group deformation of the well-known quantum harmonic oscillator, upon which the continuous position and momentum coordinates become discrete and finite.

Let $Q, \; P$ and $\hat{H} = \frac{1}{2}(P^2 + Q^2 - 1)$ be the Poisson bracket or the Schrödinger operators of position, momentum, and mode number (do not confuse with the Hamiltonian, which is $\hat{H} = \frac{1}{2} (\hat{Q} + \frac{1}{2})$), indicating by the overbar that they refer to the continuous model. On the other hand, consider the three components of quantum angular momentum, designated by the letters $\hat{Q} \equiv \hat{J}_1, \; \hat{P} \equiv \hat{J}_2$, and $K \equiv \hat{J}_3$, and compare their well-known commutation relations that characterize the oscillator and spin algebras:
The first two commutators in each line are the algebraic form of the geometric and dynamical Hamilton equations for the harmonic oscillator in phase space, under evolution by $\mathcal{H}$ and $K$, respectively. The last two commutators, however, differ and distinguish between the continuous and the finite models. In their unitary irreducible representations, the OSC spectrum of $\mathcal{Q}$ and $\mathcal{P}$ is continuous and fills the real line $\mathbb{R}$ while that of $\mathcal{H}$ is the equally spaced set $\eta n$ ($\eta$ fixed) with $n\in \{0, 1, \ldots, \infty\}$; the spectrum of the three $\mathfrak{su}(2)$ generators on the other hand, in the representation $j$ (positive integer or half-integer determined by the eigenvalue $j(j+1)$ of the Casimir invariant $j\cdot j$), is the unit-spaced set $m\mathbb{N}$ for the lowest $j$. This leads us to understand $K+j1$ as the mode number operator of a discrete oscillator system that has $2j+1$ modes $n^{\mathbb{N}}$. The oscillator Lie algebra $\text{OSc}$ of generators $1$, $\mathcal{Q}$, $\mathcal{P}$, $\mathcal{H}$ is the contraction of the algebra $\mathfrak{u}(2) = \mathfrak{su}(1) \oplus \mathfrak{su}(2)$ with generators $1$, $\mathcal{Q}$, $\mathcal{P}$, $K$, when we let $j \to \infty$ as the number and density of discrete points grow without bound [11]. This $\mathfrak{u}(2)$ can be called the mother algebra of the finite oscillator model. The wavefunctions in each model are the overlaps between the eigenfunctions of their mode generator and their position generator; they are the Hermite–Gauss (HG) functions $\Psi^{\text{HG}}_{n}(q)$ in the continuous model, and Kravchuk functions on the discrete position points of the finite model, given by quantum angular momentum theory as Wigner little-d functions [12,13], for the angle $\frac{1}{2} \pi$ between $J_{1}$ and $J_{3}$:

$$
\Psi^{(j)}_{n}(q) := d^{j}_{n,j} \left( \frac{1}{2} \pi \right) = \frac{(-1)^{n}}{2^{j}} \sqrt{\binom{2j}{n} \binom{2j}{j+q} } K_{a}(j+q; \frac{1}{2}, 2j),
$$

where $a^{j}_{n}, n^{\mathbb{N}}$, $K_{a}(s; \frac{1}{2}, 2j)$ is a symmetric Kravchuk polynomial [14], and $d^{j}_{a,b,c}$ is the Gauss hypergeometric function. These functions form multipoles under $\mathfrak{su}(2)$ that have been detailed in several papers [9,15], where they are shown to possess the desirable properties of the continuous HG modes. For the lowest $n$, the points $\Psi^{(j)}_{n}(q)$ fall closely on the continuous $\Psi^{\text{HG}}_{n}(q)$, while for higher $n$ they alternate in sign between every pair of neighbor points:

$$
\Psi^{(j)}_{n}(q) = (-1)^{n}\Psi^{(j)}_{n}(q).
$$

## 3. FOURIER ALGEBRA AND GROUP

Consider now two space dimensions $k \in \{x, y\}$, two momentum operators, the corresponding two independent mode operators, and the corresponding direct sum of the two algebra equations in Eq. (1), where the two central generators $1$ are identified into a single generator with the structure constants $\eta_{x} = \eta_{y} = 1$. This OSC Lie algebra thus generalizes Eq. (1) to

$$
[H_{k}, \bar{Q}_{k}] = -i\delta_{k,k'} \bar{P}_{k'};
$$

$$
[H_{k}, \bar{P}_{k}] = i\delta_{k,k'} \bar{Q}_{k}'.
$$

(4)

$$
\text{Out of all quadratic products of } \bar{Q}_{k} \text{ and } \bar{P}_{k}' \text{, one obtains the 10 generators of the symplectic real Lie algebra } \mathfrak{sp}(4, \mathbb{R} \text{ of paraxial optics, whose maximal compact subalgebra is the Fourier algebra } \mathfrak{u}(2)_{\mathfrak{f}} \text{. [1]} \text{. This algebra contains four up-to second degree differential operators that we identify as the generators of Fourier transformations (FTs) and other phase space rotations:}
$$

$$
\text{symmetric FT } \hat{L}_{0} := \frac{1}{4} (\hat{P}_{x}^{2} + \hat{P}_{y}^{2} + \hat{Q}_{x}^{2} + \hat{Q}_{y}^{2} - 21) = \frac{1}{2} (\hat{H}_{x} + \hat{H}_{y}),
$$

(5)

$$
\text{antisymmetric FT } \hat{L}_{1} := \frac{1}{4} (\hat{P}_{x}^{2} - \hat{P}_{y}^{2} + \hat{Q}_{x}^{2} - \hat{Q}_{y}^{2}) = \frac{1}{2} (\hat{H}_{x} - \hat{H}_{y}),
$$

(6)

$$
\text{gyration } \hat{L}_{2} := \frac{1}{2} (\hat{P}_{x} \hat{P}_{y} + \hat{Q}_{x} \hat{Q}_{y}),
$$

(7)

$$
\text{rotation } \hat{L}_{3} := \frac{1}{2} (\hat{Q}_{x} \hat{P}_{y} - \hat{Q}_{y} \hat{P}_{x}) = \frac{1}{2} \hat{M},
$$

(8)

where $M = 2L_{3}$ is the physical angular momentum operator. Their commutation relations are

$$
[L_{0}, L_{j}] = 0, \quad [L_{i}, L_{j}] = iL_{j},
$$

(9)

where the indices $i, j, k'$ are a cyclic permutation of 1, 2, 3. Abstractly, Eqs. (6)–(8) generate rotations of a 2-sphere.

For the finite oscillator model in two dimensions, we consider the direct sum of two $\mathfrak{su}(2)$ algebras, whose generators form a vector basis for $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which is (accidentally) homomorphic to the four-dimensional rotation algebra $\mathfrak{so}(4)$ [12]. We choose the representation of this algebra to be $(j_{x}, j_{y})$, determined by the values of the two independent Casimir operators in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The spectra of positions in the $x$- and $y$-directions will thus be $j_{x}^{2} + j_{y}^{2}$, as will the corresponding spectra of momenta, and modes are numbered by $n_{x}^{j_{x}}$'. We interpret the positions as the coordinates of pixels or points in a $N_{x} \times N_{y} = (2j_{x} + 1) \times (2j_{y} + 1)$ rectangular array. The two-dimensional finite harmonic oscillator functions are real, and the Cartesian products of Kravchuk functions in Eq. (2) in the two coordinates are [4]

$$
\Psi^{(j_{x}, j_{y})}_{n_{x}, n_{y}}(q_{x}, q_{y}) := \Psi^{(j_{x})}_{n_{x}}(q_{x}) \Psi^{(j_{y})}_{n_{y}}(q_{y}) = q_{x}^{j_{x}^{2}, n_{x}} \cdot q_{y}^{j_{y}^{2}, n_{y}}.
$$

(10)

There are thus $N_{x} N_{y}$ two-dimensional Kravchuk functions that can be arranged along axes of total mode $n = n_{x} + n_{y}$ and $m = n_{x} - n_{y}$ into the rhomboid pattern shown in Fig. 1. As eigenvectors of commuting operators in the Lie algebra, the Cartesian modes in Eq. (10) are orthonormal and complete under the natural inner product.
The range of gray-level densities, from black to white, is middle rhomboid, where

\[ \Psi_{nx,ny}(q_x, q_y) \]

while the rest must remain vertical. Finally, in the upper triangle, the pixels of the rhomboid contain three distinct intervals of

\[ q_x \]\

so that the upper triangle reproduces (top ↔ bottom, left ↔ right) the modes of the lower triangle but superposed with a checkerboard of changes of sign.

The images on the \( N_x \times N_y \) pixelated screen are value arrays \( F(q_x, q_y) \) that can be expanded in terms of the set of Cartesian modes Eq. (10) as

\[ F(q_x, q_y) = \sum_{n_x, n_y} F_{n_x, n_y} \Psi_{n_x, n_y}^{(l_{x, y})} (q_x, q_y), \]

\[ F_{n_x, n_y} = \sum_{q_x, q_y} F(q_x, q_y) \Psi_{n_x, n_y}^{(l_{x, y})} (q_x, q_y), \]  

(14)

as follows directly from linearity and the orthonormality and completeness of the Kravchuk basis.

4. IMPORTATION OF SYMMETRY

In the continuous case, the two-dimensional harmonic oscillator mode functions \( \Psi_{nx,ny}^{\text{HOG}}(q_x) \) can be arranged in a pattern similar to Fig. 1 but without an upper bound, forming an “inverted tower” with the same lower apex, out of which the total mode number \( n = n_x + n_y \) can grow indefinitely. Since \( L_0 \) commutes with all generators in Eqs. (5)–(8) of the \( \mathfrak{u}(2)_e \) algebra, functions with the same total mode number \( n \) will transform among themselves under the whole Fourier \( \mathfrak{u}(2)_e \) group. Functions with same total mode number \( n \) form \( \mathbf{SU}(2) \) multiplets, where the range of \( n_x - n_y = m^2 = 0 \) is spaced by two units. This is equivalent to having multiplets of spin \( \lambda = \frac{1}{2} n \), with \( \mu = \frac{1}{2} m \) playing the role of angular momentum projection on a “3”-axis.

In the finite model, however, the generators of the algebra \( \mathbf{SU}(2)_e \oplus \mathbf{SU}(2)_{2e} \) can raise and lower the modes only along the \( n_x \) or \( n_y \)-directions of Fig. 1 but not horizontally, i.e., from one value of \( \mu = \frac{1}{2} n \) to its neighbors. Symmetry importation consists in defining linear transformations among the states of the finite system using the linear combination coefficients provided by continuous models [7,8].

A. Rotations

The “physical” angular momentum operator \( \hat{M} \) in Eq. (8) generates rotations \( \mathcal{R}(\theta) := \exp(-i\theta \hat{M}) \) in the continuous model; this we now import to the finite model by simply eliminating the overbar in the notation. Therefore, because \( M = 2L_3 \), \( \mathcal{R}(\theta) \) is a rotation around the “3”-axis of the sphere by the double angle \( 2\theta \). Since the eigenvalues of \( L_0 \), \( \lambda = \frac{1}{2} m = \frac{1}{2}(n_x + n_y) \), are invariant under \( \mathcal{R}(\theta) \), the \( 2\lambda + 1 \) eigenstates of \( L_1 \), characterized by the difference eigenvalues \( \mu = \frac{1}{2}(n_x - n_y) \), will mix with linear combination coefficients given by Wigner little- \( d \) functions \( d_{\lambda\mu}^{l_{x, y}}(2\theta) \) [5,12,15]. (Note that the usual 1-2-3 numbering of axes is rotated to 2-3-1.)

To act on the Cartesian finite oscillator states \( \Psi_{n_x, n_y}^{(l_{x, y})}(q_x, q_y) \) in Eq. (10), we note the shape of the rhomboid in Fig. 1 and define their rotation (initially as a conjecture) by

\[ \mathcal{R}(\theta) \Psi_{n_x, n_y}^{(l_{x, y})}(q_x, q_y) := \sum_{n_{x'}, n_{y'}} d_{\lambda \mu}^{l_{x, y}}(2\theta) \Psi_{n_{x'}, n_{y'}}^{(l_{x, y})}(q_{x'}, q_{y'}), \]  

(15)

where the values of spin \( \lambda = \lambda(n) \) and their projections \( \mu(j_x, j_y; n_x, n_y) \) must now be examined with some care. The rhomboid contains three distinct intervals of \( n \) that should agree with the correct angular momentum \( \lambda \) of all imported \( \mathbf{SU}(2) \) multiplets in the horizontal rows of Fig. 1.
As we have assumed \( j_x > j_y \), we recognize that in each of the three intervals \( \lambda(n) \) will be

\[
\begin{align*}
\text{lower triangle:} & \quad \lambda(n) = \frac{1}{2} n, \\
0 & \leq n \leq 3 j_x, \\
\mu & = \frac{1}{2} (n_x - n_y), \\
\mu' & = \frac{1}{2} (n_x' - n_y'), \\
\text{mid rhomboid:} & \quad \lambda(n) = j_x, \\
2 j_x & < n < 2 j_x + j_y, \\
\mu & = j_x - n_y, \\
\mu' & = j_y - n_y', \\
\text{upper triangle:} & \quad \lambda(n) = j_x + j_y - \frac{1}{2} n, \\
2 j_x & \leq n \leq 2 j_x + j_y, \\
\mu & = \frac{1}{2} (n_x - n_y) - j_x + j_y, \\
\mu' & = \frac{1}{2} (n_x' - n_y') - j_x + j_y.
\end{align*}
\]  

(16)

The first two cases actually overlap for \( n = 2 j_x \) and the second two cases for \( n = 2 j_x + 1 \) \times (2 j_y + 1) screen follows from Eq. (14) and the rotation Eq. (15) of the Cartesian basis:

\[
\mathcal{R}(\theta): F(q_x, q_y) = \sum_{n_x, n_y} F_{n_x, n_y}^{(\theta)}(q_x, q_y).
\]

(17)

In Fig. 3, we show the rotation of a white on black (1 s on 0 s) image of the letter “F.” We note the inevitable “Gibbs” oscillations around the sharp edges of the figure, yet we should stress that the rotated images were obtained by successive rotations by \( \frac{\pi}{5} \). The reconstruction of the original image after six rotations by \( \frac{\pi}{5} \) would be impossible with any interpolation algorithm applied successively. The rotation by \( \frac{\pi}{5} \) in the figure shows a squeezed image of “F,” which originally contained only 1 s and 0 s but whose Gibbs-like oscillations now involve pixels with negative values; one more rotation of \( \frac{\pi}{5} \) restores a \( \pi \)-rotated (inverted) image, which is a simple permutation of pixels. In a square screen, rotations by \( \frac{\pi}{5} \) are also permutations.

**B. Symmetric and Antisymmetric Fourier Transforms**

In the continuous model, the mode number operators \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) generate fractional Fourier transforms [9] through \( \mathcal{F}_+(\beta) = \exp(-i\beta L_1) \mathcal{F} \) that multiply the continuous oscillator basis functions \( \Psi_{n_x}(q_x) \) by phases \( \exp(-i\beta q_x) \). In the finite oscillator model, the symmetric fractional Fourier–Kravchuk transform \( \mathcal{K}_+(\varphi) = \exp(-2i\varphi L_0) \) is generated by \( L_0 = \frac{1}{2}(K_x + K_y) \) as in Eq. (5). It acts on the Cartesian modes only multiplying them by phases,

\[
\mathcal{K}_+(\varphi): \Psi_{n_x, n_y}^{(\varphi)}(q_x, q_y) = \exp[-i\varphi(n_x + n_y)] \Psi_{n_x, n_y}^{(\varphi)}(q_x, q_y).
\]

(18)

and commutes with all transformations in the Fourier group.

On the other hand, a rotation by \( 2\beta \) around the 1-axis is generated by \( L_1 = \frac{1}{2}(K_x - K_y) \) in Eq. (6) to produce the antisymmetric fractional Fourier–Kravchuk transforms, \( \mathcal{K}_A(\beta) = \exp(-2i\beta L_1) \) in the group SU(2) \( \times \) SU(2):
Both using the decomposition in Eq. (14) on the pixelated screen. As in Eq. (16), Bottom: the level multiplets in each of the three blocks according to the three intervals in Eq. (16). Middle: selected states in the mid rhomboid at level \( n = 4 \) (\( \lambda = 2 \)) in the lower triangle. Top: states in the level \( n = 32 \) (\( \lambda = 2 \)).

\[
\mathcal{K}_A(\beta) \Psi_{n_x,n_y}^{(\ell_x,\ell_y)}(q_x, q_y) = \exp[-i\beta(n_x - n_y)] \Psi_{n_x,n_y}^{(\ell_x,\ell_y)}(q_x, q_y). \tag{19}
\]

As with rotations, they can be applied to arbitrary images using the decomposition in Eq. (14) on the pixelated screen. Both \( \mathcal{K}_S(\gamma) \) and \( \mathcal{K}_A(\beta) \) are “domestic” within \( \text{SU}(2) \), but they mesh appropriately with the imported rotations.

### C. Gyrotory

In the continuous model, gyrotory by \( \gamma \) around the 2-axis are generated by \( L_2 \) in Eq. (7). For \( \gamma = \frac{1}{2}\pi \), they transform Hermite–Gauss to Laguerre–Gauss modes and can be realized with simple paraxial optical setups [16–18]. They are rotations that result from a rotation by \( \frac{1}{2}\pi \) around the 3-axis (antisymmetric fractional Fourier transform by angle \( \frac{1}{2}\pi \)), a rotation \( \gamma \) around the new 1-axis, and back through \( -\frac{1}{2}\pi \) around the new 3-axis,

\[
\mathcal{G}(\gamma) \Psi_{n_x,n_y}^{(\ell_x,\ell_y)}(q_x, q_y) := e^{-i\beta n_x n_y} \sum_{n_x', n_y'} d^{\ell_x \ell_y}_{\mu \mu'}(2\gamma) \Psi_{n_x',n_y'}^{(\ell_x',\ell_y')}(q_x, q_y), \tag{21}
\]

where \( \lambda(n) \), \( \mu \), and \( \mu' \) are related to \( j_x \mu_x j_y \) \( -j_y \) through Eq. (16). This set of functions forms also, as in the Cartesian case, a complete and orthogonal basis for all images on the pixelated screen. We show the gyration of modes \( \Psi_{n_x,n_y}^{(\ell_x,\ell_y)}(q_x, q_y) \) in Fig. 4 for various values of \( 0 \leq \gamma \leq \frac{1}{2}\pi \). Note that this transformation yields complex arrays of functions, so for \( \gamma = \frac{1}{2}\pi \) we show their absolute values and phases.

For \( \gamma = \frac{1}{2}\pi \), Eq. (21) defines finite functions characterized by the total mode number \( n = n_x + n_y \) and an integer “(rectangular) angular momentum” number \( m = 2\mu = n_x - n_y, |\mu| \leq \lambda(n) \) constrained by Eq. (16), and given by

\[
\mathcal{G}(\gamma) := \mathcal{F}_{A}\left(\frac{1}{4}\pi\right) \mathcal{R}(\gamma) \mathcal{F}_{A}\left(-\frac{1}{4}\pi\right). \tag{20}
\]
\[
\tilde{N}_{n,m}^{(l)}(q_x, q_y) := e^{-i\pi(n-x)(n-y)/4} \sum_{n_x', n_y'=m} d_{n_x', n_y'}^{(l)} \left( \frac{1}{2} \pi \right) \times e^{i\pi(n_x'-n_y')/4} \psi_{n_x', n_y'}^{(l)}(q_x, q_y) = \tilde{N}_{n,m}^{(l)}(q_x, q_y).
\]  

(22)

In the square screen case, when \( j_x = j_y = j \), the functions \( \tilde{N}_{n,m}^{(l)}(q_x, q_y) \) were called Laguerre–Kravchuk modes [6], whose continuous counterparts are the well-known Laguerre–Gauss modes. Within rectangular pixelated screens, the functions Eq. (22) are also orthogonal and complete:

\[
\sum_{n,m} \tilde{N}_{n,m}^{(l)}(q_x, q_y)^* \tilde{N}_{n,m}^{(l)}(q_x', q_y') = \delta_{q_x, q_x'} \delta_{q_y, q_y'},
\]

(23)

\[
\sum_{q_x, q_y} \tilde{N}_{n,m}^{(l)}(q_x, q_y)^* \psi_{n_x', n_y'}^{(l)}(q_x, q_y) = \delta_{n,n'} \delta_{m,m'},
\]

(24)

since they are obtained from the Cartesian states through \( \mathfrak{su}(2)_F \) unitary transformations. Although the notion of their angular momentum is no longer properly valid, they still bear a recognizable resemblance, as Fig. 5 shows, where the multiplets \( \{ \tilde{N}_{n,m}^{(l)}(q_x, q_y) \}_{m=-j}^{j} \) are placed on rows for all total modes \( n \) in the three intervals in Eq. (16).

5. CONCLUDING REMARKS

In continuous systems, the elements of the \( \mathfrak{u}(2)_F = U(1) \otimes \mathfrak{su}(2)_F \) Fourier group can be parametrized by the angle \( \chi \) of the central \( U(1) \) symmetric fractional Fourier transform, which commutes with all \( \mathfrak{su}(2)_F \) transformations having Euler angles \( (\psi, \theta, \phi) \), as

\[
\tilde{D}(\chi; \psi, \theta, \phi) = \exp(-i\chi \hat{L}_z) \exp(-i\psi \hat{L}_x) \exp(-i\theta \hat{L}_y).
\]

(25)

On the finite \( N_x \times N_y \) pixelated screen, \( \mathfrak{u}(2)_F \) is correspondingly realized by subgroups of domestic Fourier–Kravchuk transformations, and imported rotations and gyrations, as

\[
\tilde{D}(\chi; \psi, \theta, \phi) = K_S \left( \frac{1}{2} \chi \right) K_{\chi} \left( \frac{1}{2} \psi \right) G \left( \frac{1}{2} \theta \right) K_{\chi} \left( \frac{1}{2} \phi \right) = K_S \left( \frac{1}{2} \chi \right) K_{\chi} \left( \frac{1}{2} \psi + \frac{1}{4} \pi \right) R \left( \frac{1}{2} \theta \right) K_{\chi} \left( \frac{1}{2} \phi - \frac{1}{4} \pi \right).
\]

(26)

The action of the Fourier group is unitary on all complex-valued images on \( N_x \times N_y \) pixelated screens, and hence there is no information loss under these transformations. We must repeat that the algorithm is not fast, but arguably the slowest, because under a generic transformation, each pixel of the new image will depend on the values of all the pixels of the original image. Also, it will necessarily involve Gibbs-like oscillations in pixelated images with sharp “discontinuities.” As the previous experience with square screens suggests [5], smoothing the original values or chopping the resulting ones can restore the visual fidelity of the image, even though unitarity will be lost. It may be that in experiments where two-dimensional beams are sampled at rectangular CCD arrays, bases of discrete functions are better suited for the task than their approximation by pointwise-sampled Hermite–Gauss oscillator wavefunctions [19].

The introduction of rectangular analogs of the Laguerre–Gauss states with “angular momentum” is the direct (but not trivial) generalization of those built for square screens in Ref. [4] and, as there, will predictably allow a unitary map to screens whose pixels are arranged along polar coordinates, as done in Ref. [20]. Finally, it should be noted that all the “discrete” functions, starting with the Wigner little-\( d_{n,j}^{(l)}(\theta) \), are actually analytic functions of continuous position \( q \) in the range \(-j - 1 < q < j + 1\), with branch-point zeros at \( q = \pm(j + 1) \) and cuts beyond. This property extends of course to the two-dimensional case for \(-j_x - 1 < q_x < j_x + 1\), \(-j_y - 1 < q_y < j_y + 1\), \( k \in \{x, y\} \). The discrete model can thus also accommodate a continuous model of modes in bounded screens, although the unitarity of the transformations holds only for the discrete integer points within.

Funding. Dirección General Asuntos del Personal Académico, Universidad Nacional Autónoma de México (DGAPA, UNAM) Óptica Matemática (PAPIIT IN101115).

REFERENCES