Canonical transformations and accidental degeneracy. I. The anisotropic oscillator

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(Received 20 November 1972)

The problem of accidental degeneracy in quantum mechanical systems has fascinated physicists for many decades. The usual approach to it is through the determination of the generators of the Lie algebra responsible for the degeneracy. In these papers we want to focus from the beginning on the symmetry Lie group of canonical transformations in the classical picture. We shall then derive its representation in quantum mechanics. In the present paper we limit our discussion to the anisotropic oscillator in two dimensions, though we indicate possible extensions of the reasoning to other problems in which we have accidental degeneracy.

1. INTRODUCTION

The subject of accidental degeneracy in quantum mechanical systems, i.e., degeneracies not associated with obvious groups of symmetries, has fascinated physicists for many decades. The two classical problems in this field have been the isotropic harmonic oscillators and the particle in a Coulomb potential. The nature of the Lie algebra responsible for the accidental degeneracy in these problems has been known for a long time.[1,2] However, the Lie groups of canonical transformations generated by these Lie algebras, apart from their geometrical invariance subgroups, have been discussed only recently.[3,4]

Besides the problems mentioned, there are others that present features of accidental degeneracy in the quantum picture. The question is then raised about general procedures for obtaining the Lie groups of canonical transformations responsible for these features.

To be able to focus on these procedures we decided to analyze systematically three simple problems in two dimensional configuration space that have accidental degeneracy: (1) the anisotropic oscillator when the ratio of the frequencies is rational,[1,5,6] (2) the isotropic oscillator constrained to a sector of the plane of angle π/q with q integer, (3) the Calogero problem[7] of particles moving in one dimension and interacting through potentials that depend both in the square and the inverse square of the distance between the particles. When we are dealing with three particles and eliminate the center of mass this problem can be reformulated in a two dimensional configuration space.

In this and the following paper we analyze cases (1) and (2), reserving the Calogero[7] problem for a later publication. While we shall be discussing very special systems, we will continuously try to keep in mind the general ideas behind these problems to see what is the information they supply on the abstract question of Lie groups of canonical transformations and accidental degeneracy.

2. ACCIDENTAL DEGENERACY IN AN ANISOTROPIC OSCILLATOR WHOSE FREQUENCIES HAVE A RATIONAL RATIO

The anisotropic oscillator whose frequencies have a rational ratio has been extensively discussed in the literature.[1,5,6] In the pioneering work of Jauch and Hill[1] the generators of the Lie algebra for both the isotropic and anisotropic oscillator (in the latter case for the two dimensional problem where the ratio of the frequencies was 1: 2) were obtained in the classical picture. Demkov[5] then discussed the different subsets of the set of states of the anisotropic oscillator that have the familiar degeneracy associated with $SU(2)$, and obtained the generators of this group in the quantum picture. Cisneros and McIntosh[6] greatly extend and complement the analysis in their search for a universal symmetry group in two dimensions.

From these and other papers it would appear that further discussion of the problem is unnecessary. The present approach differs though in that it goes directly into the determination of the canonical transformation that in the classical picture maps the anisotropic oscillator on the isotropic one. As the latter has a symmetry group of linear canonical transformations[3] that are a representation of $SU(2)$, we can combine them with those that give the mapping, to obtain the symmetry group of the anisotropic oscillator. Once the classical picture is clear we can pass to the creation and annihilation operators in the quantum picture which have different forms for the different subsets of states mentioned in the previous paragraph.[5] From them we can construct the generators of the $SU(2)$ group responsible for the accidental degeneracy of the two dimensional anisotropic oscillator whose frequencies have a rational ratio.

Besides its intrinsic interest, the present approach provides part of the ground work required in the next paper where we analyze the accidental degeneracy of the oscillator in a sector of angle π/q. It may also be useful in other problems[7] that have an energy spectrum similar to that of the anisotropic oscillators.[7]

Let us consider now a particle of mass unity moving in a plane under the influence of a quadratic potential whose frequencies in the X_i, $i = 1, 2$, directions are ω_i. The Hamiltonian is then

$$H = \tfrac{1}{2}(P_1^2 + \omega_1^2 X_1^2) + \tfrac{1}{2}(P_2^2 + \omega_2^2 X_2^2). \qquad (2.1)$$

We shall assume, furthermore, that

$$\omega_1/\omega_2 = k_2/k_1, \quad \text{or } k_1\omega_1 = k_2\omega_2 \equiv \omega, \qquad (2.2)$$

where k_1, k_2 are relatively prime integers. Without loss of generality we may take $\omega = 1$ or, equivalently,

$$\omega_i = k_i^{-1}, \quad i = 1, 2. \qquad (2.3)$$

We now introduce creation and annihilation variables under the definitions

$$\eta_i \equiv (1/\sqrt{2}),\ (k_i^{-1/2} X_i - i k_i^{1/2} P_i),$$
$$\xi_i \equiv (1/\sqrt{2}),\ (k_i^{-1/2} X_i + i k_i^{1/2} P_i), \quad i = 1, 2. \qquad (2.4)$$

In the quantum mechanical picture where $[X_i, P_j] = i\delta_{ij}$ (we take $\hbar = 1$), the variables η_i, ξ_i become operators and the Hamiltonian (2.1) takes then the form

$$H = k_1^{-1}\eta_1\xi_1 + k_2^{-1}\eta_2\xi_2 + (2k_1)^{-1} + (2k_2)^{-1}. \tag{2.5}$$

The eigenstates of H are given by

$$|\nu_1\nu_2\rangle = (\nu_1!\,\nu_2!)^{-1/2}\eta_1^{\nu_1}\eta_2^{\nu_2}|0\rangle, \tag{2.6}$$

where ν_1, ν_2 are nonnegative integers and $|0\rangle$ is the ground state $\pi^{-1/2}\exp[-\frac{1}{2}(X_1^2 + X_2^2)]$.

We now proceed to divide the set of states (2.6) into subsets characterized by a pair of indices (λ_1, λ_2) given by

$$\nu_i \equiv \lambda_i \bmod k_i, \quad \lambda_i = 0, 1, 2, \cdots k_i - 1, \quad i = 1, 2. \tag{2.7}$$

From the range of values of the λ_i we conclude that there are k_1k_2 different subsets of states, which can be characterized by the kets

$$|n_1k_1 + \lambda_1, n_2k_2 + \lambda_2\rangle \tag{2.8}$$

for given λ_1, λ_2 restricted as in (2.7) and for arbitrary nonnegative n_1, n_2. Immediately we see that the states (2.8) are eigenstates of H with eigenvalues

$$E = (n_1 + n_2) + k_1^{-1}(\lambda_1 + \tfrac{1}{2}) + k_2^{-1}(\lambda_2 + \tfrac{1}{2}), \tag{2.9}$$

and, hence, those members of the subset (2.7) for which $n_1 + n_2$ is the same have equal energy and give rise to accidental degeneracy. It is important to stress that the accidental degeneracy is present only for states within the subset labelled by (λ_1, λ_2) and not for those belonging to different subsets even if their $n_1 + n_2$ happen to be the same. We note also that each subset of states can be put into one-to-one correspondence with the full set of states of the isotropic oscillator. Thus, we can speak of k_1k_2 copies of the fundamental degeneracy pattern.

The question of what is the Lie algebra and the Lie group responsible for this accidental degeneracy then arises. We shall endeavor to answer it both in the classical and quantum picture in the next sections.

3. THE SYMMETRY LIE ALGEBRA AND GROUP FOR THE CLASSICAL ANISOTROPIC OSCILLATOR

We proceed first to analyze the classical system. The Hamiltonian then has the form (2.5) where we suppress the last two constant terms and in which η_i, ξ_i are the combinations (2.4) of classical coordinates and momenta. We shall introduce a canonical transformation which maps this Hamiltonian to another one corresponding to an isotropic harmonic oscillator where the Lie algebra of the symmetry group is well known.[3]

Before proceeding with our analysis, we note that from (2.4) the classical Poisson bracket of two observables F, G can be written as

$$\{F, G\} = \sum_i\left(\frac{\partial F}{\alpha X_i}\frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial X_i}\right) = i\sum_i\left(\frac{\partial F}{\partial\eta_i}\frac{\partial G}{\partial\xi_i} - \frac{\partial F}{\partial\xi_i}\frac{\partial G}{\partial\eta_i}\right) \tag{3.1}$$

which implies that $\{\eta_j, \xi_k\} = i\delta_{jk}$. Thus, if we have η_j', ξ_k' as functions η_j, ξ_k such that $\{\eta_j', \xi_k'\} = i\delta_{jk}$, we can be sure that X_j', P_j' defined as

$$X_j' = (1/\sqrt{2}),(\eta_j' + \xi_j'), \quad P_j' = (i/\sqrt{2}),(\eta_j' - \xi_j'), \quad j = 1, 2 \tag{3.2}$$

are canonically conjugate. If ξ_i' is also the complex conjugate of η_i', the canonical transformation is real.

We now consider the following canonical transformation in the classical picture

$$\eta_i' = k_i^{-1/2}(\eta_i\xi_i)^{(1-k_i)/2}\eta_i^{k_i}, \quad \xi_i' = \xi_i^{k_i}k_i^{-1/2}(\eta_i\xi_i)^{(1-k_i)/2}. \tag{3.3}$$

From (3.1) we see that $\{\eta_j', \xi_k'\} = i\delta_{jk}$ and, besides, as $\xi_i = \eta_i^*$ we obtain $\xi_i' = \eta_i'^*$. Furthermore,

$$H = k_1^{-1}\eta_1\xi_1 + k_2^{-1}\eta_2\xi_2 = \eta_1'\xi_1' + \eta_2'\xi_2'. \tag{3.4}$$

Thus (3.3) is a real canonical transformation that reduces the Hamiltonian (2.5) to that of an isotropic harmonic oscillator.

The symmetry group[3] for the two dimensional isotropic harmonic oscillator is the unitary group $U(2)$ whose generators are

$$\eta_i'\xi_j', \quad i, j = 1, 2 \tag{3.5}$$

and for which the Lie algebra is determined by the Poisson brackets

$$\{\eta_i'\xi_j', \eta_k'\xi_l'\} = -i(\eta_i'\xi_l'\delta_{kj} - \eta_k'\xi_j'\delta_{il}). \tag{3.6}$$

The Lie symmetry group of the anisotropic oscillator relates the creation and annihilation classical variables $\bar{\eta}_i, \bar{\xi}_i$ to η_i, ξ_i through the following steps: First, inverting the relations (3.3) and writing all variables with a bar above we get

$$\begin{aligned}\bar{\eta}_i &= k_i^{1/2}\bar{\eta}_i'^{(k_i+1)/2k_i}\bar{\xi}_i'^{(k_i-1)/2k_i},\\ \bar{\xi}_i &= k_i^{1/2}\bar{\xi}_i'^{(k_i+1)/2k_i}\bar{\eta}_i'^{(k_i-1)/2k_i}.\end{aligned} \tag{3.7a}$$

Then we note that $\bar{\eta}_i', \bar{\xi}_j'$ are related to η_i', ξ_j' by a unitary transformation generated by (3.5) and, thus, we can write[3]

$$\bar{\eta}_i' = \sum_j U_{ij}\eta_j', \quad \bar{\xi}_i' = \sum_j U_{ij}^*\xi_j', \tag{3.7b}$$

where $\|U_{ij}\|$ is a 2×2 unitary matrix and $\|U_{ij}^*\|$ is its complex conjugate. Finally, η_i', ξ_j' are related to η_i, ξ_j through (3.3), which we can also write as

$$\eta_i' = k_i^{-1/2}\eta_i^{(k_i+1)/2}\xi_i^{(1-k_i)/2}, \quad \xi_i' = k_i^{-1/2}\xi_i^{(k_i+1)/2}\eta_i^{(1-k_i)/2}. \tag{3.7c}$$

It is clear that the full transformation (3.7) leaves the Hamiltonian (3.4) invariant and, thus, is a realization of $U(2)$ which is the symmetry Lie group of the anisotropic oscillator. We can then make use of the transformations (2.4) and their inverse to express the elements of this group as real canonical transformations.

Having analyzed the classical Lie algebra and symmetry group, we turn now our attention to the quantum picture.

4. THE GENERATORS AND THE UNITARY REPRESENTATION OF THE SYMMETRY GROUP IN THE QUANTUM PICTURE

In the quantum picture the creation and annihilation variables η_i, ξ_i become operators. Therefore, η_i', ξ_i' of (3.3) must also be expressed as operators that act on the states (2.7). As η_i, ξ_i do not commute, there are ambiguities in the translation of the classical relations

(3.3) into operator form. How can we get rid of these ambiguities? We shall use the isotropic oscillator as a guide. In there $k_1 = k_2 = 1$, implying $\lambda_1 = \lambda_2 = 0$, so that we have a single set of states of the form (2.7), i.e., $|n_1 n_2\rangle$. At the same time, when $k_1 = k_2 = 1$, $\eta_i' = \eta_i$, $\xi_i' = \eta_i$, $\xi_i' = \xi_i$. Thus, in the isotropic case we have

$$\begin{aligned} \eta_1'|n_1n_2\rangle &= (n_1+1)^{1/2}|n_1+1,n_2\rangle,\\ \eta_2'|n_1n_2\rangle &= (n_2+1)^{1/2}|n_1,n_2+1\rangle,\\ \xi_1'|n_1n_2\rangle &= n_1^{1/2}|n_1-1,n_2\rangle,\\ \xi_2'|n_1n_2\rangle &= n_2^{1/2}|n_1,n_2-1\rangle. \end{aligned} \tag{4.1}$$

For the anisotropic case we require the operators η_i', ξ_j' to have the same effect on each subset of states $|n_1k_1 + \lambda_1, n_2k_2 + \lambda_2\rangle$ characterized by fixed λ_1, λ_2. This would automatically[3] guarantee that the generators $\eta_i'\xi_j'$ of $U(2)$ connect the states (of the subset of given λ_1, λ_2) for which $n_1 + n_2$ is fixed, i.e., of the same energy, showing that this symmetry group is responsible for the accidental degeneracy in the anisotropic oscillator whose frequencies have a rational ratio.

For fixed λ_1, λ_2 we shall now define the creation and annihilation operators η_i', ξ_i' as

$$\eta_i' = k_i^{-1/2}(\eta_i\xi_i - \lambda_i)^{1/2}[(\eta_i\xi_i)(\eta_i\xi_i - 1)\cdots (\eta_i\xi_i - k_i + 1)]^{-1/2}\eta_i^{k_i}, \tag{4.2a}$$

$$\xi_i' = \xi_i^{k_i}[(\eta_i\xi_i)(\eta_i\xi_i - 1)\cdots (\eta_i\xi_i - k_i + 1)]^{-1/2}(\eta_i\xi_i - \lambda_i)^{1/2}k_i^{-1/2}. \tag{4.2b}$$

We claim that (4.2) are the right quantum analogies of (3.3) when applied to the eigenstates (2.7) of the number operators $\eta_i\xi_i$, $i = 1, 2$. First, they are well defined in this basis. Second, the classical limit $\hbar \to 0$ of (4.2) is (3.3) as can be seen by keeping $\hbar$ and ω in the notation.

We now apply η_1' to the state

$$\begin{aligned} \eta_1'|n_1k_1+\lambda_1, n_2k_2+\lambda_2\rangle &= k_1^{-1/2}(\eta_1\xi_1-\lambda_1)^{1/2}[(\eta_1\xi_1)(\eta_1\xi_1-1)\cdots (\eta_1\xi_1-k_1+1)]^{-1/2}[(n_1k_1+\lambda_1)!(n_2k_2+\lambda_2)!]^{-1/2}\\ &\qquad\times \eta_1^{(n_1+1)k_1+\lambda_1}\eta_2^{n_2k_2+\lambda_2}|0\rangle\\ &= (n_1+1)^{1/2}\{[(n_1+1)k_1+\lambda_1]![n_2k_2+\lambda_2]!\}^{-1/2}\\ &\qquad\times \eta_1^{(n_1+1)k_1+\lambda_1}\eta_2^{n_2k_2+\lambda_2}|0\rangle\\ &= (n_1+1)^{1/2}|(n_1+1)k_1+\lambda_1, n_2k_2+\lambda_2\rangle. \end{aligned} \tag{4.3}$$

In a similar fashion, we can apply η_2', ξ_1', ξ_2' to (2.7) and we get kets in which, respectively, $n_1, n_2 \to n_1, n_2 + 1$; $n_1, n_2 \to n_1 - 1, n_2$; $n_1, n_2 \to n_1, n_2 - 1$, multiplied by factors $(n_2+1)^{1/2}$, $n_1^{1/2}$, $n_2^{1/2}$. It is important to stress that for each set $(\lambda_1\lambda_2)$ we have different η_i', ξ_i' as indicated in (4.2). In particular, (4.3) does not hold if the λ's for the operators and the kets do not match.

Let us introduce, for fixed λ_1, λ_2, a shorthand notation for the ket (2.7) of the form

$$|n_1k_1 + \lambda_1, n_2k_2 + \lambda_2\rangle \equiv |jm\}, \tag{4.4a}$$

in which

$$j \equiv \tfrac{1}{2}(n_1 + n_2), \qquad m \equiv \tfrac{1}{2}(n_1 - n_2). \tag{4.4b}$$

Furthermore, we denote the generators of our $SU(2)$ symmetry group by the notation

$$T_+ = T_1 + iT_2 = \eta_1'\xi_2', \qquad T_3 = \tfrac{1}{2}(\eta_1'\xi_1' - \eta_2'\xi_2'),\qquad T_- = T_1 - iT_2 = \eta_2'\xi_1'. \tag{4.5}$$

From (4.3) and similar relations, we obtain then

$$\begin{aligned} \{j'm'|T_\pm|jm\} &= [(j \mp m)(j \pm m + 1)]^{1/2}\delta_{jj'}\delta_{mm'},\\ \{j'm'|T_3|jm\} &= m\delta_{jj'}\delta_{mm'}. \end{aligned} \tag{4.6}$$

thus seeing that in each of the k_1k_2 subsets of states of the anisotropic oscillator with fixed λ_1, λ_2, the matrix elements of the generators of $SU(2)$ have the standard form.

We can now turn to the question of the unitary representation of the $SU(2)$ symmetry group in the quantum picture. In the usual way, we define[8]

$$R(\alpha,\beta,\gamma) = e^{i\alpha T_3}e^{i\beta T_2}e^{i\gamma T_3}, \tag{4.7}$$

and have

$$R(\alpha,\beta,\gamma)|jm\}_{\lambda_1\lambda_2} = \sum_{m'}|jm'\}_{\lambda_1\lambda_2}\mathfrak{D}^j_{m'm}(\alpha\beta\gamma), \tag{4.8}$$

where the $\mathfrak{D}$'s are the Wigner matrices.[8] We stress the fact that both $|jm\}$ and $|jm'\}$ [as well as the operator $R(\alpha,\beta,\gamma)$] in (4.8) correspond to the same subset characterized by a fixed λ_1, λ_2, by now adding a subscript $\lambda_1\lambda_2$ to the kets $|jm\}$.

We, furthermore, note that

$$\begin{aligned} {}_{\lambda_1'\lambda_2'}\{j'm'|jm\}_{\lambda_1\lambda_2} &= \langle n_1'k_1 + \lambda_1', n_2'k_2 + \lambda_2'|n_1k_1 + \lambda_1, n_2k_2 + \lambda_2\rangle\\ &= \delta_{n_1'n_1}\delta_{n_2'n_2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}. \end{aligned} \tag{4.9}$$

Actually, the matrix element (4.9) is different from 0 only if

$$n_ik_i + \lambda_i = n_i'k_i + \lambda_i' \quad \text{or} \quad (n_i - n_i')k_i = \lambda_i' - \lambda_i, \tag{4.10}$$

but as $\lambda_i, \lambda_i' = 0, 1, \cdots k_i - 1$, the Eq. (4.10) has a solution only when $n_i = n_i'$, $\lambda_i' = \lambda_i$.

From (4.8), (4.9) we then reach the conclusion that the unitary representation of the $SU(2)$ symmetry group responsible for the accidental degeneracy, with respect to the eigenstates (2.7) of the Hamiltonian λ, is given by

$${}_{\lambda_1'\lambda_2'}\{j'm'|R(\alpha,\beta,\gamma)|jm\}_{\lambda_1\lambda_2} = \delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\delta_{j'j}\mathfrak{D}^j_{m'm}(\alpha\beta\gamma). \tag{4.11}$$

We have obtained, from (4.2) and (4.5) the generators of the $SU(2)$ group and from (4.11) its unitary representation. In the next section we analyze the general conclusions that one can draw from the complete analysis of the group responsible for the accidental degeneracy of the anisotropic oscillator.

5. CONCLUSIONS

From the analysis of the problem of accidental degeneracy in an anisotropic oscillator system whose ratio of frequencies is rational, one possible general procedure emerges.

We first must solve fully the quantum mechanical problem and see what is the structure of the set of states that have the same energy. If this happens to be a structure that we normally associate with an $SU(2)$ group, i.e., we have sets of states that have degeneracy 1, 2, 3, 4, $\cdots$, we can look into the possibility of finding a classical canonical transformation that maps the Hamiltonian of the problem into that of a two dimensional harmonic oscillator. If the structure of accidental degeneracy is similar to that of other well-known problems in mechanics, the classical canonical transformations we may look for is the one that maps our problem into the well-known one.

Once we have this canonical transformation we can rewrite the generators of the Lie Algebra of the well-known problem in a way that makes them the generators of the symmetry group of the problem under study. We can then obtain the Lie symmetry group by a procedure similar to that in (3.7).

But our problem does not finish with the analysis in classical mechanics. We must then express the generators of our group, and frequently the creation and annihilation operators from which they are built, in the quantum picture as is done for example in (4.2). We expect this quantum mechanical formulation to reduce to the classical one in the limit $\hbar \to 0$, but we may find, as was clearly seen in Sec. 4, that the generators of the Lie algebra may have different forms in the different subsets of states of the quantum problem.

Once an explicit form of the generators is available in the quantum picture, we could pass to the determination of the unitary representation of the group of canonical transformation along the lines also discussed in Sec. 4. While the procedure outlined for finding the groups responsible for accidental degeneracy seems fairly general, we shall show in the next article that it does not apply to some other simple problems. We shall illustrate an alternative development when we discuss in the following paper the problem of the isotropic oscillator in a sector of angle π/q where q is integer.

We are indebted to Dr. P. A. Mello for many useful discussions.

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