Bol. Soc. Mat. Mex. (2015) 21:27–37
DOI 10.1007/s40590-014-0015-9

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

Kravchuk polynomials and irreducible representations of the rotation group $SO(3)$

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Received: 15 June 2013 / Accepted: 15 April 2014 / Published online: 16 May 2014

Abstract We set forth that the Kravchuk polynomials of a discrete variable are actually "encoded" within appropriate finite-dimensional irreducible representations of the group of rotations of three-dimensional space $SO(3)$. Hence, discrete irreducible representation spaces of the group $SO(3)$ can be naturally interpreted as finite (discrete) versions of the linear quantum harmonic oscillator.

Mathematics Subject Classification 33D45 · 39A70 · 47B39

1 Introduction

Classical orthogonal polynomials of a discrete variable (the Hahn, Meixner, Charlier, and Kravchuk polynomials) are extensively used in physics and mathematics [18]. In particular, discrete (finite) quantum oscillator models, which are constructed in terms of these polynomials of the hypergeometric type, find a large number of applications in signal [4,5,7] and in optical image processing [8,20]. Recall that, chronologically, this type of discrete oscillator models were introduced in the following way. One starts with a family of discrete orthogonal polynomials, governed by a difference equation with a linear spectrum. This difference equation is then interpreted as a discrete analog of

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the well-known Hamiltonian for the linear harmonic oscillator in quantum mechanics. Eigenfunctions in such a model are naturally defined in terms of the initial discrete polynomials times the square root of their orthogonality weight function. Moreover, the subsequent factorization of an ad hoc introduced "discrete Hamiltonian" enables one to construct appropriate difference analogs of the raising and lowering operators for the linear quantum harmonic oscillator and, consequently, a corresponding spectrum generating algebra. In particular, in the case of Kravchuk polynomials this spectrum generating algebra turns out to be the $su(2)$ Lie algebra [2,3], whereas in the Charlier and Meixner cases these are the $su(1,1)$ Lie algebras [6].

This work shows that there is a different way of arriving at discrete oscillator models, based on the use of the algebraic properties of the generators of a particular Lie group. As a first step in formulating this approach, we will be engaged in this work solely with the Lie group of rotations of three-dimensional space $SO(3)$ [or locally isomorphic to it, the Lie group $SU(2)$]. We contend that discrete quantum oscillator models built in terms of the Kravchuk functions are actually "encoded" within appropriate finite-dimensional irreducible representations of the rotation group $SO(3)$.

A summary of what the remaining sections of this paper contain is as follows. Section 2 recalls those difference equations from the monograph [10], that are needed in order to find the explicit forms of irreducible representations of the group of rotations of three-dimensional space $SO(3)$. Since the simplest solutions of these difference equations are expressed in terms of the Kravchuk functions, one arrives naturally at the desired discrete oscillator model. Section 3 contains conclusions and a brief discussion of some further research directions of interest. Finally, in the Appendix we have collected basic background facts about symmetric Kravchuk polynomials and, associated with them, the Kravchuk functions that are used in this work.

Throughout this exposition we employ standard notations from the theory of special functions (see, for example, [1,9,13]) and from non-relativistic quantum mechanics [15].

2 Irreducible representations of the rotation group

It is well known that in the study of representations of the rotation group $SO(3)$ [or its locally isomorphic group $SU(2)$], one employs essentially the algebraic properties of the generators of this group J_k, $k = 1, 2, 3$, which are the operators of the infinitesimal rotations about the coordinate axes (see, for example, [10,21]). These operators form the Lie algebra $so(3)$ (or $su(2)$) with the commutation relations

$$[J_1, J_2] = \mathrm{i}\, J_3, \quad [J_2, J_3] = \mathrm{i}\, J_1, \quad [J_3, J_1] = \mathrm{i}\, J_2\,, \tag{2.1}$$

where by definition $[A, B] := AB - BA$.

It is important to observe from the outset that from (2.1) it follows at once that

$$[J_1, [J_1, J_3]] = -\mathrm{i}[J_1, J_2] = J_3. \tag{2.2}$$

This observation plays a key role in this work for the following reason. Recall that quantum-mechanical analog of classical Newton's equation $m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\mathrm{d}U}{\mathrm{d}x}$ for a linear harmonic oscillator is written in terms of the position operator x and the Hamiltonian $\hat{H} := p^2/2m + m\omega^2 x^2/2$ as in [15,16]

$$\left[\hat{H}, [\hat{H}, x]\right] = (\hbar\omega)^2\, x. \tag{2.3}$$

On comparing (2.2) with (2.3) it becomes clear that, with proper normalizations, the generators J_1 and J_3 can be *also* interpreted as the position operator X and the Hamiltonian H of some *discrete (finite) model* of the linear quantum harmonic oscillator. Taking into account that the momentum operator $\hat{P}$ in quantum mechanics is defined as $\omega\hat{P} := \mathrm{i}[\hat{H}, x]$, one concludes that the association

$$J_1 \Rightarrow H, \quad J_2 \Rightarrow P, \quad J_3 \Rightarrow X, \tag{2.4}$$

enables one to interpret the commutation relations (2.1) of the Lie algebra $so(3)$ (or $su(2)$) as a closed defining algebra for a triplet X, P and H with the commutation relations

$$[X, H] = \mathrm{i}\, P, \quad [H, P] = \mathrm{i}\, X, \quad [P, X] = \mathrm{i}\, H, \tag{2.5}$$

and, consequently,

$$[H, [H, X]] \; = \; X. \tag{2.6}$$

To establish what kind of an oscillator model emerges from this interpretation (2.4) of the commutation relations (2.1), one needs to recall the following. To find the explicit form of an unitary irreducible representation of the group of rotations $SO(3)$, it is more convenient to consider the linear combinations of the generators J_1 and J_2 in the form $J_\pm = J_1 \pm \mathrm{i} J_2$. Indeed, from (2.1) it then follows that

$$[J_+, J_-] \; = \; 2\, J_3, \qquad [J_3, J_\pm] = \pm J_\pm, \tag{2.7}$$

which means that the operators $J_\pm$ are actually *step (or raising and lowering, respectively) operators*: if $\vec{f}_m$ is an eigenvector of the operator J_3, *i.e.,* $J_3 \vec{f}_m = m\, \vec{f}_m$, then the vectors $J_\pm \vec{f}_m$ represent eigenvectors $\vec{f}_{m\pm 1}$ of the same operator J_3. By using this property of the step operators $J_\pm$, one can prove, pure algebraically and, most importantly, without employing explicit forms (realizations) of the generators $J_\pm$ and J_3, that (see pages 22–25 in [10]) for *any unitary irreducible representation* of $SO(3)$ the operators $J_\pm$ and J_3 define an orthogonal basis consisting of the normalized eigenvectors of J_3 by the equations

$$J_+ \vec{f}_m^{\,(l)} = \alpha_{m+1}^{(l)}\, \vec{f}_{m+1}^{\,(l)}, \quad J_- \vec{f}_m^{\,(l)} = \alpha_m^{(l)}\, \vec{f}_{m-1}^{\,(l)}, \quad J_3\, \vec{f}_m^{\,(l)} = m\, \vec{f}_m^{\,(l)}, \tag{2.8}$$

where $m = -l, -l+1, \ldots, l$, the weight l of the corresponding irreducible representation is an integer or half an odd integer, and $\alpha_m^{(l)} := \sqrt{(l+m)(l-m+1)}$.

In terms of the initial generators $J_k,\ k = 1, 2, 3$, the equations (2.8) in *the canonical basis* $\vec{f}_m^{(l)}$ can be written as (*cf* formula (19) on page 26 in [10])

$$J_1 \vec{f}_m^{(l)} = \frac{1}{2}\left[\alpha_{m+1}^{(l)}\, \vec{f}_{m+1}^{(l)} + \alpha_m^{(l)}\, \vec{f}_{m-1}^{(l)}\right], \tag{2.8'}$$
$$J_2 \vec{f}_m^{(l)} = \frac{1}{2\mathrm{i}}\left[\alpha_{m+1}^{(l)}\, \vec{f}_{m+1}^{(l)} - \alpha_m^{(l)}\, \vec{f}_{m-1}^{(l)}\right],$$
$$J_3\, \vec{f}_m^{(l)} = m\, \vec{f}_m^{(l)}.$$

Notice that since in a finite-dimensional space each linear operator is given by a matrix, from (2.8′), it is evident that the generators J_k in the canonical basis $\vec{f}_m^{(l)}$, $m = -l, 1-l, 2-l, \ldots, l$, are represented by $(2l+1)\times(2l+1)$-matrices of the form (*cf.* (21) on page 28 in [10]):

$$||(J_1)_{m,\,m'}^{(l)}|| = \frac{1}{2}\begin{pmatrix} 0 & \alpha_{1-l}^{(l)} & 0 & \cdots & 0 & 0 \\ \alpha_{1-l}^{(l)} & 0 & \alpha_{2-l}^{(l)} & \cdots & 0 & 0 \\ 0 & \alpha_{2-l}^{(l)} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_l^{(l)} \\ 0 & 0 & 0 & \cdots & \alpha_l^{(l)} & 0 \end{pmatrix}, \tag{2.9}$$

$$||(J_2)_{m,\,m'}^{(l)}|| = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \alpha_{1-l}^{(l)} & 0 & \cdots & 0 & 0 \\ -\alpha_{1-l}^{(l)} & 0 & \alpha_{2-l}^{(l)} & \cdots & 0 & 0 \\ 0 & -\alpha_{2-l}^{(l)} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_l^{(l)} \\ 0 & 0 & 0 & \cdots & -\alpha_l^{(l)} & 0 \end{pmatrix}, \tag{2.10}$$

$$||(J_3)_{m,\,m'}^{(l)}|| = \begin{pmatrix} -l & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-l & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2-l & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & l-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & l \end{pmatrix}, \tag{2.11}$$

The problem of finding the form of an unitary irreducible representation of the rotation group $SO(3)$ thus reduces to that of solving the equations (2.8), that is, finding explicit forms for the triplet of operators J_k, satisfying the commutation relations (2.1) and exhibiting properties (2.8′). Well-known examples of continuous unitary representations[1] of the group $SO(3)$, constructed in this way and often used in various applications, are those formulated in terms of functions on the sphere. In this case, the canonical basis consists of the spherical functions

[1] Recall that the representation $g \to T_g$ is called *continuous* if the elements of the matrix T_g are continuous functions of g (see, for example, [10]).

$$Y_l^m(\theta,\varphi) := \frac{1}{\sqrt{2\pi}}\, e^{\mathrm{i}m\varphi}\, P_l^{\,m}(\cos\theta), \tag{2.12}$$

where $P_l^m(x)$ are *the normalized associated Legendre functions* (see (26), p.46 in [10]). The generators $J_\pm$ and J_3 in this basis are then realized as partial differential operators

$$J_\pm := e^{\pm\mathrm{i}\varphi}\left(\pm\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\,\frac{\partial}{\partial\varphi}\right), \quad J_3 := -\mathrm{i}\frac{\partial}{\partial\varphi}\,, \tag{2.13}$$

in the variables θ and φ.

But note that the Eq. (2.8′) do have another type of solutions in terms of the *discrete eigenvectors* $\vec{f}_m^{(l)}$ of J_3. Notice that contrary to the case of spherical functions (2.12), this discrete basis does not depend explicitly on the group parameters. So, upon using those solutions of (2.8′), one actually arrives at the explicit form of a discrete version of the linear quantum harmonic oscillator in the following way.

Recall that a physical system in quantum mechanics is described by the eigenfunctions of Hamiltonian H, whereas either the position operator X is associated with the multiplication by the coordinate x (and this case referred to as *coordinate realization*), or the momentum operator P is associated with the multiplication by the momentum p (*momentum realization*). Therefore, to interpret the generators J_1 and J_3 as in the association scheme (2.4), one should build a basis of an irreducible representation of $SO(3)$ (or $SU(2)$) in terms of the eigenvectors of the generator J_1, rather than of the generator J_3 as in (2.8′). This can be regarded as introducing a *dual canonical basis*, defined as follows. Let us first change slightly the enumeration of the basis vectors $\vec{f}_m^{(l)}$, $-l \le m \le l$, by arranging them as

$$\vec{u}_n^{(l)} := \vec{f}_{n-l}^{(l)}, \quad n = 0, 1, 2, \ldots, 2l. \tag{2.14}$$

Then, the *dual canonical basis vectors* $\vec{v}_n^{(l)}$ with components $\left(\vec{v}_n^{(l)}\right)_k$, $k = 0, 1, \ldots, 2l$, are defined in terms of the canonical basis vectors $\vec{u}_n^{(l)}$ as

$$\left(\vec{v}_n^{(l)}\right)_k := \left(\vec{u}_k^{(l)}\right)_n, \quad k, n = 0, 1, 2, \ldots, 2l. \tag{2.15}$$

It is not hard to verify that there are solutions of Eq. (2.8′) with the generators J_i, $i = 1, 2, 3$, explicitly given by (2.8′) as *difference operators*, and with basis vectors $\vec{f}_m^{(l)}$, whose $2l+1$ components $\left(\vec{f}_m^{(l)}\right)_n$, $n = 0, 1, 2, \ldots, 2l$, are defined in terms of the Kravchuk functions $\psi_n^{(l)}(x)$ [see (4.12) in the Appendix] as

$$\left(\vec{f}_m^{(l)}\right)_n := \psi_n^{(l)}(m), \quad -l \le m \le l. \tag{2.16}$$

The advantage of this simple solution (2.16) is that the associated dual canonical basis vectors, defined as in (2.15), coincide with the canonical basis vectors in (2.16) due to the self-duality property of the Kravchuk functions (4.12). Therefore it becomes quite transparent why the commutation relations (2.1) admit the association scheme (2.4). In particular, the generator J_3, represented in the canonical basis $\left\{\vec{f}_{-l}^{(l)}, \vec{f}_{-l+1}^{(l)}, \ldots, \vec{f}_{l-1}^{(l)}, \vec{f}_{l}^{(l)}\right\}$ by the diagonal matrix (2.11) with the diagonal elements $\{-l, -l+1, \ldots, l-1, l\}$, can be also interpreted, but in the dual canonical basis (2.15), as a matrix of the multiplication by the discrete coordinate $x = \{-l, -l+1, \ldots, l-1, l\}$. That is why the generator J_3 can be regarded as the operator X. Besides, from the difference equation (4.18) it follows at once that the canonical basis vectors (2.16) are actually eigenvectors of the generator J_1; this explains why $J_1 \Rightarrow H$ in the association scheme in (2.4). Finally, from the previous two, $J_1 \Rightarrow H$ and $J_3 \Rightarrow X$, it is plain that [*cf* the second entry in (2.4)]

$$J_2 = \mathrm{i}\,[J_1, J_3] \quad \Rightarrow \quad \mathrm{i}\,[H, X] = P. \tag{2.17}$$

In conclusion, it should be pointed out that in the three-volume encyclopedic monograph by N. Ja. Vilenkin and A. U. Klimyk the Kravchuk polynomials had been attributed to matrix elements of irreducible representations of the Lie group $SU(2)$, treated “as functions of column index” (see page 346 in [19]). So the algebraic reasoning in this section reveals that those “matrix elements as functions of column index” are simply matrix elements in *the dual canonical basis*, thus making it transparent how the former ones emerge from the group-theoretical point of view.

3 Concluding remarks

We have demonstrated that the generators of the group of rotations of *three-dimensional space* $SO(3)$ can be interpreted as the triplet of the operators $\{X, P, H\}$, which define a discrete (finite) model of *the linear quantum harmonic oscillator* in terms of the Kravchuk functions. It is worthwhile to mention that the necessary prerequisite of observing this circumstance had been presumably available at least since the appearance of the monograph by Gel’fand et al. [10], first published in Russian in 1953 (recall that Kravchuk polynomials [14] had been known since 1929). So it seems to us that the discrete Kravchuk oscillator did not attract much attention earlier because it is realized on *discrete* function bases for the finite-dimensional spaces of unitary irreducible representations of the rotation group $SO(3)$, while the attention of the authors of [10] and many other contemporary and later authors has been on bases of functions that are continuous functions (similar to spherical harmonics) over the manifold of the sphere.

The approach followed in this work seems to be quite general. In particular, it can be used for constructing discrete oscillator models, associated with the discrete series of irreducible representations of the Lie group $SU(1, 1)$. This will be dealt with elsewhere.

A final remark concerns the possibility of studying the group-theoretic properties of families of discrete polynomials that are *not* associated with some Lie algebra. A recent work by Kalnins et al. [12] discussed the generic three-parameter second-order superintegrable system $S9$ in two dimensions in detail (see also [11,17]). It turns out that this superintegrable model is closely interconnected with hypergeometric orthogonal polynomials from the Askey scheme [13]. In particular, various function space realizations of the quadratic Racah–Wilson algebra, which is the symmetry algebra behind the $S9$ superintegrable model, can be associated with *all* hypergeometric polynomials in the Askey scheme. These remarkable works [11,12] thus reveal the group-theoretic context of such intricate orthogonal families as the Wilson and Racah polynomials that satisfy second-order difference equations with quadratic spectra.

Acknowledgments Discussions with E.I.Jafarov, W.Miller, Jr. and L.Vinet are gratefully acknowledged. The participation of MKA in this work has been partially supported by the SEP-CONACYT project 168104 "Operadores integrales y pseudodiferenciales en problemas de física matemática" and the *Proyecto de Redes* PROMEP (México). NMA and KBW have been supported by the DGAPA-UNAM IN105008-3 and SEP-CONACYT 79899 projects "Óptica Matemática".

Appendix: Kravchuk polynomials and functions

Kravchuk polynomials [14] in the real variable x are defined in [13] (see pages 237–241) as

$$K_n(x;\, p, N) \;:=\; {}_2F_1\left(-n, -x; -N;\, p^{-1}\right), \quad n = 0, 1, 2, \ldots, N, \tag{4.1}$$

where p is a positive real number smaller than unit, $0 < p < 1$, and N is some positive integer number.

Since Gauss's hypergeometric function ${}_2F_1\,(a, b; c; z)$ is symmetric with respect to the parameters a and b, *i.e.*, ${}_2F_1\,(a, b; c; z) \;=\; {}_2F_1\,(b, a; c; z)$, from the definition (4.1) it follows at once that the Kravchuk polynomials $K_n(x;\, p, N)$ are *self-dual*:

$$K_n(m;\, p, N) \;=\; K_m(n;\, p, N), \quad m = 0, 1, 2, \ldots, N. \tag{4.2}$$

Observe also that from Euler's (or the Pfaff or Pfaff–Kummer) transformation formula (see (1.7.2) in [13]),

$${}_2F_1\,(a, b; c; z) \;=\; (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{4.3}$$

it follows that

$${}_2F_1\left(-n, -x; -N;\, p^{-1}\right) = \left(\frac{p-1}{p}\right)^n {}_2F_1\left(-n, x-N; -N; \frac{1}{1-p}\right). \tag{4.4}$$

This means that

$$K_n(x;\, p, N) \;=\; \left(\frac{p-1}{p}\right)^n K_n(N-x;\, 1-p, N), \tag{4.5}$$

reflecting the fact that the Kravchuk polynomials $K_n(x;\,p,N)$ for generic values of the parameter p do not exhibit any *reflection symmetry property* with respect to the change $x \Rightarrow N-x$. Only exceptional value of $p=1/2$ represents the case when the Kravchuk polynomials $K_n(x;\,1/2,N)$ are either symmetric (when n is even) or skew-symmetric (when n is odd) with respect to the change $x \Rightarrow N-x$:

$$K_n(x;\,1/2,N) \,=\, (-1)^n\, K_n(N-x;\,1/2,N). \tag{4.6}$$

From *the discrete orthogonality relation* for the Kravchuk polynomials $K_n(x;\,p,N)$ for the generic values of the parameter p (see (9.11.2) in [18]) it follows that

$$\sum_{l=0}^{N}\binom{N}{l}\,K_n(l;\,1/2,N)\,K_{n'}(l;\,1/2,N) \,=\, \frac{2^N}{\binom{N}{n}}\,\delta_{nn'}, \tag{4.7}$$

where $\binom{m}{n} := m!/n!(m-n)!$ is the binomial coefficient. Very often it turns out to be more convenient to represent (4.7) in a symmetric form with respect to the variable x by denoting $N=2M$, that is,

$$\sum_{m=-M}^{M}\binom{2M}{M+m}\,k_n^{(M)}(m)\,k_{n'}^{(M)}(m) \,=\, \frac{4^M}{\binom{2M}{n}}\,\delta_{nn'}\,, \tag{4.8}$$

where the polynomials $k_n^{(M)}(x), \quad n=0,1,2,\ldots,2M$, are defined as

$$k_n^{(M)}(x) := K_n(x+M;\,1/2,2M) = {}_2F_1\,(-n,-x-M;-2M;2)\,. \tag{4.9}$$

The polynomials $k_n^{(M)}(x)$ will be referred to as *the symmetric Kravchuk polynomials*.

The next natural step is to define orthonormalized *Kravchuk functions*, which can be interpreted as *discrete (finite) analogs* of *the Hermite functions* $H_n(x)\exp(-x^2/2)$, where $H_n(x)$ are the classical Hermite polynomials.[2] Since $k_0^{(M)}(x)=1$ by definition, from (4.8) it follows that

$$4^{-M}\sum_{m=-M}^{M}\binom{2M}{M+m} \equiv \sum_{m=-M}^{M}\left[\psi_0^{(M)}(m)\right]^2 \,=\, 1, \tag{4.10}$$

[2] We remind the reader that the properly normalized Hermite functions are interpreted in quantum mechanics as *the linear harmonic oscillator wave functions* [15].

where

$$\psi_0^{(M)}(x) := \frac{1}{2^M}\sqrt{\frac{(2M)!}{\Gamma(x+M+1)\Gamma(M-x+1)}}, \quad x \in [-M, M]. \tag{4.11}$$

This means that Kravchuk functions $\psi_n^{(M)}(x)$, defined as

$$\psi_n^{(M)}(x) := \binom{2M}{n}^{1/2} k_n^{(M)}(x)\, \psi_0^{(M)}(x), \quad n = 0, 1, 2, \ldots, 2M, \tag{4.12}$$

do satisfy *the discrete orthogonality relation* of the form

$$\sum_{m=-M}^{M} \psi_n^{(M)}(m)\, \psi_{n'}^{(M)}(m) \;=\; \delta_{nn'}, \tag{4.13}$$

which is essentially just another way of writing down the orthogonality relation (4.8).

Evidently, all other main properties of the Kravchuk polynomials, which follow from their three-term recurrence relation, difference equation, and so on, can be readily reformulated in terms of the Kravchuk functions (4.12). For instance, the three-term recurrence relation (*cf.* (9.11.3) in [13])

$$2x\, k_n^{(M)}(x) = (n-2M)\, k_{n+1}^{(M)}(x) - n\, k_{n-1}^{(M)}(x) \tag{4.14}$$

for the symmetric Kravchuk polynomials $k_n^{(M)}(x)$ entails a corresponding relation

$$-\,2x\, \psi_n^{(M)}(x) = \sqrt{(n+1)(2M-n)}\, \psi_{n+1}^{(M)}(x) + \sqrt{n(2M-n+1)}\, \psi_{n-1}^{(M)}(x) \tag{4.15}$$

for the Kravchuk functions (2.12).

Similarly, using the difference equation (*cf.* (9.11.5) in [13])

$$2(M-n)\, k_n^{(M)}(x) = (M-x)\, k_n^{(M)}(x+1) + (M+x)\, k_n^{(M)}(x-1) \tag{4.16}$$

for the symmetric Kravchuk polynomials $k_n^{(M)}(x)$ and taking into account that

$$\psi_0^{(M)}(x+1) = \sqrt{\frac{M-x}{M+x+1}}\, \psi_0^{(M)}(x)\,, \tag{4.17}$$

one arrives at the difference equation

$$\begin{aligned} 2(M-n)\, \psi_n^{(M)}(x) = {} & \sqrt{(x+M+1)(M-x)}\, \psi_n^{(M)}(x+1) \\ & + \sqrt{(x+M)(M-x+1)}\, \psi_n^{(M)}(x-1) \end{aligned} \tag{4.18}$$

for the Kravchuk functions (4.12). This difference equation is regarded as defining the discrete model of the quantum oscillator,

$$\hat{H}^{(FO)}\,\psi_n^{(M)}(x)\;=\;(M-n)\,\psi_n^{(M)}(x), \tag{4.18'}$$

governed by the "Hamiltonian"

$$\hat{H}^{(FO)}=\frac{1}{2}\left[\sqrt{(x+M+1)(M-x)}\,T_{+}\;+\;\sqrt{(x+M)(M-x+1)}\,T_{-}\right], \tag{4.19}$$

where $T_{\pm}f(x):=f(x\pm 1)$.

The Kravchuk polynomials (4.1) are known to reduce to the Hermite polynomials $H_n(x)$ in the limit as $N\to\infty$. In particular, for the symmetric Kravchuk polynomials $k_n^{(M)}(x)$ one has (*cf.* (9.11.15) in [13])

$$\lim_{M\to\infty}\,M^{n/2}\,k_n^{(M)}\left(\sqrt{M}x\right)\;=\;\frac{(-1)^n}{2^n}\,H_n(x). \tag{4.20}$$

By using this limit property and the asymptotical behavior $\sqrt{\frac{2\pi}{x}}\,\left(\frac{x}{e}\right)^x$ of the gamma function $\Gamma(x)$ for large x, it is not hard to verify that the Kravchuk functions (4.12) reduce to the Hermite functions in the limit as $M\to\infty$.

Finally, observe that the self-duality (4.2) of the Kravchuk polynomials $K_n(x;\,p,\,N)$ entails the following duality property of the Kravchuk functions (4.12):

$$\psi_n^{(M)}(k-M)\;=\;\psi_k^{(M)}(n-M),\quad 0\le n,k\le 2M. \tag{4.21}$$

It is immediate to show that this duality property enables the transformation of the discrete orthogonality relation (4.13) into *discrete closure (completeness) relation* for the Kravchuk functions (4.12), which has the form

$$\sum_{n=0}^{2M}\psi_n^{(M)}(m)\,\psi_n^{(M)}(m')=\delta_{mm'}, \tag{4.22}$$

and *vice versa*. In other words, contrary to the continuous case when the two properties of the type (4.13) and (4.22) require separate proofs, here (4.13) and (4.22) are mutually linked because of the self-duality property (4.21).

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