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Kravchuk oscillator revisited

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Abstract. The study of irreducible representations of Lie algebras and groups has traditionally considered their action on functions of a continuous manifold (e.g. the ‘rotation’ Lie algebra $so(3)$ on functions on the sphere). Here we argue that functions of a discrete variable —Kravchuk functions— are on equal footing for that study in the case of $so(3)$. They lead to a discrete quantum model of the harmonic oscillator, and offer a corresponding set of special function relations. The technique is applicable to other special function families of a discrete variable, which stem from low-dimensional Lie algebras and are stationary solutions for the corresponding discrete quantum models.

1. Introduction

The representation theory of Lie algebras and groups has provided a royal road to study families of special functions [1, 2]. Here we follow this road in the reverse direction: starting from a family of discrete special functions —Kravchuk polynomials— that satisfy a three-term recurrence relation and a difference equation, we find the Lie algebra in which they appear as bases for irreducible representations —in this case the rotation algebra $so(3)$. On the way, seen from the perspective of Hamiltonian systems, we identify the discrete physical system for which this family provides the stationary wave functions.

The strategy developed here opens the way to propose other known discrete special function families of (as yet unknown) Lie-algebraic import, and find the discrete (or continuous) Hamiltonian system that will realize them as proper wave functions. In Sect. 2 we condense the definition and relevant properties of the Kravchuk polynomials, and the Kravchuk functions that are orthonormal and complete in finite-dimensional vector spaces. Their recurrence relation and difference properties are made explicit in Sect. 3, and used in Sect. 4 to endow the family with a Hamiltonian and a position operator realized by matrices; when subject to obey the Hamilton equations, their commutator produces a momentum operator and provides the potential of the discrete Hamiltonian system whose stationary solutions are the proposed family of discrete functions. They will inherit a rich set of properties: in this $so(3)$ case they participate with the Wigner little- d functions in the fractional Fourier-Kravchuk transform [3] and define in Sect. 5 a *canonical* dual basis of momentum. In Sect. 6 we offer some applications, conclusions and comments on other discrete special functions that may be subject to this analysis.



2. Kravchuk polynomials and functions

Kravchuk polynomials were originally introduced as a generalization of Hermite polynomials [4], replacing their orthogonality relation under an integral over the real line x with Gaussian weight $\exp(-x^2)$, by a summation over $N + 1$ points with a binomial distribution for weight. Here we find it more useful to represent them as terminating hypergeometric series [5, pp. 237–241], i.e., polynomials of degree n in x ,

$$K_n(x; p, N) := {}_2F_1(-n, -x; -N; p^{-1}) \quad (1)$$

$$= \left(\frac{p-1}{p}\right)^n {}_2F_1\left(-n, x-N; -N; \frac{1}{1-p}\right) \quad (2)$$

$$= \left(\frac{p-1}{p}\right)^n K_n(N-x; 1-p, N), \quad (3)$$

for $n \in \{0, 1, 2, \dots, N\}$, $0 < p < 1$, where $N \in \mathbb{Z}_0^+$ is some nonnegative integer number, and where we passed from (1) to (2) by using the Euler transformation formula [5, Eq. (1.7.2)].

Since the hypergeometric function in (1) is symmetric with respect to first two parameters it follows that the Kravchuk polynomials are *self-dual*:

$$K_n(m; p, N) = K_m(n; p, N), \quad m, n \in \{0, 1, 2, \dots, N\}. \quad (4)$$

For generic values of the parameter p , the Kravchuk polynomials do not exhibit any special property under reflection of the argument m across the midpoint of its interval, $m \leftrightarrow N-m$. Only for $p = \frac{1}{2}$ do they possess definite parity:

$$K_n(m; \frac{1}{2}, N) = (-1)^n K_n(N-m; \frac{1}{2}, N). \quad (5)$$

We are also informed [5] of the discrete orthogonality relation satisfied by the Kravchuk polynomials over the range of their argument:

$$\sum_{m=0}^N \binom{N}{m} K_n(m; \frac{1}{2}, N) K_{n'}(m; \frac{1}{2}, N) = 2^N \binom{N}{n}^{-1} \delta_{n,n'}, \quad (6)$$

where $\binom{r}{s} := r!/s!(r-s)!$ is the binomial coefficient. Indeed, there is a corresponding orthogonality relation for $K_n(m; p, N)$ when the parameter p has a generic value [5, Eq. (9.11.2)]. Of course, being polynomials in m , the Kravchuk polynomials are defined for any real or complex value of m . To have explicit parity, we will shift the orthogonality interval so that it be symmetric under inversions, denoting $x_m := \frac{1}{2}N - m$ for the $N+1$ equidistant points in the symmetric interval $x \in [-\frac{1}{2}N, \frac{1}{2}N]$. When N grows without bound while that interval and the density of points grow as $\sim \sqrt{N}$, one recovers the Hermite polynomials [5, Eq. (9.11.15)],

$$H_n(x) = \lim_{N \rightarrow \infty} (2N)^{n/2} K_n\left(\sqrt{\frac{1}{2}N} \left(\sqrt{\frac{1}{2}N} - x\right); \frac{1}{2}, N\right). \quad (7)$$

Out of the Kravchuk polynomials with definite parity $K_n(m; \frac{1}{2}, N)$ and orthogonality (6), it is convenient to define the Kravchuk *functions*:

$$\psi_n^{(M)}(x_m) := c_{n,m}^{(M)} K_n(x_m + M; \frac{1}{2}, 2M), \quad (8)$$

of the argument x_m that ranges over the symmetric set of points $x_m = m - \frac{1}{2}N$, and where we denote $M := \frac{1}{2}N$; the coefficients $c_{n,m}^{(M)}$ we choose so as to have the orthonormality relation in its simplest form,

$$\sum_{x_m=-M}^M \psi_n^{(M)}(x_m) \psi_{n'}^{(M)}(x_m) = \delta_{n,n'}. \quad (9)$$

To find $c_{n,m}^{(M)}$ in (8), we note that $K_0(m; \frac{1}{2}, N) = 1$, so we can sum (9) using $\sum_{m=0}^N \binom{N}{m} = 2^N$, and conclude that $\psi_0^{(M)}(x)$ is the square root of the binomial distribution,

$$\psi_0^{(M)}(x) = \frac{1}{2^M} \sqrt{\binom{2M}{M+x}} = \frac{1}{2^M} \sqrt{\frac{(2M)!}{\Gamma(M+x+1)\Gamma(M-x+1)}}. \quad (10)$$

We shall call $\psi_0^{(M)}(x)$ the *ground state* in analogy with the quantum harmonic oscillator case. As we argued above, the argument x need not be an integer, because the ratio of gamma functions analytically extends the binomial to the complex plane, with poles on real line at $\pm x_k = M + k$, $k \in \{1, 2, \dots\}$; to avoid them, let us keep $x \in [-M, M]$.

Now, writing $c_{n,m}^{(M)} = d_n^{(M)} \psi_0^{(M)}(m)$ and replacing again in (9) to find $d_n^{(M)}$, we conclude that for $n \in \{0, 1, \dots, 2M\}$,

$$\psi_n^{(M)}(x_m) = \frac{(-1)^n}{2^M} \sqrt{\binom{2M}{n} \binom{2M}{M+x_m}} K_n(M+x_m; \frac{1}{2}, 2M) \quad (11)$$

provides an orthonormal basis for the $(2M+1)$ -dimensional vector space of functions of the discrete variable $x_m = M - m$ in the symmetric interval $x_m \in \{-M, -M+1, \dots, M\}$; the sign in (11) has been added so that $\psi_n^{(M)}(-M) > 0$, following the standard sign convention for the quantum harmonic oscillator wavefunctions $\Psi_n(x)$. Indeed, it has been known [6] that as M grows without bound, while the orthogonality interval and density of points grow as $\sim \sqrt{M}$ with $x\sqrt{M}$ integer, their limit is

$$\lim_{M \rightarrow \infty} (-1)^{x\sqrt{M}} M^{1/4} \psi_n^{(M)}(x\sqrt{M}) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x) = \Psi_n(x). \quad (12)$$

A consequence of the self-duality (4) of the symmetric Kravchuk polynomials and of their parity (5) is that, for $0 \leq n, m \leq N$,

$$\psi_n^{(M)}(x_m) = \psi_{M-m}^{(M)}(M-n) = (-1)^n \psi_n^{(M)}(-x_m), \quad (13)$$

and also that the basis of Kravchuk functions $\psi_n^{(M)}(x_m)$ is *complete*:

$$\sum_{n=0}^{2M} \psi_n^{(M)}(x_m) \psi_n^{(M)}(x_{m'}) = \delta_{m,m'}. \quad (14)$$

3. Three-term recurrence relations and difference equations

Two important properties satisfied by the Kravchuk polynomials is a three-term recurrence relation, which is inherited to the Kravchuk functions $\psi_n^{(M)}(x_m)$ in (11), and a difference equation, which in discrete systems takes the place of the Schrödinger equation. The factorization of this difference equation leads to the raising and lowering operators, which in turn determine the *mother* Lie algebra of the discrete system.

The three-term recurrence relation for Kravchuk polynomials [5, Eq. (9.11.3)] yields the corresponding relation for the Kravchuk functions (11),

$$-2x_m \psi_n^{(M)}(x_m) = \sqrt{(n+1)(2M-n)} \psi_{n+1}^{(M)}(x_m) + \sqrt{n(2M-n+1)} \psi_{n-1}^{(M)}(x_m). \quad (15)$$

Similarly, the difference equation for Kravchuk polynomials [5, Eq. (9.11.5)], with $x_m = M - m$ so that $x_{m\mp 1} = x_m \pm 1$, defines the ground state

$$\psi_0^{(M)}(x_m \pm 1) = \sqrt{\frac{M-x_m}{M+x_m \mp 1}} \psi_0^{(M)}(x_m), \quad (16)$$

and a relation between three neighboring points x_m of the Kravchuk functions,

$$2(M-n)\psi_n^{(M)}(x_m) = \sqrt{(M+x_m+1)(M-x_m)}\psi_n^{(M)}(x_m+1) + \sqrt{(M+x_m)(M-x_m+1)}\psi_n^{(M)}(x_m-1), \tag{17}$$

which we identify as the ‘Schrödinger’ difference equation that defines the discrete model of the quantum harmonic oscillator, as we shall proceed to show below.

The distinction between the three-term recurrence relation (15) and the difference equation (17), is that the latter can be slid along x , i.e., it remains valid if we replace $x_m \mapsto x_m + \varepsilon$, for any $-1 < \varepsilon < 1$; the limits are imposed by the poles that $\psi_n^{(M)}(x)$ inherits from (16) at $\pm x = M+k$, $k \in \{1, 2, \dots\}$. On the other hand, the recurrence relation (15) is between three successive states of this oscillator model, n and $n \pm 1$, except for the self-duality (4).

4. Geometry and dynamics in the $so(3)$ algebra

We can regard the difference equation (17) as an eigenvector equation for the Kravchuk functions $\psi_n^{(M)}(x_m)$, when these are accommodated into $2M+1$ column vectors $\boldsymbol{\psi}_n^{(M)}$ numbered by n , of rows $x_m = M-m$, with $n, m \in \{0, 1, \dots, 2M\}$. This is then a matrix equation that reads

$$\mathbf{H}^{(M)} \boldsymbol{\psi}_n^{(M)} = (M-n) \boldsymbol{\psi}_n^{(M)}, \quad M-n \in \{-M, -M+1, \dots, M\}, \tag{18}$$

governed by the $(2M+1) \times (2M+1)$ shifted Hamiltonian matrix $\mathbf{H}^{(M)} = \{H_{m,m'}^{(M)}\}_{m,m'=-M}^M$,

$$\mathbf{H}^{(M)} = \frac{1}{2} \begin{pmatrix} 0 & \alpha_{1-M}^{(M)} & 0 & \dots & 0 & 0 \\ \alpha_{1-M}^{(M)} & 0 & \alpha_{2-M}^{(M)} & \dots & 0 & 0 \\ 0 & \alpha_{2-M}^{(M)} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \alpha_M^{(M)} \\ 0 & 0 & 0 & \dots & \alpha_M^{(M)} & 0 \end{pmatrix}, \tag{19}$$

where from (17) we identify

$$\alpha_m^{(M)} := \sqrt{(x_m+M)(M-x_m+1)}, \quad x_m \in \{-M, -M+1, \dots, M\}. \tag{20}$$

Since the matrix (19) is symmetric and real, $H_{m,m'}^{(M)} = H_{m',m}^{(M)}$, it is self-adjoint. We have called it ‘shifted Hamiltonian’ because it is a second-difference operator, thus a finite analogue of the quantum second-differential Hamiltonian operators, and its equally-spaced spectrum is shifted from $\{\frac{1}{2}, \frac{3}{2}, \dots\}$ to $[-M, M]$. For short, we will henceforth call it simply ‘Hamiltonian’.

Having written the difference equation for Kravchuk functions as an eigenvalue problem for a Hamiltonian matrix, we can easily define a position matrix \mathbf{X} such that its eigenvalues count the rows of the vectors in that basis,

$$\mathbf{X}^{(M)} \boldsymbol{\psi}^{(M)}(x_m) = x_m \boldsymbol{\psi}^{(M)}(x_m), \quad x_m = M-m \in \{-M, -M+1, \dots, M\}. \tag{21}$$

This matrix is of course diagonal,

$$\mathbf{X}^{(M)} = \begin{pmatrix} -M & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-M & 0 & \dots & 0 & 0 \\ 0 & 0 & 2-M & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & M-1 & 0 \\ 0 & 0 & 0 & \dots & 0 & M \end{pmatrix}, \tag{22}$$

with elements x_m , and is also obviously self-adjoint and real.

Now that we have Hamiltonian and position matrices, \mathbf{H} and \mathbf{X} , we can investigate the geometry and dynamics afforded by the Lie commutator form of the Hamilton equations of mechanics. The first Hamilton equation is geometric: it defines the *momentum* matrix $\mathbf{P}^{(M)}$ as the vector tangent to the evolution trajectory,

$$\mathbf{P}^{(M)} := i[\mathbf{H}^{(M)}, \mathbf{X}^{(M)}], \quad (23)$$

where the factor i is required for the momentum matrix to be self-adjoint. It is

$$\mathbf{P}^{(M)} = \frac{i}{2} \begin{pmatrix} 0 & \alpha_{1-M}^{(M)} & 0 & \cdots & 0 & 0 \\ -\alpha_{1-M}^{(M)} & 0 & \alpha_{2-M}^{(M)} & \cdots & 0 & 0 \\ 0 & -\alpha_{2-M}^{(M)} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_M^{(M)} \\ 0 & 0 & 0 & \cdots & -\alpha_M^{(M)} & 0 \end{pmatrix}, \quad (24)$$

with elements also given in terms of the $\alpha_m^{(M)}$'s in (20). This matrix is an analogue of the momentum operator in quantum mechanics, $-i d/dx$; instead of a derivative, (24) acts as a weighted finite difference on $f(x_m)$ that yields $-i[\alpha_{m+1}^{(M)} f(x_{m+1}) - \alpha_{m-1}^{(M)} f(x_{m-1})]$.

The second Hamilton equation is dynamic and determines how the tangent momentum of the trajectory evolves under the Hamiltonian of the system. One finds that the Lie-commutator form of the second Hamilton equation returns the diagonal position matrix (22),

$$[\mathbf{H}^{(M)}, \mathbf{P}^{(M)}] = i\mathbf{X}^{(M)}. \quad (25)$$

The two Hamilton equations, (23) and (25), are those of a harmonic oscillator—classical or quantum—and combine to yield its Newton equation,

$$[\mathbf{H}^{(M)}, [\mathbf{H}^{(M)}, \mathbf{X}^{(M)}]] = \mathbf{X}^{(M)}. \quad (26)$$

The generated trajectories of position and momentum will thus follow harmonic motion.

Granted that we have a Hamiltonian system that is harmonic, we should lastly investigate the remaining commutator between the above position and momentum matrices. Instead of the quantum-mechanically expected unit matrix, we find

$$[\mathbf{X}^{(M)}, \mathbf{P}^{(M)}] = -i\mathbf{H}^{(M)}. \quad (27)$$

This is a consequence of the difference equation (17) satisfied by the Kravchuk functions $\psi_n^{(M)}(x)$ in (11), and the construction of the diagonal position matrix in (22). The three commutators (23), (25) and (27) close with the structure constants that characterize the rotation Lie algebra $so(3)$. This algebra is different from the 4-parameter noncompact *oscillator* algebra (which includes $\mathbf{1}$) that we would obtain had we started with the Hermite functions $\Psi_n(x)$ and their differential equation.

The orthogonality and completeness of the Kravchuk functions over a $(2M+1)$ -dimensional vector space, Eqs. (9) and (14), insure that the matrices are self-adjoint on that finite-dimensional space. We identify the self-adjoint irreducible representations to which the three matrices belong through the value of the Casimir invariant,

$$(\mathbf{H}^{(M)})^2 + (\mathbf{X}^{(M)})^2 + (\mathbf{P}^{(M)})^2 = M(M+1)\mathbf{1}^{(M)}, \quad (28)$$

i.e., it corresponds to generic 'spin' M .

5. The algebra $so(3)$ and dual canonical bases

In the previous section we presented matrices that we called Hamiltonian, position, and momentum, and which were built to act as difference operators on the vectors (that can be called states, or signals, or wavefunctions) in a complex vector space of integer dimension $2M+1$. We can identify these operators with the common notation for the generators of $so(3)$ as follows:

$$J_1 \leftrightarrow H, \quad J_2 \leftrightarrow P, \quad J_3 \leftrightarrow X, \quad (29)$$

(dropping the index (M)) with commutation relations $[J_i, J_j] = iJ_k$ (where i, j, k is a cyclic permutation of 1, 2, 3) that characterize $so(3) = su(2)$ and generate the corresponding Lie groups of spin, $SO(3)$ if M is integer and $SU(2)$ if it is half-integer.

The elements of $SO(3)$ are, with Euler angle parametrization,

$$\begin{aligned} \mathcal{R}(\alpha, \beta, \gamma) &= \exp(-i\alpha J_3) \exp(-i\beta J_2) \exp(-i\gamma J_3) \\ &\leftrightarrow \exp(-i\alpha X) \exp(-i\beta P) \exp(-i\gamma X), \end{aligned} \quad (30)$$

and their action $\mathcal{R} : \phi = \phi_R$ on any vector $\phi = \{\phi(x_m)\}_{m=0}^{2M}$ is given by a linear combination with Wigner *big-D* coefficients [7, Sect. 3.6],

$$\phi_R(x_m) := \left(\mathcal{R}(\alpha, \beta, \gamma) : \phi \right)(x_m) = \sum_{x_{m'}=-M}^M D_{x_m, x_{m'}}^M(\alpha, \beta, \gamma) \phi(x_{m'}). \quad (31)$$

In particular, an eigenvector of the position generator $J_3 \leftrightarrow X$, namely $\xi_{m'} = \{\delta_{x_{m'}, x_m}\}_{x_m=-M}^M$ (with fixed $x_{m'} = M - m' \in \{-M, -M+1, \dots, M\}$) can be transformed into the Kravchuk eigenvector of the Hamiltonian $J_1 \leftrightarrow H$, namely $\psi_{m'} = \{\psi_{m'}(x_m)\}_{x_m=-M}^M$ through a rotation of $\frac{1}{2}\pi$ around the axis of momentum, $J_2 \leftrightarrow P$. The Wigner matrix function $D_{x_m, x_{m'}}^M(0, \frac{1}{2}\pi, 0)$ in (31) reduces then to a *little-d* Wigner matrix function $d_{x_m, x_{m'}}^M(\frac{1}{2}\pi)$ while the sum over x_m collapses to the single $x_{m'}$. Thus we find the relation

$$\psi_{m'}^{(M)}(x_m) = d_{x_m, x_{m'}}^M(\frac{1}{2}\pi), \quad (32)$$

between the Kravchuk functions and the Wigner *d*-functions of angle $\frac{1}{2}\pi$.

Now a rotation $\mathcal{R}(0, 0, \frac{1}{2}\pi)$ by $\frac{1}{2}\pi$ around the $J_3 \leftrightarrow X$ axis, will bring the Hamiltonian $J_1 \leftrightarrow H$ to momentum $J_2 \leftrightarrow P$. The momentum eigenvectors are obtained from the Kravchuk eigenvectors of the Hamiltonian through the Wigner *big-D* matrix $D_{x_m, x_{m'}}^M(0, 0, \frac{1}{2}\pi) = \delta_{m, m'} \exp(-i\frac{1}{2}\pi x_{m'})$. This defines the momentum eigenvectors in the position basis to be

$$\tilde{\psi}_{m'}^{(M)}(x_m) := (-i)^{x_m} \psi_{m'}^{(M)}(x_m). \quad (33)$$

This basis is the *canonical dual* to the position eigenbasis of Kronecker deltas, $\xi_{m'}$ at $x_m = x'_{m'}$. Performing the rotation (33) twice, the $J_1 \leftrightarrow H$ axis is inverted, which inverts the Hamiltonian eigenvalues; this exchanges the bottom ψ_0 and top ψ_N states, and generally the states $n \in \{0, 1, \dots, N\}$ and $N - n$, with an alternating sign between every two positions,

$$\psi_{N-n}^{(M)}(x_m) := (-1)^{x_m} \psi_n^{(M)}(x_m). \quad (34)$$

Finally, a rotation by θ around the $J_1 \leftrightarrow H$ axis represents Hamiltonian evolution, rotating the momentum axis $J_2 \leftrightarrow P$ towards the position axis $J_3 \leftrightarrow X$, exactly as the harmonic oscillator does in quantum mechanics [8], which is the (inverse) fractional Fourier transform. This rotation is

$$\mathcal{F}(-\theta) := \exp(-i\theta H) = \mathcal{R}(-\frac{1}{2}\pi, \theta, \frac{1}{2}\pi), \quad (35)$$

and its action on any vector in the position basis $\phi(x_m)$, is then given by

$$\phi_\theta(x_m) := (\mathcal{F}(\theta) : \phi)(x_m) = \sum_{x_{m'}=-M}^M \exp(-i\theta(x_m - x_{m'})) d_{x_m, x_{m'}}^M(\theta) \phi(x_{m'}). \quad (36)$$

This is the fractional Fourier-Kravchuk transform [3]. In particular, acting on the Kravchuk functions,

$$(\mathcal{F}(\theta) : \psi_n^{(M)})(x_m) = \exp(-in\theta) \psi_n^{(M)}(x_m), \quad (37)$$

which is the same property enjoyed by the quantum harmonic oscillator wavefunctions under the fractional Fourier integral transform [8].

We close this section by reminding the reader that coherent states in the finite oscillator model can be obtained by rotating the ground state $\psi_0^{(M)}(x_m)$ to any (θ, ϕ) on the sphere with (31) [9]. These coherent states will perform harmonic motion under evolution by $H \leftrightarrow J_1$, although they are not shape-invariant as their continuous counterparts are. The rotation of points on the sphere projected on any plane is of course a harmonic motion, which is geometric rather than physical under the oscillator potential.

6. Conclusions

Starting with the family of Kravchuk functions, with their recurrence and finite difference properties, we have built the generators of the group of rotations of three-dimensional space $SO(3)$. The three generators thus define the discrete and finite model of the quantum harmonic oscillator, which shares with the continuous model both the geometry and dynamics of a Hamiltonian system. We should point out that the necessary prerequisites to follow this strategy were presumably available at least since the appearance of the monograph by Gel'fand, Minlos and Shapiro [10], published in Russian in 1953 —and of the Kravchuk polynomials [4], which were known since 1929.

It seems to us that the discrete Kravchuk oscillator did not attract attention earlier because it realizes *discrete* function bases for the finite-dimensional spaces of the irreducible representations of the rotation group $SO(3)$, while the attention of the authors of Ref. [10], and many contemporary and later authors, has been on bases of functions such as spherical harmonics, that are continuous functions over the manifold of the sphere.

It should be also pointed out that the Kravchuk polynomials *do* appear in the three-volume encyclopedic monograph by N.Ja. Vilenkin and A.U. Klimyk via the matrix elements of irreducible representations of the Lie group $SU(2)$, treated as functions of the column index (see [11, page 346]). Observe that in our terminology this corresponds to matrix elements in the dual canonical basis; this provides an algebraic reasoning on why matrix elements, as functions of the column index, are of clear interest and explains how they emerge from a group-theoretic point of view.

The approach followed in this work seems to be quite general. In particular, it can be used for constructing a discrete model for free Hamiltonian systems associated with the irreducible representations of the Euclidean group $ISO(2)$, of the Lie group $SU(1,1)$ in the discrete and continuous series of irreducible representations for models of discrete radial harmonic and repulsive oscillators [12]. This will be dealt with elsewhere.

One final remark must be made, however. During the discussion of this work at the VIII Symposium on Quantum Theory and Symmetries (Mexico City, August 5–9, 2013) we became aware, thanks to W. Miller, Jr., about the recent progress in establishing close links between the generic 3-parameter two-dimensional 2nd-order superintegrable system S_9 [13, 14] and hypergeometric orthogonal polynomials from the Askey scheme [5]. It turns out that various function space realizations of the quadratic Racah–Wilson algebra, which is the

symmetry algebra behind this superintegrable model S_9 , can be put into correspondence with *all* hypergeometric polynomials in the Askey scheme. Of course, these remarkable works [13, 14] thus reveal the physical interpretation and group-theoretic context of such intricate orthogonal families as the Wilson and Racah polynomials, which satisfy 2nd-order difference equations with the quadratic spectra [5]. But the Kravchuk polynomials are known to be solutions of the 2nd-order difference equation with the linear spectrum [5] and this fact is essential for constructing unitary irreducible representations of the rotation Lie algebra $so(3)$ in terms of the Kravchuk polynomials. Therefore it seems to us that in this particular case when one aims at revealing a physical interpretation and fundamental group-theoretic properties of the Kravchuk polynomials it quite suffices to exploit the much simpler Lie algebra $so(3)$ rather than the more elaborate algebraic approach of Kalnins, Miller, Jr. and Post.

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