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Bivariate raising and lowering differential operators for eigenfunctions of a 2D Fourier transform

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Abstract

We define a two-dimensional (2D) Fourier transform that self-reproduces a one-parameter family of bivariate Hermite functions; these are eigenfunctions of a Hamiltonian differential operator of second order, whose exponential is that transform. We find explicit forms of the bivariate raising and lowering partial differential operators of first degree for the eigenfunctions of this 2D Fourier transform.

Keywords: bivariate Fourier transform, bivariate Hermite polynomials, raising and lowering partial differential operators, eigenfunctions of the bivariate Fourier transform, two-dimensional harmonic oscillator

(Some figures may appear in colour only in the online journal)

1. Introduction

The Fourier transform, the quantum harmonic oscillator and the associated Lie algebraic relations, have been an unusually fruitful field of research, because beyond the classical definitions and theory, several extensions have been developed which continue to display their intimate connections, applying to diverse physical systems. These include signal analysis and image processing in their discrete and finite versions, which are fundamental for most nuclear models, and particularly their raising and lowering operators become realized in photonic devices that harbor entangled states of light. Deformations of the original mathematical frame have also yielded interesting results when the main components retain their roles: similar integral kernels, Hermite functions, and their raising and lowering operators closing into a Heisenberg–Weyl Lie algebra.

The interest of mathematical physics researchers in multivariate special functions has increased in connection with various possible generalizations of the well-known one-variable special functions to more than one-dimensions (1D) [1] that are not separable as Cartesian products. For instance, the multivariable generalization of Bessel functions [2], originally introduced by Appell and Kampé de Feriet [3], which arise in the problem of the elliptic motion of planets [4]. Correspondingly, the multivariate generalization of classical orthogonal polynomials has attracted great interest recently [5–7]; the Hermite polynomials of several variables appear quite naturally in the description of quantum systems governed by quadratic Hamiltonians, in the kinetic theory of gases, of fluctuations, and in optical systems (see e.g. [8–10] and references therein).

The subject of this work is the study of a new one-parameter family of two-dimensional (2D) generalizations of the classical Fourier transform, their Hermite function eigenvectors bound through raising and lowering operators. The basic and well-known frame is laid out in section 2 to serve as comparison for the expressions that are thereafter subject to generalization. Section 3 includes the definition of the multivariate Hermite polynomials and functions, which goes back to Charles Hermite [11] in 1864, and some of their integral and differential properties; however, their systematic use for the study of mathematical physics problems of Hamiltonian phase spaces is much more recent. In section 4 we introduce the definition of a bivariate Fourier transform that reproduces this particular family of bivariate Hermite functions. The bivariate raising and lowering differential operators for this set of deformed 2D oscillator eigenstates are given in section 5, and confirm that, here too, the bivariate Fourier transform is the exponential of the Hamiltonian operator. The concluding section 6 offers a brief discussion of some further research directions that may be of interest.

2. The harmonic oscillator and Fourier transform

The 1D Hermite polynomials and their properties are very well known [12, section 5.5]; here we mention only those germane to the solution of the harmonic oscillator system in quantum mechanics, whose wave functions in the dimensionless variable x are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\,\sqrt{\pi}}} H_n(x) \exp\left(-x^2/2\right), \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where $H_n(x)$ are the Hermite polynomials [17, (10.13.9)]

$$H_n(x) = (2x)^n {}_2F_0\left(\left. \begin{matrix} (1-n)/2 & -n/2 \\ - & \end{matrix} \right| -\frac{1}{x^2} \right) = (2x)^n \sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s}}{s!\,(n-2s)!}, \tag{2}$$

orthogonal with respect to the weight function $\exp(-x^2)$,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x) H_m(x) \mathrm{d}x = 2^n n!\, \delta_{n,m},$$

and where $\delta_{n,m}$ denotes the Kronecker delta.

In quantum mechanics they are eigenfunctions of the linear harmonic oscillator Hamiltonian

$$\mathbf{H}\,\psi_n(x) = \frac{1}{2}\left(x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\psi_n(x) = \left(n + \frac{1}{2}\right)\psi_n(x), \tag{3}$$

which is a self-adjoint second order differential operator. This is an orthonormal set of functions in the Hilbert space $\mathcal{L}^2(\mathbb{R})$ defined by closure with respect to the sesquilinear inner product

$$(\psi_n, \psi_m)_{\mathbb{R}} = \int_{\mathbb{R}} \psi_n(x)\psi_m(x)\mathrm{d}x = \delta_{n,m},$$

where integration is over the full real line $x \in \mathbb{R}$. The basis of functions $\{\psi_n\}_{n=0}^{\infty}$ is dense in $\mathcal{L}^2(\mathbb{R})$: any function $f(x) \in \mathcal{L}^2(\mathbb{R})$ can be approximated in the norm by the expansion $f(x) = \sum_{n=0}^{\infty} c_n\,\psi_n(x)$ and the expansion coefficients $\{c_n\}_{n=0}^{\infty}$ are determined by the integrals $c_n = \left(\psi_n, f\right)_{\mathbb{R}}$ for $n = 0, 1, \ldots$.

The Fourier transform of Lebesgue square-integrable functions on the real line $\mathbb{R}$ is the unitary map of the Hilbert space $\mathcal{L}^2(\mathbb{R})$ given by

$$f(x) \to \widetilde{f}(x) = (\mathbf{F}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(\mathrm{i}xx')f(x')\mathrm{d}x'. \tag{4}$$

Among its many well-known properties is that of being the fourth root of the unit operator, $\mathbf{F}^4 = 1$, and that it transforms the quantum harmonic oscillator wave functions (1) into themselves, multiplying them by the phase i^n,

$$\left(\mathbf{F}\psi_n\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(\mathrm{i}xx')\,\psi_n(x')\mathrm{d}x' = \mathrm{i}^n\,\psi_n(x). \tag{5}$$

Passing now to 2D of space, x, y, the isotropic 2D harmonic oscillator is characterized by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\left(x^2 + y^2\right).$$

The associated Schrödinger equation $\mathcal{H}\psi = \mathcal{E}\psi$ separates in Cartesian coordinates, yielding the 2D oscillator wave functions

$$\psi_{n,m}(x, y) = \frac{1}{\sqrt{2^{n+m}\pi n!m!}}\exp\left(-\frac{x^2 + y^2}{2}\right)H_n(x)H_m(y), \tag{6}$$

with energy eigenvalues $\mathcal{E} = n + m + 1$.

The Cartesian product of two ordinary Fourier transforms (4) in x and y is the well-known 2D Fourier transform

$$(\mathcal{F}f)(x, y) = \widetilde{f}(x, y) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} f(x', y')\exp(\mathrm{i}(xx' + yy'))\mathrm{d}x'\mathrm{d}y'. \tag{7}$$

The 2D harmonic oscillator wavefunctions (6) are naturally its eigenfunctions

$$\left(\mathcal{F}\psi_{n,m}\right)(x, y) = \mathrm{i}^{n+m}\,\psi_{n,m}(x, y). \tag{8}$$

The 2D Fourier transform (7) represents a *bivariate extension* of (4) in separated variables, characterized by the integral kernel

$$K(x, y; x', y') = \exp(\mathrm{i}(xx' + yy')). \tag{9}$$

Evidently, similar expressions are valid for the isotropic three- and higher-dimensional harmonic oscillators [5, 13, 14].

3. Bivariate Hermite polynomials and functions

Other ways of generalizing classical univariate Hermite polynomials [15] to bivariate expressions were proposed by Hermite himself [11], who introduced a class of polynomials in several variables with generating functions that are exponentials of quadratic forms. We briefly recall here some properties of the bivariate Hermite polynomials that were analyzed later by Appell and Kampé de Feriet [3].

Consider the positive definite quadratic form defined by

$$\varphi_\Lambda(x, y) = \mathbf{x}^t\Lambda\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2, \tag{10}$$

where $\Lambda = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $\mathbf{x} = (x \;\; y)^t$, $a, c > 0$, and $\delta = \det(\Lambda) = ac - b^2 > 0$. Then, as introduced by Hermite [3, p 373], and studied by means of a Rodrigues-type formula, the bivariate hypergeometric Hermite polynomials are defined by

$$H_{n,m}(x, y; \Lambda) = (-1)^{n+m} \exp\Big(\varphi_\Lambda(x, y)\Big)\frac{\partial^{n+m}}{\partial x^n\, \partial y^m}\Big[\exp\Big(-\varphi_\Lambda(x, y)\Big)\Big], \quad n, m = 0, 1, 2, \dots . \tag{11}$$

One particular case of (11), which has turned out to be a useful tool in the study of quantum-mechanical harmonic oscillator entangled states [16], corresponds to an antidiagonal Λ matrix, with $a = 0 = c$ and $b = 2/\tau$. In that case, the incomplete two-variable Hermite polynomials are defined by

$$\begin{aligned} h_{n,m}(x, y|\tau) &= \tau^{n+m} \exp\left(-\frac{xy}{\tau}\right)\frac{\partial^{n+m}}{\partial x^n\, \partial y^m}\left[\exp\left(\frac{xy}{\tau}\right)\right] \\ &= n!\, m! \sum_{r=0}^{\min(n,m)} \frac{\tau^r x^{n-r} y^{m-r}}{r!(n-r)!(m-r)!} \\ &= \min(n, m)!\, x^{\frac{1}{2}(|n-m|+m-n)}\, y^{\frac{1}{2}(|n-m|-m+n)}\, \tau^{\min(n,m)} L_{\min(n,m)}^{|n-m|}\left(-\frac{xy}{\tau}\right), \end{aligned} \tag{12}$$

where

$$L_n^\alpha(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!},$$

are the associated Laguerre polynomials [10].

Finally, when Λ is the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ($a = c = 1$ and $b = 0$), these bivariate polynomials return to the product of two univariate Hermite polynomials in separated variables

$$H_{n,m}(x, y; \mathbf{I}) = H_n(x)H_m(y), \tag{13}$$

where $H_n(x)$ and $H_m(y)$ are defined as in (2).

For general values of the parameters a, b, c, the polynomials (11) can be expressed in terms of $H_n(x)$ by the bilinear expression [3, p 370, equation (21)]

$$H_{n,m}(x, y; \Lambda) = \sum_{k=0}^{\min(n,m)} (-2)^k\, k!\, \binom{m}{k}\binom{n}{k} a^{\frac{n-k}{2}}\, b^k\, c^{\frac{m-k}{2}}\, H_{n-k}\left(\frac{ax+by}{\sqrt{a}}\right) H_{m-k}\left(\frac{bx+cy}{\sqrt{c}}\right). \tag{14}$$

These bivariate Hermite polynomials $H_{n,m}(x, y; \Lambda) = u(x, y)$ are of total degree $n + m$, and in particular $H_{0,0}(x, y; \Lambda) = 1$. They also satisfy the following second-order partial differential equation [17, p 288, equation (10)]

$$c\,\frac{\partial^2 u(x, y)}{\partial x^2} - 2b\frac{\partial^2 u(x, y)}{\partial x\,\partial y} + a\,\frac{\partial^2 u(x, y)}{\partial y^2} - 2\delta\left(x\frac{\partial u(x, y)}{\partial x} + y\frac{\partial u(x, y)}{\partial y}\right) + 2\delta\,(n+m)u(x, y) = 0, \tag{15}$$

where we recall that $\delta = ac - b^2 > 0$. This is a bivariate second-order linear partial differential equation that is admissible, potentially self-adjoint and of hypergeometric type [18]. Finally, a generating function for these bivariate Hermite polynomials has been given in [3, p 370, equation (18)].

The bivariate Hermite polynomials (11) of distinct total degrees are mutually orthogonal [15, 19] with respect to the weight function $\exp(-\varphi_\Lambda(x, y))$,

$$\iint_{\mathbb{R}^2} \exp\big(-\varphi_\Lambda(x, y)\big) H_{n,j}(x, y; \Lambda) H_{m,k}(x, y; \Lambda)\,\mathrm{d}x\,\mathrm{d}y = \Omega(n, j, m, k)\delta_{N,M}, \tag{16}$$

where $N = n + j$, $M = m + k$, and

$$\Omega(n, j, m, k) = \frac{2^{n+j}\pi n!\, a^m b^{k-j} c^j}{\sqrt{ac - b^2}} \sum_{s=0}^{j} (-1)^{j+s}(-k)_{j-s}(m - s + 1)_s \binom{j}{s}\left(\frac{b^2}{ac}\right)^s > 0,$$

where $(u)_m := \Gamma(u + m)/\Gamma(u)$ is the Pochhammer symbol. The orthogonality relation (16) is readily derived from (14) and (2) by using repeatedly the formula [20, equation (3.462.2), p 337]. Observe carefully that the orthogonality relation (16) implies that a polynomial $H_{n,j}(x, y; \Lambda)$ of total degree $n + j$ and $b \neq 0$, is *not* orthogonal to any other polynomial $H_{m,k}(x, y; \Lambda)$ of the same total degree $m + k = n + j$. Only in the 'Cartesian' case $b = 0$ orthogonality holds between *any two* bivariate Hermite polynomials.

We remind the reader that it is well known from the book by Appell and Kampé de Feriet [3] that the bivariate Hermite polynomials (11) are also biorthogonal. This existence of two types of orthogonality relations is intimately related with the fact that the spectrum of the second-order linear partial differential equation (16), that governs the bivariate Hermite polynomials (11), is degenerate.

Properties of the zeros of the bivariate Hermite polynomials (11) have been obtained recently [19] by considering the affine transformation

$$s = \frac{ax + by}{\sqrt{a}}, \qquad t = \frac{bx + cy}{\sqrt{c}}, \tag{17}$$

which enables one to transform (14) into

$$\widetilde{H}_{n,m}(s, t; \Lambda) = \sum_{k=0}^{\min(n,m)} (-2)^k\, k!\, \binom{m}{k}\binom{n}{k} a^{\frac{n-k}{2}}\, b^k c^{\frac{m-k}{2}} H_{n-k}(s) H_{m-k}(t), \tag{18}$$

for $n, m \geqslant 0$.

Notice in the definition of the quadratic form $\varphi_\Lambda(x, y)$ in (10), that the parameters a and c essentially produce only re-scalings in the directions of the x, y coordinates axes; we shall thus fix $a = 1 = c$, keeping only the parameter b in what follows, writing

$$\widetilde{H}_{n,m}(s, t; b) = \sum_{k=0}^{\min(n,m)} (-2)^k k! \binom{m}{k}\binom{n}{k} b^k H_{n-k}(s)H_{m-k}(t), \tag{19}$$

with $\delta = 1 - b^2 > 0$ for $n, m \geqslant 0$. These polynomials are solutions of the partial differential equation

$$\left[\frac{\partial^2}{\partial s^2} + 2b\frac{\partial^2}{\partial s\,\partial t} + \frac{\partial^2}{\partial t^2} - 2\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right) + 2(n+m)\right]\widetilde{H}_{n,m}(s, t; b) = 0, \tag{20}$$

satisfy the differential shift relations

$$\frac{\partial \widetilde{H}_{n,m}(s, t; b)}{\partial s} = 2n\,\widetilde{H}_{n-1,m}(s, t; b), \qquad \frac{\partial \widetilde{H}_{n,m}(s, t; b)}{\partial t} = 2m\,\widetilde{H}_{n,m-1}(s, t; b), \tag{21}$$

and exhibit the orthogonality relation

$$\iint_{\mathbb{R}^2} \exp\left(\frac{2bst - s^2 - t^2}{1 - b^2}\right)\widetilde{H}_{n,j}(s, t; b)\widetilde{H}_{m,k}(s, t; b)\mathrm{d}s\,\mathrm{d}t = \widetilde{\Omega}(n, j, m, k)\delta_{N,M}, \tag{22}$$

with $N = n + j$, $M = m + k$, and

$$\widetilde{\Omega}(n, j, m, k) = 2^{n+j}\pi\, n!\sqrt{1 - b^2}\, b^{k-j} \sum_{s=0}^{j}(-1)^{j+s}\binom{j}{s} \times (-k)_{j-s}(m - s + 1)_s\, b^{2s} > 0. \tag{23}$$

In particular, for $m = n$, $k = j$ and $b \in (-1, 1)$, one has the normalization in the form

$$\iint_{\mathbb{R}^2} \widetilde{H}_{n,j}(s, t; b)\widetilde{H}_{n,j}(s, t; b)\exp\left(\frac{2bst - s^2 - t^2}{1 - b^2}\right)\mathrm{d}s\,\mathrm{d}t = d_{n,j}(b)$$
$$d_{n,j}(b) = 2^{n+j}\pi\sqrt{1 - b^2}\, n!\, j!\; {}_2F_1\left(-j, -n; 1; b^2\right) > 0. \tag{24}$$

Finally, when $b = 0$, the case of separated variables described above, is recovered.

Following the common quantum mechanical oscillator functions (1), which incorporate the measure and normalization of the Hermite polynomials, attending (22) it is natural to define the *bivariate Hermite functions* associated to the bivariate Hermite polynomials $\widetilde{H}_{n,m}(s, t; b)$, as

$$\psi_{n,m}(s, t; b) = \frac{1}{\sqrt{d_{n,m}(b)}} \exp\left(-\frac{s^2 - 2bst + t^2}{2\left(1 - b^2\right)}\right)\widetilde{H}_{n,m}(s, t; b), \tag{25}$$

for $n, m \geqslant 0$. In particular, the ground state wave function is

$$\psi_{0,0}(s, t; b) = \frac{1}{\sqrt{\pi\sqrt{1 - b^2}}} \exp\left(-\frac{s^2 - 2bst + t^2}{2\left(1 - b^2\right)}\right). \tag{26}$$

This is a 2D Gaussian function, squeezed with the ratio $(1 + b)/(1 - b)$ in the direction given by the angle $\frac{1}{4}\pi\,\mathrm{sign}(b)$. In figure 1 we show three ground states corresponding to different values of b.

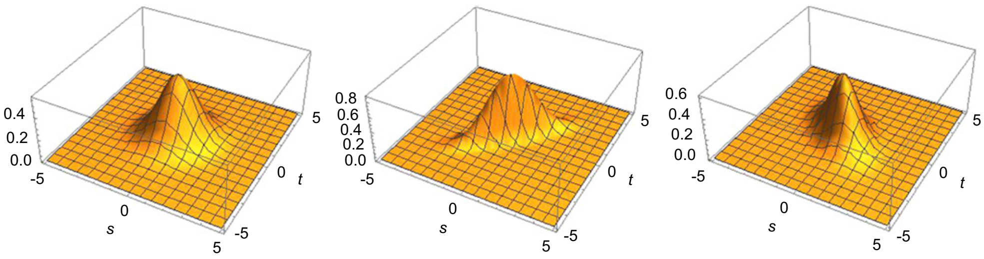

Figure 1. Ground states $\psi_{0,0}(s, t; b)$, for values of $b = 1/10$, $9/10$ and $-1/2$.

From (21) one readily derives the differential shift relations

$$\left(\frac{\partial}{\partial s} + \frac{s - bt}{1 - b^2}\right)\psi_{n,m}(s, t; b) = 2n\sqrt{\frac{d_{n-1,m}(b)}{d_{n,m}(b)}}\,\psi_{n-1,m}(s, t; b), \tag{27}$$

$$\left(\frac{\partial}{\partial t} + \frac{t - bs}{1 - b^2}\right)\psi_{n,m}(s, t; b) = 2m\sqrt{\frac{d_{n,m-1}(b)}{d_{n,m}(b)}}\,\psi_{n,m-1}(s, t; b). \tag{28}$$

As a consequence, the bivariate Hermite functions will shift both indices as

$$\left(\frac{\partial}{\partial s} + \frac{s - bt}{1 - b^2}\right)\left(\frac{\partial}{\partial t} + \frac{t - bs}{1 - b^2}\right)\psi_{n,m}(s, t; b) = 4nm\sqrt{\frac{d_{n-1,m-1}(b)}{d_{n,m}(b)}}\,\psi_{n-1,m-1}(s, t; b).$$

The bivariate Hermite functions are solutions of the partial differential equation (20), that we can cast in the form of a Hamiltonian second-order partial differential eigenfunction equation, which entangles the s and t directions of a 2D harmonic oscillator as

$$\mathcal{H}^{(b)}\,\psi_{n,m}(s, t; b) = (n + m + 1)\,\psi_{n,m}(s, t; b), \tag{29}$$

where the Hamiltonian operator is

$$\mathcal{H}^{(b)} := \frac{1}{2}\left[\frac{s^2 - 2bst + t^2}{1 - b^2} - \frac{\partial^2}{\partial s^2} - 2b\frac{\partial^2}{\partial s\,\partial t} - \frac{\partial^2}{\partial t^2}\right]. \tag{30}$$

Observe that, as in the case of ordinary Cartesian separation ($b = 0$), the spectrum of the bivariate second-order differential Hamiltonian operator $\mathcal{H}^{(b)}$ is finitely degenerate: $N + 1$ functions $\psi_{n,m}(s, t; b)$, $n + m = N$, with indices $(n, m) = \{(N, 0);\ (N - 1, 1);\ \ldots;\ (1, N - 1);\ (0, N)\}$ correspond to the same eigenvalue N of the operator $\mathcal{H}^{(b)}$.

The inner product of two bivariate Hermite functions corresponding to the 1D case (3) is, from (22)

$$\iint_{\mathbb{R}^2} \psi_{n,j}(s, t; b)\psi_{m,k}(s, t; b)\mathrm{d}s\,\mathrm{d}t = \delta_{n+j,m+k}\frac{\widetilde{\Omega}(n, j, m, k)}{\sqrt{d_{n,j}(b)d_{m,k}(b)}}, \tag{31}$$

where $\widetilde{\Omega}(n, j, m, k)$ is defined as in (23), which implies that each of these functions is indeed normalized

$$\iint_{\mathbb{R}^2} \left(\psi_{n,j}(s, t; b)\right)^2 \mathrm{d}s\,\mathrm{d}t = 1. \tag{32}$$

Moreover, if $n + j > m + k$ $(N > M)$, then

$$\iint_{\mathbb{R}^2} \psi_{n,j}(s,\, t;\, b)\psi_{m,k}(s,\, t;\, b)\mathrm{d}s\, \mathrm{d}t = 0. \tag{33}$$

4. The bivariate Fourier transform and its eigenfunctions

We define the *bivariate Fourier transform* as the following integral transform

$$\left(\mathcal{F}_b f\right)(x,\, y) = \widetilde{f}\,(s,\, t;\, b) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} f(x,\, y)\, K(s,\, t;\, x,\, y;\, b)\, \mathrm{d}x\, \mathrm{d}y, \tag{34}$$

with the kernel

$$K(s,\, t;\, x,\, y;\, b) = \frac{1}{\sqrt{1 - b^2}} \exp\left(\frac{\mathrm{i}}{\delta}[sx + ty - b(sy + tx)]\right), \tag{35}$$

and $\delta = 1 - b^2$; the exponent can be written as

$$sx + ty - b\,(sy + tx) = (s,\, t)\begin{pmatrix} 1 & -b \\ -b & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{36}$$

When $b = 0$ this kernel reduces to ordinary Fourier kernel (9).

One can verify that the bivariate Fourier transform $\mathcal{F}_b$ reproduces the ground state $\psi_{0,0}(x,\, y;\, b)$ in (26)

$$\begin{aligned}
\mathcal{F}_b\left(\psi_{0,0}(x,\, y;\, b)\right) &= \mathcal{F}_b\left(\frac{1}{\sqrt{\pi\sqrt{1 - b^2}}} \exp\left(-\frac{x^2 - 2bxy + y^2}{2\left(1 - b^2\right)}\right)\right) \\
&= \frac{1}{2\pi^{3/2}\left(1 - b^2\right)^{3/4}} \iint_{\mathbb{R}^2} \exp\left(-\frac{x^2 - 2bxy + y^2}{2\left(1 - b^2\right)}\right) K(s,\, t,\, x,\, y;\, b)\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{2\pi^{3/2}\left(1 - b^2\right)^{3/4}} \iint_{\mathbb{R}^2} \exp\left(\frac{2\mathrm{i}y(-t + b(s + \mathrm{i}x)) + x(2\mathrm{i}bt - 2\mathrm{i}s + x) + y^2}{2\left(b^2 - 1\right)}\right) \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{\sqrt{2}\,\pi\left(1 - b^2\right)^{1/4}} \int_{\mathbb{R}} \exp\left(\frac{b^2(t + \mathrm{i}y)^2 - 2bst + s^2 + y(y - 2\mathrm{i}t)}{2\left(b^2 - 1\right)}\right) \mathrm{d}y \\
&= \frac{1}{\sqrt{\pi}\left(1 - b^2\right)^{1/4}} \exp\left(\frac{-2bst + s^2 + t^2}{2\left(b^2 - 1\right)}\right) = \psi_{0,0}(s,\, t;\, b).
\end{aligned} \tag{37}$$

Then, for the all other functions $\psi_{n,m}(s,\, t;\, b)$ defined in (25), for any n and m nonnegative integers, from (18) and (25) it follows by integration that

$$\left(\mathcal{F}_b\, \psi_{n,m}\right)(x,\, y;\, b) = \mathrm{i}^{n+m}\, \psi_{n,m}(s,\, t;\, b). \tag{38}$$

Finally, we note that equation (29) indicates that the integral bivariate Fourier transform (34) can also be written in operator form as

$$\mathcal{F}_b = \exp\left(\mathrm{i}\frac{\pi}{2}\left(\mathcal{H}^{(b)} - 1\right)\right). \tag{39}$$

This type of exponential relation is well known to hold for the ordinary Fourier transform and the quantum harmonic oscillator Hamiltonian [21].

5. Bivariate raising and lowering operators

To construct raising and lowering operators for eigenfunctions of the Hamiltonian $\mathcal{H}^{(b)}$ in (30) and of the bivariate Fourier transform $\mathcal{F}_b$, we derive first some differential identities for its kernel (35), noting that one can variously write the exponent as

$$sx + ty - b(sy + tx) = (s - bt)x + (t - bs)y = (x - by)s + (y - bx)t,$$

so that its partial derivatives easily yield

$$\frac{\partial}{\partial s} K(s, t; x, y; b) = \frac{\mathrm{i}}{\delta}(x - by)\, K(s, t; x, y; b), \tag{40}$$

$$\frac{\partial}{\partial t} K(s, t; x, y; b) = \frac{\mathrm{i}}{\delta}(y - bx)\, K(s, t; x, y; b), \tag{41}$$

and

$$s\, K(s, t; x, y; b) = \frac{1}{\mathrm{i}}\left(\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) K(s, t; x, y; b), \tag{42}$$

$$t\, K(s, t; x, y; b) = \frac{1}{\mathrm{i}}\left(b\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) K(s, t; x, y; b). \tag{43}$$

With the aid of these identities the following relations for the bivariate Fourier transform (34) are proven

$$\frac{\partial}{\partial s}\tilde{f}(s, t; b) = \frac{\mathrm{i}}{\delta}\iint_{\mathbb{R}^2} K(s, t; x, y; b)(x - b\, y) f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{44}$$

$$\frac{\partial}{\partial t}\tilde{f}(s, t; b) = \frac{\mathrm{i}}{\delta}\iint_{\mathbb{R}^2} K(s, t; x, y; b)(y - b\, x) f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{45}$$

$$s\tilde{f}(s, t; b) = \mathrm{i}\iint_{\mathbb{R}^2} K(s, t; x, y; b)\left(\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{46}$$

$$t\tilde{f}(s, t; b) = \mathrm{i}\iint_{\mathbb{R}^2} K(s, t; x, y; b)\left(b\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{47}$$

Out of these relations one deduces that some linear combinations of the variables s and t and their partial derivatives $\partial/\partial s$ and $\partial/\partial t$ are reproduced by the bivariate Fourier transform $\mathcal{F}_b$. In particular, for the two linear combinations of the form

$$\mathbf{A} := \frac{1}{\sqrt{2}}\left[s + t + (1 + b)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\right],$$

$$\mathbf{A}^{\dagger} := \frac{1}{\sqrt{2}}\left[s + t - (1 + b)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\right], \tag{48}$$

one obtains their intertwining relations with this transform

$$\mathbf{A}\,\mathcal{F}_b = \mathrm{i}\,\mathcal{F}_b\,\mathbf{A}, \qquad \mathbf{A}^\dagger\mathcal{F}_b = -\mathrm{i}\,\mathcal{F}_b\,\mathbf{A}^\dagger. \tag{49}$$

This means that these two linear combinations (48) actually represent the bivariate lowering and raising differential operators for the eigenfunctions of the bivariate Fourier transform operator $\mathcal{F}_b$.

It is important to notice that although the identities (49) exactly reproduce similar identities between the univariable lowering $\mathbf{a} := \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$ and raising $\mathbf{a}^\dagger := \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)$ operators, and the standard Fourier transform operator (4) on $\mathbb{R}$, the action of the bivariate lowering $\mathbf{A}$ and raising $\mathbf{A}^\dagger$ differential operators on the eigenfunctions (25) of the bivariate Fourier transform operator $\mathcal{F}_b$ is distinct from the one-dimensional case. Indeed, by using the differential shift relations (27) and (28) it is not hard to evaluate that

$$\begin{aligned}\mathbf{A}\,\psi_{n,m}(s,\,t;\,b) = \sqrt{2}\,(1+b)\Bigg[& n\sqrt{\frac{d_{n-1,m}(b)}{d_{n,m}(b)}}\,\psi_{n-1,m}(s,\,t;\,b)\\ &+ m\sqrt{\frac{d_{n,m-1}(b)}{d_{n,m}(b)}}\,\psi_{n,m-1}(s,\,t;\,b)\Bigg],\end{aligned} \tag{50}$$

$$\begin{aligned}\mathbf{A}^\dagger\psi_{n,m}(s,\,t;\,b) = \frac{1}{\sqrt{2}}\Bigg[& \sqrt{\frac{d_{n+1,m}(b)}{d_{n,m}(b)}}\,\psi_{n+1,m}(s,\,t;\,b)\\ &+ \sqrt{\frac{d_{n,m+1}(b)}{d_{n,m}(b)}}\,\psi_{n,m+1}(s,\,t;\,b)\Bigg].\end{aligned} \tag{51}$$

Thus, if a function $\psi_{n,m}(s,\,t)$ is an eigenfunction of the operator $\mathcal{F}_b$, associated with the eigenvalue i^{n+m}, then the functions $\mathbf{A}\psi_{n,m}(s,\,t)$ and $\mathbf{A}^\dagger\psi_{n,m}(s,\,t)$ are definite linear combinations of *two* eigenfunctions of $\mathcal{F}_b$, associated with its eigenvalues i^{n+m-1} and i^{n+m+1}, respectively.

It is also readily verified that $\mathbf{A}$ and $\mathbf{A}^\dagger$ obey, after a simple renormalization of both, the standard Heisenberg commutation relation:

$$\left[\mathbf{A},\,\mathbf{A}^\dagger\right] = \mathbf{A}\mathbf{A}^\dagger - \mathbf{A}^\dagger\mathbf{A} = 2(1+b)\mathbf{I}, \tag{52}$$

where $\mathbf{I}$ is the unit operator. Finally, in complete analogy with the one-variable case one can define a *bivariate number operator* as

$$\mathcal{N}^{(b)} = \mathbf{A}^\dagger\mathbf{A} = \frac{1}{2}\left[(s+t)^2 - (1+b)^2\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)^2\right] - (1+b). \tag{53}$$

However, from (50) and (51) one then obtains that the number operator is *not* diagonal

$$\begin{aligned}\mathcal{N}^{(b)}\,\psi_{n,m}(s,\,t;\,b) &= (1+b)\Bigg[(n+m)\psi_{n,m}(s,\,t;\,b)\\ &+ n\sqrt{\frac{d_{n-1,m+1}(b)}{d_{n,m}(b)}}\,\psi_{n-1,m+1}(s,\,t;\,b) + m\sqrt{\frac{d_{n+1,m-1}(b)}{d_{n,m}(b)}}\,\psi_{n+1,m-1}(s,\,t;\,b)\Bigg].\end{aligned}$$

Nevertheless, the bivariate operators $\mathcal{N}^{(b)}$ and $\mathcal{H}^{(b)}$ *do* commute, as can be readily verified by using explicit forms (53) and (30) of the operators $\mathcal{N}^{(b)}$ and $\mathcal{H}^{(b)}$, respectively. This circumstance is a consequence of the degeneracy of the spectrum of the bivariate second-order differential Hamiltonian operator $\mathcal{H}^{(b)}$.

Observe also that, as it is evident from (49), the bivariate number operator $\mathcal{N}^{(b)}$ *commutes* with the bivariate Fourier transform $\mathcal{F}_b$, i.e., $[\mathcal{N}^{(b)}, \mathcal{F}_b] = 0$. Moreover, using the relations (47) it is not difficult to verify that the Hamiltonian operator $\mathcal{H}^{(b)}$ defined in (30) also commutes with the bivariate Fourier transform, $[\mathcal{H}^{(b)}, \mathcal{F}_b] = 0$, thus confirming that this bivariate Fourier transform has been defined consistently. All these expressions reproduce the well-known ordinary 2D Fourier transform when $b = 0$.

6. Concluding comments and outlook

We have studied a one-parameter family of 2D extensions of the bivariate Hermite functions and explicitly constructed a corresponding bivariate Fourier transform which reproduces them. Raising and lowering operators for the eigenfunctions of this bivariate Fourier transform are also explicitly found.

We may surmise that the existence of raising and lowering operators hints at the existence of an su(1,1) dynamical algebra in 2D space. It also seems possible to extend this construction to a fractionalization of the bivariate Fourier transform (34) by means of bilinear generating functions in the spirit of Namias [22], or within a larger symplectic group, such as was developed by Moshinsky and Quesne [23] through an integral transform representation. Both of these lines of research will be pursued in future work.

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