

## Dependence of $s$ -waves on continuous dimension: The quantum oscillator and free systems

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Wavefunctions with rotational symmetry (i.e., zero angular momentum) in  $D$  dimensions, are called  $s$ -waves. In quantum quadratic systems (free particle, harmonic and repulsive oscillators), their radial parts obey Schrödinger equations with a fictitious centrifugal (for integer  $D \geq 4$ ) or centripetal (for  $D = 2$ ) potential. These Hamiltonians close into the three-dimensional Lorentz algebra  $so(2, 1)$ , whose exceptional interval corresponds to the critical range of continuous dimensions  $0 < D < 4$ , where they exhibit a one-parameter family of self-adjoint extensions in  $\mathcal{L}^2(\mathfrak{R}^+)$ . We study the characterization of these extensions in the harmonic oscillator through their spectra which – except for the Friedrichs extension – are not equally spaced, and we build their time evolution Green function. The oscillator is then contracted to the free particle in continuous- $D$  dimensions, where the extension structure is maintained in the limit of continuous spectra. Finally, we compute the free time evolution of the expectation values of the Hamiltonian, dilatation generator, and square radius between three distinct sets of ‘heat’-diffused localized eigenstates. This provides a simple group-theoretic description of the purported contraction/expansion of Gaussian-ring  $s$ -waves in  $D > 0$  dimensions.

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### 1 Introduction: $s$ -waves

Rotationally symmetric states of central potentials, including the free particle, are called  $s$ -waves for short; they are of interest especially in low dimensions  $D = 1, 2, 3$  and 4, both for quantum and wave phenomena [1]. In this paper we study the dependence of  $s$ -waves on *continuous* dimension  $D > 0$ . We note that the range  $0 < D < 4$  is especially interesting because it corresponds to the *exceptional interval* – identified by Bargmann [2] – of unitary irreducible representations of the three-dimensional Lorentz group  $SO(2, 1)$ . In this interval, the radial quantum harmonic oscillator and free particle Hamiltonians admit one-parameter families of self-adjoint extensions. In this context we examine the Green functions that drive time evolution unitarily; this analysis is made on the quantum harmonic oscillator, which is then contracted to the free limit. The expectation values of  $so(2, 1)$  generators are computed between normalized ‘radial Gaussians’ that are obtained through ‘heat’ diffusion of their  $\delta$ -type eigenfunctions. In particular, we find the generic behavior of the expectation value of the square radius.

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Models of wave and diffusion phenomena are ruled by the Laplacian which, in  $D$ -dimensional spherical coordinates  $\vec{q}(\rho, \Omega) \in \mathbb{R}^D$ , is

$$\nabla_{\vec{q}}^2 = \frac{\partial^2}{\partial \rho^2} + \frac{D-1}{\rho} \frac{\partial}{\partial \rho} + \frac{L(\Omega)^2}{\rho^2}, \quad (1)$$

where  $L^2(\Omega)$  is the square angular momentum operator. The  $s$ -waves are independent of  $\Omega$ , so  $\Psi(\vec{q}) = \Psi(\rho)$  depend only on the radial coordinate  $\rho = |\vec{q}|$ , and  $L^2(\Omega) \Psi(\rho) = 0$ .

The Laplacian  $\nabla^2$  and its Fourier transform, the square-position operator  $Q^2$ , can be linearly combined and commuted to yield the operators

$$J_0 := \frac{1}{4}(-\nabla^2 + Q^2), \quad J_1 := \frac{1}{4}(-\nabla^2 - Q^2), \quad J_2 := -i\frac{1}{4}(Q \cdot \nabla + \nabla \cdot Q), \quad (2)$$

which form a vector basis for the three-dimensional Lorentz Lie algebra  $\mathfrak{so}(2, 1)$ , with the well-known commutation relations,

$$[J_0, J_1] = iJ_2, \quad [J_2, J_0] = iJ_1, \quad [J_1, J_2] = -iJ_0. \quad (3)$$

Also relevant for our study are the linear combinations

$$J_+ := J_0 + J_1 = -\frac{1}{2}\nabla^2, \quad J_- := J_0 - J_1 = \frac{1}{2}Q^2, \quad (4)$$

and the Casimir invariant

$$J^2 := J_1^2 + J_2^2 - J_0^2. \quad (5)$$

When the Casimir invariant is a multiple  $k(1-k)1$  of the unit operator, it essentially determines (modulo a label related to the spectrum) the irreducible representations of the algebra, and the Lie group to which it exponentiates. For  $k$  a positive integer, this is the proper pseudo-orthogonal Lorentz group  $\mathbf{SO}(2, 1)$ ; for  $k$  half-integer, one has its double cover  $\mathbf{SU}(1, 1) = \mathbf{Sl}(2, \mathbb{R}) = \mathbf{Sp}(2, \mathbb{R})$  (pseudo-unitary, real linear, and real symplectic groups); for  $k$  a quarter-integer, the four-fold cover is the metaplectic group  $\mathbf{Mp}(2, \mathbb{R})$ , and the universal cover  $\overline{\mathbf{Sp}}(2, \mathbb{R})$  results when  $k$  is irrational [3, Sect. 9.4]. The fundamental representation of  $\mathbf{SO}(2, 1)$  is by  $3 \times 3$  matrices, and  $2 \times 2$  matrices for its double cover ( $\mathbf{Mp}(2, \mathbb{R})$  and higher covers have no faithful finite matrix realization).

Further properties pertain the action of the algebras and groups on a Hilbert space of functions. Most of the early work on  $\mathfrak{so}(2, 1)$  used the realization of the algebra by first-order differential operators on the Hilbert space of Lebesgue square-integrable functions on the circle. Here the realization is given by the *second-order* differential operators (2) and (4), acting on functions of the *radial* coordinate relevant for  $s$ -waves, in the Hilbert space  $\mathcal{L}^2(\mathbb{R}^+)$  of square-integrable functions on the positive half-line. This realization of the algebra  $\mathfrak{so}(2, 1)$  on  $\mathcal{L}^2(\mathbb{R}^+)$  was examined in [4] and [5], and was exponentiated to unitary *radial canonical* integral transforms, forming the group  $\mathbf{Sp}(2, \mathbb{R})$  in even- $D$  dimensions, and  $\mathbf{Mp}(2, \mathbb{R})$  in odd- $D$  dimensions. The kernel of these integral transforms include the Green functions for the quantum harmonic and repulsive oscillators (generated by  $J_0$  and  $J_1$ ), the free particle (generated by  $J_+$ ), scaling and chirp phase (generated by  $J_2$  and  $J_-$ ). Second-order differential operators have extra properties over the first-order ones; one of them is the possibility that operators have a manifold of self-adjoint extensions.

We should emphasize that in adopting the group theoretical approach for the description of  $\mathfrak{so}(2, 1) = \mathfrak{sp}(2, \mathbb{R})$  quadratic systems, we are taking advantage of the isomorphism that exists between their  $\mathbf{Sp}(2, \mathbb{R})$  transformations in classical and quantum mechanics. Thus we refer to the square-radius operator and observable, and not to its ‘square root’ radius, as other authors have done and treated with Taylor expansions for time development. In this regard, our results are exact.

In Sect. 2 we perform this analysis for the  $D$ -dimensional quantum harmonic oscillator. The results are contracted to the free particle in Sect. 3, so that we can examine the Green function in Sect. 4. The free

time evolution of the expectation values of the square radius  $J_1$  is addressed in Sect. 5 as a radial canonical transformation among the three  $\mathfrak{so}(2, 1)$  generators; its matrix representation succinctly leads to closed-form results for three families of Gaussian-ring-type initial conditions. As shown by Andreatta and Dodonov [6], different powers of the radius and/or of its measure lead to different quantifications of the shrinking and/or expansion of the radial wavepackets. The appeal of the square-radius expectation value for  $s$ -states is the simplicity of its time evolution under all quadratic potentials with a  $\gamma/r^2$  term that realizes isotropic systems in continuous- $D$  dimensions. In Sect. 6 we present some relevant conclusions.

## 2 The $D$ -dimensional harmonic oscillator

It is well known that the physical eigenfunctions of the harmonic oscillator Schrödinger equation, for mass  $m$ , energy  $E$ , and angular frequency  $\omega$  (Hooke constant  $\frac{1}{2}m\omega^2$ ), are the solutions to

$$H^{\text{HO}} \Psi(\vec{q}) := \left( -\frac{\hbar^2}{2m} \nabla_{\vec{q}}^2 + \frac{m\omega^2}{2} \rho^2 \right) \Psi(\vec{q}) = E \Psi(\vec{q}). \tag{6}$$

For  $s$ -waves  $\Psi(\rho) = \Psi(|\vec{q}|)$ , we change functions and variables through

$$\Psi(\rho) =: \left( \frac{m\omega}{\hbar} \right)^{1/4} \rho^{-(D-1)/2} \psi(r), \quad \rho =: \sqrt{\frac{\hbar}{m\omega}} r, \quad E =: 2\hbar\omega \mu, \tag{7}$$

so that the wavefunctions  $\psi(r)$  satisfy a dimensionless form of (6),

$$K_0 \psi(r) := \frac{1}{4} \left( -\frac{d^2}{dr^2} + \frac{\gamma}{r^2} + r^2 \right) \psi(r) = \mu \psi(r), \tag{8}$$

$$r > 0, \quad \gamma := \frac{1}{4}(D-1)(D-3). \tag{9}$$

They are normalized under the  $D$ -independent form of the inner product that defines the Hilbert space  $\mathcal{L}^2(\mathfrak{R}^+)$ ,

$$(\phi, \psi)_{\mathcal{L}^2(\mathfrak{R}^+)} := \int_0^\infty dr \phi(r)^* \psi(r) = \frac{1}{S_{D-1}} (\Phi, \Psi)_{\mathcal{L}^2(\mathfrak{R}^D)}, \tag{10}$$

where  $S_{D-1} := 2\pi^{D/2}/\Gamma(\frac{1}{2}D)$  is the  $(D-1)$ -dimensional surface of the unit sphere and  $\mathcal{L}^2(\mathfrak{R}^D)$  is the usual Hilbert space of  $D$ -dimensional quantum mechanics.

### 2.1 The radial Lorentz algebra

Eq. (8) is the realization of the generator  $J_0$  in (2) on  $\mathcal{L}^2(\mathfrak{R}^+)$ , i.e.,  $K_0 := J_0|_{\mathcal{L}^2(\mathfrak{R}^+)}$ . On this space, its companion generators  $J_1, J_2$  are

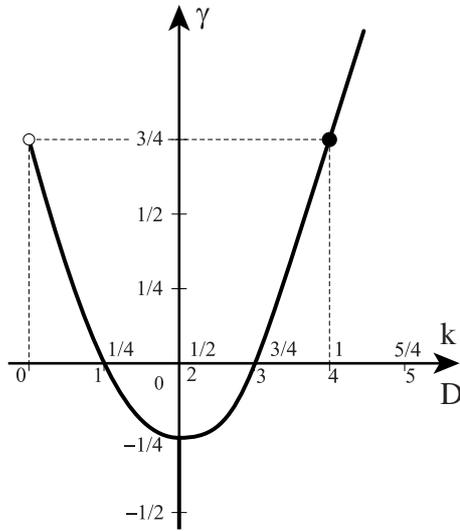
$$K_1 := \frac{1}{4} \left( -\frac{d^2}{dr^2} + \frac{\gamma}{r^2} - r^2 \right), \quad K_2 := -i\frac{1}{2} \left( r \frac{d}{dr} + \frac{1}{2} \right). \tag{11}$$

The free-particle Hamiltonian and the square radius are

$$K_+ := K_0 + K_1 = \frac{1}{2} \left( -\frac{d^2}{dr^2} + \frac{\gamma}{r^2} \right), \quad K_- := K_0 - K_1 = \frac{1}{2} r^2. \tag{12}$$

In this realization, the quadratic Casimir invariant is a number,

$$K^2 := K_1^2 + K_2^2 - K_0^2 = \frac{3}{16} - \frac{1}{4}\gamma = \frac{1}{4}D(1 - \frac{1}{4}D) =: k(1-k), \tag{13}$$



**Fig. 1** The exceptional interval of dimensions  $0 < D < 4$ . The horizontal axis is the Bargmann parameter  $k = \frac{1}{4}D$ , the vertical axis is the ‘interference’ parameter  $\gamma \in \Re$ . The potential  $\gamma/r^2$  is a barrier for  $\gamma > 0$ , and a well for  $\gamma < 0$ . Inside the exceptional interval, barriers and wells are *weak*; outside, they are *strong*. The *pure* harmonic half-oscillator  $\gamma = 0$  sits at  $k = \frac{1}{4}$  and at  $\frac{3}{4}$  ( $D = 1$  and  $3$ ). For strong wells  $\gamma = -\frac{1}{4} - \kappa^2 < -\frac{1}{4}$ , the Bargmann parameter is complex,  $k = \frac{1}{2}(1 \pm i\kappa)$ .

where  $k$ , the *Bargmann index* [2], is

$$k(\gamma) := \frac{1}{2}(1 \pm \sqrt{\gamma + \frac{1}{4}}) = \begin{cases} \frac{1}{4}D, \\ 1 - \frac{1}{4}D, \end{cases} \quad (14)$$

$$\gamma(k) = (2k-1)^2 - \frac{1}{4} = \gamma(1-k), \quad (15)$$

$$\gamma \in \Re \Rightarrow \begin{cases} \gamma \geq -\frac{1}{4} \Rightarrow k > 0 \text{ real,} \\ \gamma < -\frac{1}{4} \Rightarrow k = \frac{1}{2}(1 \pm i\kappa), \kappa > 0 \text{ real.} \end{cases} \quad (16)$$

The Bargmann index  $k$  characterizes the self-adjoint irreducible representations of the algebra [7]; it is plotted with  $\gamma$  in Fig. 1. The interval where the function  $k(\gamma)$  is two-valued,  $0 < k < 1$ , is called *exceptional*; it corresponds to the critical interval of dimensions  $0 < D < 4$ .

The term  $\gamma/r^2$  in (8) is due to geometry, so it is a ‘fictitious’ potential, its nature is centrifugal or centripetal, according to the sign of  $\gamma$ . It has been called *interference potential* by Schleich and coworkers [1, 8], who consider it to be part of the *kinetic* energy, since it originates in the Laplacian (1) and derives from the Sommerfeld superposition of quantum plane waves in  $D$ -dimensional spaces. This fictitious potential term is absent in  $D = 1$  and  $3$  dimensions ( $k = \frac{1}{4}, \frac{3}{4}$ ), for which  $\gamma = 0$ ; it attains its most attractive value in  $D = 2$  dimensions, with  $\gamma = -\frac{1}{4}$  ( $k = \frac{1}{2}$ ). We shall treat  $\gamma/r^2$  as a *bona fide* potential for the full range  $\gamma \in \Re$ . Thus we shall distinguish between centrifugal *barriers*  $\gamma > 0$ , centripetal *wells*  $\gamma < 0$ , and the *pure* oscillator  $\gamma = 0$ . In the exceptional interval  $-\frac{1}{4} \leq \gamma < \frac{3}{4}$  these potentials will be called *weak*, and outside, *strong*.

## 2.2 Solutions proper at infinity

The differential equation (8) has one regular singular point at the origin  $r = 0$ , and an irregular one at infinity  $r \rightarrow \infty$ ; this characterizes the confluent hypergeometric equation, whose independent solutions are  ${}_1F_1(a; c; x)$  and  $x^{1-b} {}_1F_1(1+a-b; 2-b; x)$ . The asymptotic behavior  $r \rightarrow \infty$  can be found keeping only the second-derivative and  $r^2$  terms; the solutions are  $\sim 1/\sqrt{r} \exp(\mp \frac{1}{2}r^2)$ , and only the decreasing Gaussian is acceptable in a spatially constraining potential. This is the defining property of the Whittaker

functions of the second kind,  $W_{\alpha,\beta}(x) \underset{x \rightarrow \infty}{\sim} e^{-x/2} x^\alpha$  [9, Sect. 13.1], [10, Eq. 9.227], so that

$$\begin{aligned} \psi_\mu^k(r) &= c_\mu^k \frac{1}{\sqrt{r}} W_{\mu,k-1/2}(r^2) \\ &= c_\mu^k e^{-r^2/2} \left( \frac{\Gamma(1-2k)}{\Gamma(1-k-\mu)} r^{2k-1/2} {}_1F_1\left(\begin{matrix} k-\mu \\ 2k \end{matrix}; r^2\right) \right. \\ &\quad \left. + \frac{\Gamma(2k-1)}{\Gamma(k-\mu)} r^{-2k+3/2} {}_1F_1\left(\begin{matrix} 1-k-\mu \\ 2-2k \end{matrix}; r^2\right) \right). \end{aligned} \tag{17}$$

Here  $c_\mu^k$  is the normalization constant obtained in Appendix A,

$$c_\mu^k := \sigma_\mu^k \sqrt{\frac{2 \sin 2\pi k}{\pi} \frac{\Gamma(1-k-\mu) \Gamma(k-\mu)}{\psi(1-k-\mu) - \psi(k-\mu)}} = c_\mu^{1-k}, \tag{18}$$

where  $\sigma_\mu^k$  is a sign that may be adjusted to convention when necessary,  $\psi(z)$  is the *digamma* function (logarithmic derivative of the  $\Gamma$ -function [9, Sect. 6.3]), and we note that the radicand is real and positive both for  $k > 0$  and for  $k = \frac{1}{2}(1 + i\kappa)$ ,  $\kappa \in \mathfrak{R}$ .

When  $k$  is integer or half-integer, (18) is apparently singular because  $\sin 2\pi k = 0$ ; these cases are found as well-behaved limits in the Appendix, Eqs. (129)–(132). Particularly, when  $k = \frac{1}{2}$  – the crucial case of  $D = 2$  dimensions – both (17) and (18) will merit special considerations. We note that the reflection  $k \leftrightarrow 1 - k$  across  $k = \frac{1}{2}$  ( $\gamma = -\frac{1}{4}$ ,  $D = 2$ ) formally exchanges the two summands of (17).

### 2.3 Solutions proper at the origin

The behavior of the solutions (17) at the origin  $r \rightarrow 0^+$  determines whether they belong to  $\mathcal{L}^2(\mathfrak{R}^+)$  or not. We recall that the power function  $r^\alpha$  is square-integrable on  $(0, R)$ ,  $0 < R < \infty$ , when

$$\int_0^R dr |r^\alpha|^2 = \frac{r^{2\alpha+1}}{2\alpha+1} \Big|_0^R < \infty, \quad \text{i.e., for } \text{Re } \alpha > -\frac{1}{2}. \tag{19}$$

Since  ${}_1F_1(a; c; 0) = 1$ , near to the origin (17) behaves as

$$\psi_\mu^k(r) \underset{r \rightarrow 0^+}{\sim} c_\mu^k \left( \frac{\Gamma(1-2k)}{\Gamma(1-k-\mu)} r^{2k-1/2} + \frac{\Gamma(2k-1)}{\Gamma(k-\mu)} r^{-2k+3/2} \right). \tag{20}$$

Also, to respect the quantum mechanical interpretation, the kinetic energy (12) of the  $s$ -wave should be finite,

$$(\psi_\mu^k, K_+ \psi_\mu^k)_{\mathcal{L}^2(\mathfrak{R}^+)} = \frac{1}{2} \int_0^\infty dr \left| \frac{d\psi_\mu^k(r)}{dr} \right|^2 + \frac{\gamma}{2} \int_0^\infty dr \frac{|\psi_\mu^k(r)|^2}{r^2} < \infty, \tag{21}$$

so we write the behavior of the derivative at the origin,

$$\frac{d\psi_\mu^k(r)}{dr} \underset{r \rightarrow 0^+}{\sim} c_\mu^k \left( \left(2k - \frac{1}{2}\right) \frac{\Gamma(1-2k)}{\Gamma(1-k-\mu)} r^{2k-3/2} + \left(-2k + \frac{3}{2}\right) \frac{\Gamma(2k-1)}{\Gamma(k-\mu)} r^{-2k+1/2} \right). \tag{22}$$

The first summand in (20) is square-integrable for all  $\text{Re } k > 0$ , and thus all dimensions  $D > 0$  ( $\gamma \geq -\frac{1}{4}$ ), including  $k = \frac{1}{2}(1 + i\kappa)$  for the strong well case  $\gamma < -\frac{1}{4}$ ; it is a constant for  $k = \frac{1}{4}$  ( $D = 1$ ) and singular for  $0 < k < \frac{1}{4}$  ( $0 < D < 1$ ). The second summand in (20) is square-integrable in the exceptional interval  $0 < \text{Re } k < 1$  of dimensions  $0 < D < 4$  ( $-\frac{1}{4} \leq \gamma < \frac{3}{4}$ ) and in strong wells; it is constant at  $k = \frac{3}{4}$  ( $D = 3$ ) and singular for  $\frac{3}{4} < k < 1$  ( $3 < D < 4$ ). The behavior of the derivatives (22) is more restrictive:

the first term is square-integrable for  $k > \frac{1}{2}$  ( $D > 2$ ), constant for  $k = \frac{3}{4}$  ( $D = 3$ ) and singular for  $\frac{1}{2} < k < \frac{3}{4}$  ( $2 < D < 3$ ), while the second is square-integrable for  $0 < k < \frac{1}{2}$  ( $0 < D < 2$ ), constant for  $k = \frac{1}{4}$  ( $D = 1$ ) and singular for  $\frac{1}{4} < k < \frac{1}{2}$  ( $1 < D < 2$ ). Although these ranges exclude the crucial value  $\text{Re } k = \frac{1}{2}$  corresponding to dimension  $D = 2$ , cancellation occurs between the summands of both (20) and (22) because  $\lim_{\varepsilon \rightarrow 0} \Gamma(\varepsilon)/\Gamma(-\varepsilon) = -1$ .

#### 2.4 Equally-spaced spectra

In the strong-barrier interval ruled by Eq. (8) for  $\gamma \geq \frac{3}{4}$  ( $k \geq 1$ ,  $D \geq 4$ ), the behavior of the first summand in (20) is adequate for square-integrability; however, the second one is not, so its coefficient must be vanish. This occurs for values of  $\mu \in \mathfrak{R}$  where in the denominator  $\Gamma(k-\mu)$  has poles, namely when  $k - \mu$  is zero or a negative integer. This reasoning leads to the well-known equally-spaced spectrum of  $K_0$  in (8),

$$\mu = k + m, \quad m \in \mathcal{Z}_0^+, \quad \Sigma(K_0) = \{k, k+1, \dots\}. \quad (23)$$

For these values of  $\mu$ , the remaining  ${}_1F_1$  series truncates to an associated Laguerre polynomial, and the normalized wavefunctions are

$$\psi_{k+m}^k(r) = \sqrt{\frac{2m!}{\Gamma(2k+m)}} e^{-r^2/2} r^{2k-1/2} L_m^{(2k-1)}(r^2), \quad (24)$$

with the choice of sign  $\sigma_{k+m}^k = (-1)^m$  in (18).

We note that the associated Laguerre polynomials  $L_m^{(\alpha)}(r^2)$  are well defined for the range  $\alpha > -1$ , i.e., for  $k > 0$ , corresponding to continuous dimensions  $D > 0$ , and the range  $\gamma \geq -\frac{1}{4}$ . This includes strong barriers and, in the exceptional interval, weak barriers and wells, as shown in Fig. 1. In particular, for the lowest integer dimensions, one has the well-known results

$$D = 1 : \psi_{1/4+m}^{1/4}(r) = (-1)^m e^{-r^2/2} H_{2m}(r) / 2^{2m-1/2} \sqrt{m! \Gamma(\frac{1}{2}+m)}, \quad (25)$$

$$D = 2 : \psi_{1/2+m}^{1/2}(r) = \sqrt{2r} e^{-r^2/2} L_m(r^2) / \sqrt{m!}, \quad (26)$$

$$D = 3 : \psi_{3/4+m}^{3/4}(r) = (-1)^m e^{-r^2/2} H_{2m+1}(r) / 2^{2m+1/2} \sqrt{m! \Gamma(\frac{3}{2}+m)}. \quad (27)$$

We noted above that in the exceptional interval, where  $\gamma(k)$  is two-valued, the reflection  $k \leftrightarrow 1-k$  across  $k = \frac{1}{2}$  only exchanges the two summands in (17). For a fixed value  $\gamma$  of the potential, if we restrict  $\frac{1}{2} < k < 1$  as done by Bargmann [2] and shown in Fig. 2, we have thus a second branch of solutions for (9), characterized by the spectrum

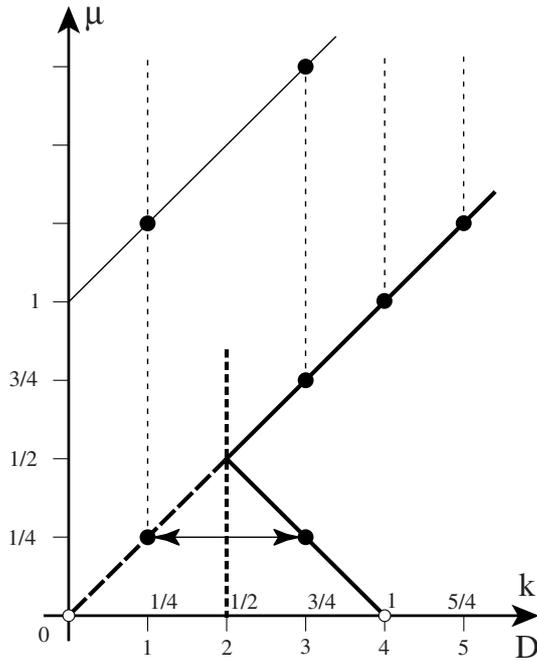
$$\mu = 1 - k + m, \quad m \in \mathcal{Z}_0^+, \quad \Sigma(K_0) = \{1-k, 2-k, \dots\}, \quad (28)$$

and the two branches coalesce at  $k = \frac{1}{2}$ .

The 'half'-harmonic oscillator on  $r > 0$  with  $\gamma = 0$ , corresponds to  $s$ -states in one dimension ( $k = \frac{1}{4}$ ) and in three ( $k = \frac{3}{4}$ ). The first contains the even states of the full oscillator (on  $r \in \mathfrak{R}$ ) while the second contains the odd ones, which vanish at  $r = 0$ . That they belong to distinct values of  $k$  is usually overlooked (and is unnecessary) in elementary treatments of the oscillator. The full oscillator is recovered introducing the two-element group  $\mathcal{O}(1)$  of identity and inversion, to connect the two  $\mathcal{L}^2(\mathfrak{R}^+)$  subspaces into one  $\mathcal{L}^2(\mathfrak{R})$ .

#### 2.5 Non-equally-spaced spectra

Standard quantum mechanics seems ambiguous when the Hamiltonian operator has a family of self-adjoint extensions [11, 12], because this leads to spectra that are *not unique*. The Aharonov-Bohm model [13]



**Fig. 2** Equally-spaced spectra  $\{\mu\}$  of oscillators with barriers and weak wells for the Bargmann index range  $k > 0$  and continuous dimensions  $D > 0$ . In the exceptional interval, the same value  $\gamma$  of the central potential corresponds to both  $k$  and  $1 - k$  (joined by the two-headed arrow). The Bargmann index unravels the bivaluation of  $k(\gamma)$  in (15). The  $\bullet$ 's above  $k = \frac{1}{4}$  ( $D = 1$ ) and the distinct  $\frac{3}{4}$  ( $D = 3$ ), correspond to the  $r > 0$ -half of even and odd states in the pure one-dimensional harmonic oscillator on  $\mathfrak{R}$ .

incorporates this feature successfully to describe the physics of  $\mathfrak{R}^3$ -space pierced by a magnetic line. In our case however, this interpretation is not sought; rather, we aim at understanding the role of these extensions in the time evolution of  $s$ -waves in oscillators and free particles within the exceptional interval of continuous dimension  $0 < D < 4$ .

We remind the reader that an operator  $H$  is self-adjoint in a complex Hilbert space  $\mathcal{H}$  with a sesquilinear inner product  $(\phi, \psi)_{\mathcal{H}}$ , when

$$(H^\dagger \phi, \psi)_{\mathcal{H}} := (\phi, H \psi)_{\mathcal{H}} = (H \phi, \psi)_{\mathcal{H}}, \quad \text{and} \quad (29)$$

$$\text{the domain of } H^\dagger \text{ is equal to the domain of } H. \quad (30)$$

Elementary consequences of self-adjointness are that the spectra are real and that eigenfunctions corresponding to distinct eigenvalues are orthogonal. So, when integrability and boundary conditions allow for more than one set of mutually orthogonal eigenfunctions, there are correspondingly more than one self-adjoint extensions of the same formal operator  $H(k)$ .

We can find the inner product of two solutions (17),  $\psi_\nu^k(r)$  and  $\psi_\mu^k(r)$ ,  $\nu \neq \mu$ , by means of their Wronskian  $\mathcal{W}(\psi_\nu^{k*}, \psi_\mu^k)$ , considering

$$\begin{aligned} (\mu - \nu) \psi_\nu^k(r)^* \psi_\mu^k(r) &= \psi_\nu^k(r)^* H \psi_\mu^k(r) - (H \psi_\nu^k(r))^* \psi_\mu^k(r) \\ &= -\frac{1}{2} \frac{d}{dr} \left( \psi_\nu^k(r)^* \frac{d\psi_\mu^k(r)}{dr} - \frac{d\psi_\nu^k(r)^*}{dr} \psi_\mu^k(r) \right) \\ &=: -\frac{1}{2} \frac{d}{dr} \mathcal{W}(\psi_\nu^{k*}, \psi_\mu^k). \end{aligned} \quad (31)$$

Integrating from 0 to  $\infty$  we have the inner product (10); at infinity, the Wronskian is zero because of the Gaussian decrease of the Whittaker wavefunctions (17). Hence, the inner product of two  $s$ -waves is given by

$$(\psi_\nu^k, \psi_\mu^k)_{\mathcal{L}^2(\mathfrak{R}^+)} = \frac{1}{2(\mu - \nu)} \mathcal{W}(\psi_\nu^{k*}, \psi_\mu^k) \Big|_{r \rightarrow 0^+}. \quad (32)$$

This can be computed both when  $k^* = k > 0$  is real, and also in the strong well case, where  $k = \frac{1}{2}(1 \pm i\kappa)$  and  $k^* = 1 - k$ . Using (20) and (22), we arrive at

$$\mathcal{W}(\psi_\nu^{k^*}, \psi_\mu^k) \Big|_{r \rightarrow 0^+} = c_\nu^{k^*} c_\mu^k \frac{\pi}{\sin 2\pi k} \times \left( \frac{1}{\Gamma(k-\nu)\Gamma(1-k-\mu)} - \frac{1}{\Gamma(1-k-\nu)\Gamma(k-\mu)} \right). \tag{33}$$

The limit  $k \rightarrow \frac{1}{2}^\pm$  yields the Wronskian for  $D = 2$  dimensions,

$$\mathcal{W}(\psi_\nu^{1/2^*}, \psi_\mu^{1/2}) \Big|_{r \rightarrow 0^+} = c_\nu^{1/2^*} c_\mu^{1/2} \frac{\pi}{\sin 2\pi k} \frac{\psi(\frac{1}{2} - \nu) - \psi(\frac{1}{2} - \mu)}{\Gamma(\frac{1}{2} - \nu)\Gamma(\frac{1}{2} - \mu)}. \tag{34}$$

Now we can choose one wavefunction  $\psi_\nu^k$  – with a fixed  $\nu$ , generally distinct from  $k + n$  or  $1 - k + n$  – and search for the all the wavefunctions  $\psi_\mu^k$  that are orthogonal to it. This is achieved when the Wronskian (33) vanishes, i.e., when

$$f_k(\mu) = \frac{1}{f_{1-k}(\mu)} := \frac{\Gamma(k - \mu)}{\Gamma(1 - k - \mu)} = \frac{\Gamma(k - \nu)}{\Gamma(1 - k - \nu)} = \text{constant}. \tag{35}$$

For  $k \rightarrow \frac{1}{2}^\pm$ , the vanishing condition becomes

$$\psi(\frac{1}{2} - \mu) = \psi(\frac{1}{2} - \nu) = \text{constant}, \tag{36}$$

in terms of the digamma function. The roots  $\{\mu\}$  of these transcendental equations determine the spectrum  $\Sigma(K_0, \nu)$  of  $K_0$  (one-half the oscillator Hamiltonian) in the Hilbert space  $\mathcal{H}_{k|\nu|}$  spanned by the orthonormal basis  $\{\psi_\mu^k\}_{\mu \in \Sigma(K_0, \nu)}$ .

The function  $f_k(\mu)$  in (35) is shown in Fig. 3 for three values of  $k > \frac{1}{2}$  and for  $k = \frac{1}{2}$  in (36); the horizontal lines  $f_k(\nu)$  intersect the graphs at the spectrum points  $\{\mu\} \in \Sigma(K_0, \nu)$  that include the chosen  $\nu$ . There is one negative eigenvalue  $\nu = \mu_{\min} < 0$  when the constant in (35) is  $f_k > \Gamma(k)/\Gamma(1-k)$  for  $\frac{1}{2} < k < 1$ , and also for  $0 < k < \frac{1}{2}$  when  $0 < f_k < \Gamma(k)/\Gamma(1-k)$ . The only equally-spaced spectra occur for  $\psi_{k+n}^k$ ,  $n \in \mathcal{Z}_0^+$ , when  $f_k(k) = \pm\infty$ , belonging to the spectrum (23), and for  $\psi_{1-k+m}^k$  when  $f_k(1-k) = 0$ , belonging to the distinct spectrum (28). These are the two *Friedrichs* self-adjoint extensions of the  $\text{so}(2, 1)$  generator  $K_0$ , characterized for having finite kinetic energy (21), as can be determined from (22).

Strong wells  $\gamma < -\frac{1}{4}$  correspond to complex Bargmann indices  $k = \frac{1}{2}(1 \pm i\kappa)$ ,  $\kappa \in \mathfrak{R}^+$ , so  $k^* = 1 - k$ . In this case  $|f_k(\mu)| = 1$ , so we find the spectrum using its phase,  $f_k(\mu) = \exp(i\eta_k(\mu))$ ,

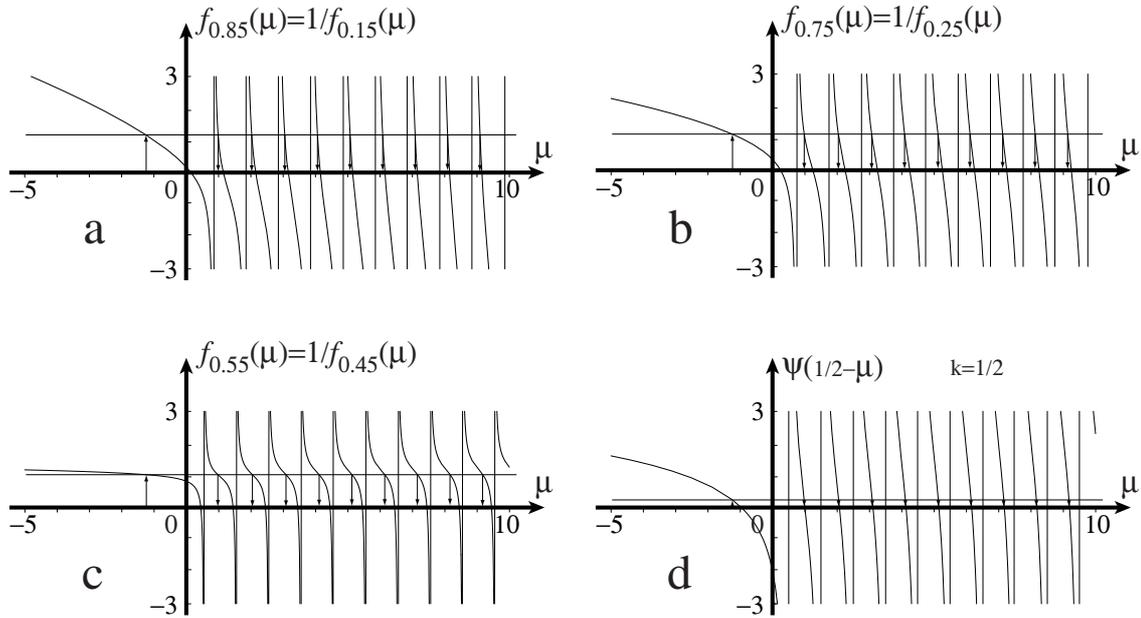
$$\eta_k(\mu) = -\eta_{1-k}(\mu) = 2 \arg \Gamma(\frac{1}{2}[1 + i\kappa] - \mu) = \text{constant}, \tag{37}$$

which is shown in Fig. 4 on a cylinder cut open. Again, by choosing a fixed  $\nu$ , the companion  $\mu$ 's in the spectrum are found at the intersections of the phase  $\arg f_k(\nu)$  with the graph of  $\eta_k(\mu)$ .

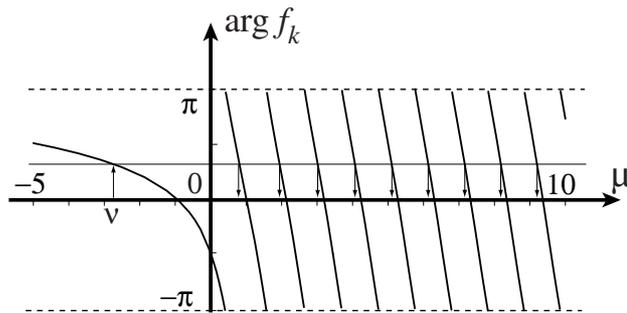
Finally, we note that eigenfunctions of the two Friedrichs extensions  $k$  and  $1 - k$  ( $k \neq \frac{1}{2}$ ) are *not* orthogonal; their Wronskian is also  $r$ -independent and nonvanishing,

$$(\psi_{k+n}^{k^*}, \psi_{1-k+m}^{1-k})_{\mathcal{L}^2(\mathfrak{R}^+)} = \frac{2 \sin 2\pi k}{\pi} \frac{\sqrt{\Gamma(2k+m)\Gamma(2-2k+n)}}{\sqrt{n!m!} (1 - 2k + n - m)}, \tag{38}$$

as can be checked using [10, Eq. 7.414.9]. For  $k = \frac{1}{2}$ , the right-hand side of (38) is zero unless  $n = m$ , because the two solutions are the same; for reasons to be seen below, we need not detail the second, logarithmic solution to the differential equation (8).



**Fig. 3** The spectra  $\{\mu\} \in \Sigma(K_0, \nu)$  of oscillators with weak barriers and wells are found by drawing the horizontal line  $f_k(\nu)$  through the graphs of  $f_k(\mu)$ . (a) for  $k = 0.85$  ( $D = 3.4$ , a barrier  $\gamma = 0.24$ ). (b) For  $k = 0.75 = \frac{3}{4}$  ( $D = 3$ , pure oscillator  $\gamma = 0$ ). (c) For  $k = 0.55$  ( $D = 2.2$ , a well  $\gamma = -0.24$ ). (d) For  $k = \frac{1}{2}$  ( $D = 2$ , a well  $\gamma = -\frac{1}{4}$ ). Having chosen  $\nu = -1.25$ , marked by the ascending arrow, one thus finds the spectrum. In (a)–(c), the reflection  $k \leftrightarrow 1 - k$  also yields spectra for  $k = 0.15, 0.25, 0.45$  ( $D = 0.6, 1, 1.8$ ) using the reciprocal scale  $f_{1-k}(\mu) = 1/f_k(\mu)$ .



**Fig. 4** The spectra  $\mu \in \Sigma(K_0, \nu)$  of oscillators with strong wells, shown for  $k = \frac{1}{2}(1 + i)$ , corresponding to  $\gamma = -1.25$ . This graph differs from the previous ones in that here we plot the phase  $\arg f_k(\mu)$ , which is periodic modulo  $2\pi$ . The spectrum here includes the value  $\nu$ , here chosen as  $\nu = -1.25$ . In this strong well case, the spectra are *not* bounded from below.

### 2.6 Green function for oscillator systems

The time-dependent Schrödinger equation of the system ruled by the Hamiltonian given in the left-hand side of Eq. (6), is found replacing the right-hand side with the diffusive-oscillatory time derivative  $i\hbar\partial/\partial t$ . Separated solutions are  $\Psi_{E(\rho)} \exp(-iEt/\hbar)$  for energies  $E = 2\hbar\omega\mu$  in the spectrum of the operator. Translated to the dimensionless form for  $K_0$  in Eq. (9), with  $r$  and  $\tau := \omega t$ , the eigenfunctions satisfy

$$2K_0 \psi_\mu^k(r, \tau) = i \frac{\partial}{\partial \tau} \psi_\mu^k(r, \tau), \quad \psi_\mu^k(r, \tau) = \psi_\mu^k(r) \exp(-2i\mu\tau), \quad (39)$$

where according to the value  $k$  (and in the exceptional interval also by the self-adjoint extension label  $\nu$ ), the spectra are (23), (28), or (35)–(36).

The time evolution of  $s$ -waves in the quantum harmonic oscillator is obtained applying the exponentiated operator  $K_0$  to the initial waveform  $\psi_0(r) = \psi(r, 0)$ ; this yields generally integral transforms where the

kernel is the Green function of the system:

$$\begin{aligned}\psi(r, \tau) &= \exp(-2i\tau K_0) \psi_0(r) \\ &= \int_0^\infty dr' G_k^{\text{HO}}(r, r'; \tau) \psi_0(r'),\end{aligned}\quad (40)$$

$$G_k^{\text{HO}}(r, r'; \tau) := \sum_{\mu \in \Sigma(K_0, \nu)} \psi_\mu^k(r) e^{-2i\mu\tau} \psi_\mu^k(r')^*.\quad (41)$$

The Green function is the time development of an initial Dirac  $\delta(r - r')$ , and is a solution to the time-dependent Schrödinger equation (39) in  $(r, \tau)$  and in  $(r', \tau)$ . Time evolution is a one-parameter Lie group of unitary transformations, so the Green function must satisfy

$$\text{composition: } \int_0^\infty dr' G_k(r, r'; \tau_1) G_k(r', r''; \tau_2) = G_k(r, r''; \tau_1 + \tau_2).\quad (42)$$

$$\text{unit: } G_k(r, r'; 0) = \delta(r - r'),\quad (43)$$

$$\text{inverse: } G_k(r, r'; -\tau) = G_k(r', r; \tau)^*.\quad (44)$$

For equally-spaced spectra (i.e. real  $k > 0$ , and Friedrichs extensions in the exceptional interval), the series (41) is one of the bilinear generating functions of the Laguerre polynomials [10, Eq. 8.976.1] in (24),

$$G_k^{\text{HO}}(r, r'; \tau) = \sum_{m \in \mathbb{Z}_0^+} \psi_{k+m}^k(r) e^{-2i(k+m)\tau} \psi_{k+m}^k(r')^* \quad (45)$$

$$= e^{-i\pi k} \frac{\sqrt{r r'}}{\sin \tau} \exp\left(i \frac{r^2 + r'^2}{2 \tan \tau}\right) J_{2k-1}\left(\frac{r r'}{\sin \tau}\right) \quad (46)$$

$$= C_k \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} (r, r'). \quad (47)$$

The expression (46) is a well-known result from [4]; it is the fractional Hankel transform built from the eigenfunction set (45) (see [14] and [15, pp. 175–180]). In (47) we write the *radial canonical* integral transform kernel following [16, Chap. 9] and [3, App. C]. Recall that for every  $\text{Sp}(2, \mathfrak{R})$  matrix  $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , canonical transforms  $\mathcal{C}(\mathbf{M})$  act linearly on the phase space operators of position  $\vec{Q}$  and momentum  $\vec{P}$ ; on  $\mathcal{L}^2(\mathfrak{R}^+)$ -functions they act through

$$\mathcal{C}(\mathbf{M}) : f(r) = \int_0^\infty dr' C_k(\mathbf{M})(r, r') f(r'), \quad (48)$$

$$C_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (r, r') = e^{-i\pi k} \frac{\sqrt{r r'}}{b} \exp\left(i \frac{dr^2 + ar'^2}{2b}\right) J_{2k-1}\left(\frac{r r'}{b}\right), \quad (49)$$

$$C_k \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} (r, r') = a^{2k-1} \exp\left(i \frac{cr^2}{2a}\right) \delta(r' - r/a). \quad (50)$$

The trigonometric arguments of the Green function (46) show that  $\tau$  has a fundamental periodicity interval of  $2\pi$ ; the phase  $\exp(-i\pi k)$  is important because it determines the cover of the oscillator cycle realized on  $\text{SO}(2) \subset \overline{\text{SO}(2, 1)}$ . For rational  $k = p/q$  ( $p$  and  $q$  relatively prime), one has a  $q$ -fold cover of the evolution subgroup  $\text{SO}(2)$ , with only a phase difference between values of  $\tau$  separated by  $2\pi$ . The two pure half-oscillators at  $k = \frac{1}{4}, \frac{3}{4}$  ( $D = 1$  and  $3$ ), intertwined with parity to the full line  $r \in \mathfrak{R}$ , yield the one-dimensional canonical transform kernel [16, 17] with the proper metaplectic phase  $e^{-i\pi/4} - \text{Mp}(2, \mathfrak{R})$  covers  $\text{SO}(2, 1)$  four times – where the difference between its values at  $\tau = 2\pi$  and  $4\pi$  is a sign.

There is thus a correspondence between three distinct mathematical objects: canonical integral transforms,  $\text{Sp}(2, \mathfrak{R})$  matrices, and the exponentials of the  $\text{sp}(2, \mathfrak{R})$  Lie algebra generators  $K_\alpha$  in Eqs. (8), (11) and (12). Particularly,

$$\begin{aligned} \mathcal{C} \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} &= \exp(-2i\tau K_0), & \mathcal{C} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} &= \exp(-i\beta K_+), \\ \mathcal{C} \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} &= \exp(-2i\alpha K_2), & \mathcal{C} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} &= \exp(-i\gamma K_-). \end{aligned} \tag{51}$$

The non-equally spaced spectra obtained for the non-Friedrichs extensions in the exceptional interval  $0 < k < 1$ , break the Hamiltonian away from the Lorentz algebra  $\mathfrak{so}(2, 1)$ , because the companion raising and lowering operators  $J_1 \pm iJ_2$  would map the domains  $\mathcal{H}_{k[\nu]}$  out of themselves. A Green function that we could build according to the general bilinear generating formula (41), but with non-equally spaced powers of  $e^{2i\tau}$ , is not calculable in closed form. We surmise that only if we initially were to write  $\psi_0(r)$  as  $\sum_{\mu \in \Sigma(K_0, \nu)} A_\mu^k \psi_\mu^k(r')$ , could we state that the coefficients will evolve in time as  $A_\mu^k \exp(-2i\mu\tau)$ . The time evolution operator  $\exp(-2i\mu K_0)$  would be then representable by a diagonal matrix acting on column vectors  $\{A_\mu^k\}_{\mu \in \Sigma(K_0, \nu)}$ , but *not* by an integral kernel. We should recall that Bargmann’s treatment of the representations of the three-dimensional Lorentz group [2] used various Hilbert spaces, among them  $\mathcal{L}^2(\mathcal{S}_1)$  on the circle, in which the *first*-order differential operators representing the algebra are self-adjoint, and their spectra equally-spaced. Such operators do not have self-adjoint extension families with non-equal spacing; this refinement appears with *second*-order differential ones. Our work in [18] consisted in providing a *two-component*  $\mathcal{L}^2(\mathfrak{R}^+)$  Hilbert space to restore equal spacing in all representation series. It turns out that this refinement is maintained in the contraction limit of the oscillator to the free particle, where the spectra are continuous, and where the impediment of using non-Friedrichs extensions to form Green functions will become more evident.

### 3 Contraction of the oscillator to the free particle

Having seen the spectra of *s*-waves in the *D*-dimensional quantum harmonic oscillator, we shall now contract this system, its Hamiltonian operator, its spectrum and its eigenfunctions, to those of the free particle. This is our method to analyze spectra that are continuous and common to the whole family of self-adjoint extensions of free Hamiltonians having the interference potential  $\gamma/r^2$  – centrifugal barrier or centripetal well. To this end, we first determine the asymptotic behavior of the the non-equally spaced spectra at high energies.

#### 3.1 Asymptotic spectra in the exceptional interval

The non-equal spacing between energy levels of the non-Friedrichs extensions of the oscillator Hamiltonians in the exceptional interval  $0 < k < 1$  ( $0 < D < 4$ ), is most visible between the lowest levels. As we go to higher energies  $\mu \rightarrow \infty$ , the behavior of the function  $f_k(\mu)$  in (35) can be traced using the reflection and Stirling formulas [9, Eqs. 6.1.17 and 6.1.37]), to be

$$\begin{aligned} f_k(\mu) &= \frac{\Gamma(k - \mu)}{\Gamma(1 - k - \mu)} = \frac{\sin \pi(k + \mu)}{\sin \pi(k - \mu)} \frac{\Gamma(k + \mu)}{\Gamma(1 - k + \mu)} \\ \widetilde{\mu \rightarrow +\infty} &= -\frac{\sin \pi(k + \mu)}{\sin \pi(k - \mu)} \mu^{1-2k}, \quad \text{for } \frac{1}{2} < k < 1. \end{aligned} \tag{52}$$

As we can see in Fig. 3, as  $\mu \rightarrow \infty$ , the distance between two adjoining levels decreases asymptotically to unity. Setting  $\mu = k + M + \varepsilon_M$  with integer  $M \rightarrow \infty$ , we inquire into the difference  $\varepsilon_M$ , finding

$$f_{\frac{1}{2} < k < 1}(\mu) \widetilde{M \rightarrow \infty} = \frac{\sin(2k-1)\pi}{\pi \varepsilon_M} M^{1-2k} \tag{53}$$

$$\Rightarrow \varepsilon_M \widetilde{M \rightarrow \infty} \frac{\sin(2k-1)\pi}{\pi f_k(\nu)} \frac{1}{M^{2k-1}} \rightarrow 0^+. \quad (54)$$

On the other hand, for the second branch  $0 < k < \frac{1}{2}$  and  $\mu = 1 - k + M + \varepsilon_M$ , we obtain

$$f_{0 < k < \frac{1}{2}}(\mu) \widetilde{M \rightarrow \infty} \frac{\sin 2\pi k}{\pi \varepsilon_M} M^{2k-1} \quad (55)$$

$$\Rightarrow \varepsilon_M \widetilde{M \rightarrow \infty} \frac{\sin 2\pi k}{\pi f_k(\nu)} \frac{1}{M^{1-2k}} \rightarrow 0^+. \quad (56)$$

For the special case  $k = \frac{1}{2}$ ,  $f_k(\mu)$  is replaced by its derivative, leading to the spectrum given by the roots of the digamma function in (36). We then take  $\mu = \frac{1}{2} + M + \varepsilon_M$  with integer  $M \rightarrow \infty$ , and use the reflection and asymptotic formulas for the digamma function [9, Eqs. 6.3.7 and 6.3.18]), finding

$$\psi\left(\frac{1}{2} - \nu\right) = \psi\left(\frac{1}{2} - \mu\right) \widetilde{\mu \rightarrow +\infty} \ln(M + \varepsilon_M - 1) + \pi \cot \pi \varepsilon_M \Rightarrow \varepsilon_M \rightarrow 0^-, \quad (57)$$

irrespective of the value or sign of  $\psi\left(\frac{1}{2} - \nu\right)$ .

The asymptotic spectra in strong wells (37) can be found also with relative ease. For positive energies  $\mu \rightarrow \infty$ , the recurrence relation

$$\eta_k(\mu + 1) = \eta_k(\mu) + 2 \arctan \frac{\kappa}{2\mu + 1} \widetilde{\mu \rightarrow +\infty} \eta_k(\mu) + \frac{\kappa}{\mu} \text{ modulo } 2\pi, \quad (58)$$

shows that the separation between neighboring spectrum points at higher energies decreases asymptotically to unity, and so tends to equal spacing. For negative energies  $\mu \rightarrow -\infty$ , the behavior of this phase can be found using [9, Eq. 6.1.39],

$$\eta_k(\mu) = 2 \arg \Gamma\left(\frac{1}{2}[1 + i\kappa] - \mu\right) \widetilde{\mu \rightarrow -\infty} \kappa \ln |\mu| \text{ modulo } 2\pi, \quad (59)$$

which grows ever slower with  $|\mu|$ , but without a lower bound.

### 3.2 Scaling parameters, coordinates and functions

To proceed with the contraction, we now consider a sequence of harmonic oscillators (6) with decreasing spring constants; their angular frequencies  $\{\omega_N\}_{N=1}^{\infty}$  are chosen to be  $\omega_N := \omega_1/N$ , for any starting frequency  $\omega_1 \equiv \omega$ ; the limit of the sequence will be the quantum free particle. Recalling that the dimensional variables of radius  $\rho$  and energy  $E$  in (6) were converted to the dimensionless  $r$  and  $\mu$  through (7), we scale the latter to suit our sequence of oscillators as follows:

$$r_N := \sqrt{\frac{\omega_N}{\omega_1}} r_1 = \frac{r_1}{\sqrt{N}} =: \frac{\frac{1}{2}pr}{\sqrt{\mu}} > 0, \quad r_1 \equiv r, \quad (60)$$

$$E = \hbar\omega_1 \frac{p^2}{2} \approx 2\hbar\omega_N \mu = 2\hbar\omega_1 \frac{\mu}{N}, \quad (61)$$

$$\text{where } p := \sqrt{\frac{2E}{\hbar\omega_1}} \approx 2\sqrt{\frac{\mu}{N}} \text{ defines momentum.} \quad (62)$$

[We write  $E$  and  $p$  as ‘approximately equal’ in (61) and (62), because the oscillators in the sequence have their energies quantized to  $2\hbar\omega_N\mu$ .] When  $N \rightarrow \infty$  and  $\omega_N \rightarrow 0$ , all finite energy levels collapse to zero, and we must go to ever larger  $|\mu|$ ’s concurrently with  $N \rightarrow \infty$ , in the dimensionless proportion  $\mu/N \approx \frac{1}{4}p^2$  to match the chosen finite energy  $E$ .

The dimensionless differential equation (8) for the  $N^{\text{th}}$  oscillator in the sequence is written in terms of the variable  $r_N$ ; in terms of  $r$  through (60), it reads

$$\frac{1}{2} \left( -\frac{d^2}{dr^2} + \frac{\gamma}{r^2} + \frac{r^2}{N^2} \right) \psi_{\mu}^k \left( \frac{r}{\sqrt{N}} \right) = \frac{2\mu}{N} \psi_{\mu}^k \left( \frac{r}{\sqrt{N}} \right) = \frac{1}{2} p^2 \psi_{\mu}^k \left( \frac{r}{\sqrt{N}} \right). \quad (63)$$

When  $N \rightarrow \infty$ , this becomes the quantum free particle radial equation in  $D$  dimensions (12), namely

$$H^F \phi_p^k(r) := \frac{1}{2} \left( -\frac{d^2}{dr^2} + \frac{\gamma}{r^2} \right) \phi_p^k(r) = \frac{1}{2} p^2 \phi_p^k(r). \tag{64}$$

The limit of the oscillator wavefunctions (17) is found from

$$\phi_p^k(r) := \frac{1}{2} p \lim_{N \rightarrow \infty} \sqrt{N} \psi_\mu^k \left( \frac{r}{\sqrt{N}} \right), \quad \text{for } \mu \approx \frac{1}{4} N p^2 = \frac{N E}{\hbar \omega_1}. \tag{65}$$

### 3.3 Contraction for equally-spaced spectra

For equally-spaced spectra  $\mu = k + m$ ,  $m \in \mathbb{Z}_0^+$ , where  $\mu$ ,  $N$  and  $p$  are related by (65), we adapt standard limit formulas for the confluent hypergeometric to Bessel functions (cf. [9, Eq. 13.3.2]), finding

$${}_1F_1 \left( \begin{matrix} -m \\ 2k \end{matrix}; \frac{r^2}{N} \right) = \frac{\Gamma(2k) m!}{\Gamma(2k+m)} L_m^{(2k-1)} \left( \frac{r^2}{N} \right) \xrightarrow{N \rightarrow \infty} \Gamma(2k) \left( \frac{1}{2} pr \right)^{1-2k} J_{2k-1}(pr). \tag{66}$$

The prefactors of this expression in Eq. (17), using (134) from Appendix A, follow

$$c_{k+m}^k \frac{\Gamma(1-2k)}{\Gamma(1-k-m)} e^{-r^2/2N} \left( \frac{r}{\sqrt{N}} \right)^{2k-1/2} \xrightarrow{N \rightarrow \infty} \sqrt{\frac{2}{\mu}} \frac{\left( \frac{1}{2} pr \right)^{2k-1/2}}{\Gamma(2k)}, \tag{67}$$

so the limit functions, which in (65) are multiplied by  $\frac{1}{2} p \sqrt{N} = \sqrt{\mu}$ , become

$$\phi_p^k(r) = \sqrt{pr} J_{2k-1}(pr) = \phi_r^k(p), \tag{68}$$

for all  $p > 0$  and  $k > 0$ . We recognize here the kernel of the Hankel transform, which is real, orthogonal, and Dirac-normalized,

$$(\phi_p^k, \phi_{p'}^k)_{\mathcal{L}^2(\mathbb{R}^+)} = \sqrt{pp'} \int_0^\infty dr r J_{2k-1}(pr) J_{2k-1}(p'r) = \delta(p - p'). \tag{69}$$

In particular, for  $D = 1$  and  $3$  dimensions there is no barrier nor well ( $\gamma = 0$ ,  $k = \frac{1}{4}$  and  $\frac{3}{4}$ ), and we have the even and odd positive halves of the solutions to the full ( $r \in \mathbb{R}$ ) free Schrödinger equation,

$$\phi_p^{1/4}(r) = \sqrt{pr} J_{-1/2}(pr) = \sqrt{2/\pi} \cos pr = (e^{ipr} + e^{-ipr})/\sqrt{2\pi}, \tag{70}$$

$$\phi_p^{3/4}(r) = \sqrt{pr} J_{+1/2}(pr) = \sqrt{2/\pi} \sin pr = (e^{ipr} - e^{-ipr})/i\sqrt{2\pi}, \tag{71}$$

for  $p \geq 0$ . In dimension  $D = 2$ , the weak well limit to  $\gamma = -\frac{1}{4}$ , i.e.  $k = \frac{1}{2}$ , is uneventful:  $s$ -waves  $\sqrt{pr} J_0(pr)$  are well known.

The dimensional form of the free Schrödinger equation for a particle of mass  $m$ , energy  $E = \hbar v$ , and  $\vec{q}(\rho, \Omega) \in \mathbb{R}^D$ , is regained from (64) after the transformation that undoes (7), namely

$$-\frac{\hbar^2}{2m} \nabla_{\vec{q}}^2 \Phi_v(\rho) = E \Phi_v(\rho), \quad E =: \hbar v, \tag{72}$$

$$\Phi_v(\rho) := \left( \frac{\hbar}{mv} \right)^{1/4} \rho^{-\frac{1}{2}(D-1)} \phi_p(r), \quad \rho =: \sqrt{\frac{\hbar}{mv}} \frac{pr}{2}, \tag{73}$$

where it is understood that the  $s$ -waves  $\Phi_v(\rho)$  are spherically symmetric.

### 3.4 Contraction in exceptional interval

In the exceptional dimension range  $0 < D < 4$ , self-adjoint extensions are characterized, as we saw above, by the value  $\nu$  of some eigenvalue in its spectrum through the constant  $f_k(\nu) \in \mathfrak{R}$  in (35). The asymptotic relations (54) and (56) imply that for large  $|\mu|$ 's the eigenvalues become equally spaced and tend to  $1 - k + M$  for  $0 < k \leq \frac{1}{2}$ , or to  $k + M$  for  $\frac{1}{2} \leq k < 1$ , with  $M \in \mathcal{Z}_0^+$ . Yet the value of  $\nu$  persists in the ratio of the two summands in (17), which after the contraction we write as the linear combination

$$\phi_p^{k[\nu]}(r) = \sqrt{pr} [\cos \theta_{[\nu]}^k J_{2k-1}(pr) + \sin \theta_{[\nu]}^k J_{1-2k}(pr)], \quad (74)$$

with the angle  $\theta_{[\nu]}^k$ . This angle is related to the constant  $f_k(\nu)$  that characterizes the self-adjoint extension in (35), through

$$\cot \theta_{[\nu]}^k = \frac{\Gamma(1-2k)}{\Gamma(2k-1)} \frac{\Gamma(k-\mu)}{\Gamma(1-k-\mu)} = \frac{\Gamma(1-2k)}{\Gamma(2k-1)} f_k(\nu) = \tan \theta_{[\nu]}^{1-k}. \quad (75)$$

We note that the wavefunctions (74) are real and have the  $k \leftrightarrow 1-k$  symmetry  $\phi_p^{k[\nu]}(r) = \phi_p^{1-k, [\nu]}(r) = \phi_p^{k[\nu]}(r)^*$ . The ratio (75) is well defined throughout the exceptional interval, including the case  $k = \frac{1}{2}$  ( $D = 2$ ), where  $\lim_{\varepsilon \rightarrow 0} \Gamma(\varepsilon)/\Gamma(-\varepsilon) = -1$ . As we approach the Friedrichs extensions,  $f_k(\nu) \rightarrow 0$  and  $\rightarrow \pm\infty$ , one of the two terms of (74) disappears as  $\theta_{[\nu]}^k \rightarrow 0$  and  $\rightarrow \pm\frac{1}{2}\pi$ , but for all others both are present, and are square-integrable. One of them is singular at the origin: the first for  $0 < k < \frac{1}{4}$ , and the second for  $\frac{3}{4} < k < 1$ . And as in the oscillator case, the eigenfunction spaces belonging to different self-adjoint extensions of the same free Hamiltonian, are not orthogonal, except for dimensions  $D = 4k = 1$  and  $3$ . While all oscillator spectra in the exceptional interval are lower-bound, with at most one finite negative value, after contraction all self-adjoint extensions of the free Hamiltonian will have the same spectrum  $\frac{1}{2}p^2 \geq 0$ . The distinction between self-adjoint extensions in the free Hamiltonians would not have been evident had we not studied oscillators first, with their discrete and distinct spectra, characterized by  $f_k(\nu) \in \mathfrak{R}$ .

### 3.5 Contraction in the strong well case

Although the dimensional interpretation breaks down in the strong well case [ $\gamma < -\frac{1}{4}$ ,  $k = \frac{1}{2}(1 + i\kappa)$ ,  $\kappa > 0$ ] because  $D = 2(1 + i\kappa)$  is complex, the limit of the oscillator to the free wavefunctions, (66)–(67) leading to (74), is surprisingly insensitive to this situation. For positive energies,  $|f_k(\mu)| = 1$  and the self-adjoint extension in (37) are characterized by the phase  $\eta_k(\nu) = \arg f_k(\nu)$ . Asymptotically, the eigenvalues become equally spaced by (58), and the indices of the Bessel functions are  $2k - 1 = i\kappa$ . Analogous to (74), but with phases  $e^{\pm i\zeta_k(\nu)}$ , we now have

$$\phi_p^{k[\nu]}(r) = \sqrt{\frac{1}{2}pr} \left( e^{i\zeta_k(\nu)} J_{i\kappa}(pr) + e^{-i\zeta_k(\nu)} J_{-i\kappa}(pr) \right), \quad (76)$$

with the phase  $\zeta_k(\nu)$  fixed as before from the ratio of the two (now complex conjugate) coefficients in (17),

$$\zeta_k(\nu) = \frac{1}{2} \arg \left( \frac{\Gamma(-i\kappa)}{\Gamma(i\kappa)} \frac{\Gamma(k-\mu)}{\Gamma(1-k-\mu)} \right) = \frac{1}{2} \eta_k(\nu) - \arg \Gamma(i\kappa). \quad (77)$$

On the other hand, for the negative energy levels  $E < 0$  of the strong well potential, whose asymptotic spacing is  $\sim \kappa \ln |\mu|$  according to the result (59) and Fig. 4, the linear ratio  $\mu/N \approx E/\hbar\omega_1$  cannot accumulate them to a continuum – as the positive- $\mu$  levels do. The contraction process (60)–(62) for positive levels would seem to obliterate the negative ones from the spectrum of a free particle in a strong well. It is possible that a  $\sqrt{\delta}$ -type trapped state accumulates at infinite negative energy, as in the one-dimensional hydrogen atom [19]; we need not treat this case further in the present paper however, since it corresponds to unphysical, complex dimensions.

### 4 Green function for free systems

The contraction process of the harmonic oscillator to the free particle in  $D$  dimensions applies also to the time-dependent systems. The sequence of oscillator frequencies  $\omega_N = \omega/N$ , led to a rescaling of the position coordinates through  $r_N = r/\sqrt{N}$  in (60); the physical time, being  $t = \tau/\omega$ , leads thus to the change of scale in the dimensionless time variable  $\tau_N = \tau/N$ .

#### 4.1 Green function from equally-spaced spectra

The Green function of quantum oscillators (45)–(46) contracts to the Green function of free systems when we write it in terms of the variables  $r_N, r'_N$ , and  $\tau_N$ , and let  $N \rightarrow \infty$ . The case of equally-spaced spectra for dimensions  $D = 4k > 0$  ( $\gamma \geq -\frac{1}{4}$ ), including strong barriers and the Friedrichs extension of the Hamiltonians in the exceptional interval, is straightforward. The arguments of the Gaussian and Bessel factors have the asymptotic behavior

$$\frac{r_N^2 + r'_N{}^2}{2 \tan \tau_N} \underset{N \rightarrow \infty}{\sim} \frac{r^2 + r'^2}{2\tau}, \quad \frac{r_N r'_N}{\sin \tau_N} \underset{N \rightarrow \infty}{\sim} \frac{r r'}{\tau}, \tag{78}$$

with the prefactor  $\sqrt{r_N r'_N} / \sin \tau_N$  leading to Dirac normalization (65). Alternatively, the Green function is the generating function, an integral instead of a sum, over the momentum variable  $p$  of two wavefunctions (68). Both ways yield an expression valid for  $k > 0$ ,

$$G_k^F(r, r'; \tau) = \int_0^\infty dp \phi_p^k(r) e^{-ip^2\tau/2} \phi_p^k(r')^* \tag{79}$$

$$= \sqrt{r r'} \int_0^\infty dp p J_{2k-1}(pr) e^{-ip^2\tau/2} J_{2k-1}(pr') \tag{80}$$

$$= e^{-i\pi k} \frac{\sqrt{r r'}}{\tau} \exp\left(i \frac{r^2 + r'^2}{2\tau}\right) J_{2k-1}\left(\frac{r r'}{\tau}\right) \tag{81}$$

$$= C_k \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} (r, r'), \tag{82}$$

where in the last line we write again the radial canonical transform kernel. Represented by matrices, the contraction process is

$$\lim_{N \rightarrow \infty} \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1/\sqrt{N} \end{pmatrix} \begin{pmatrix} \cos \tau/N & \sin \tau/N \\ -\sin \tau/N & \cos \tau/N \end{pmatrix} \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1/\sqrt{N} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}. \tag{83}$$

As its harmonic oscillator counterpart, the free Green function solves the time-dependent free Schrödinger differential equations in  $(r, \tau)$  and in  $(r', \tau)$ , and represents the time evolution of the initial condition  $\delta(r-r')$ ; and it must satisfy the group properties (43)–(44). It is interesting that to verify the group composition property (42), one uses the same integral (80)–(81), replacing  $p$  by  $r'$ , and other substitutions for  $r, r''$  and  $\tau$ . The integral (80)–(81) is valid for all  $k$ 's where  $\int_0^R dp \dots$  is finite, i.e., for all  $\text{Re } k > 0$ , including the strong well case with the Bessel factor  $J_{i\kappa}(rr'/\tau)$ .

We draw attention again to the particular cases of  $D = 1$  and 3 dimensions, where the wavefunctions (70)–(71) are the even and odd parts of the one-dimensional free waves  $\sim e^{ipr}$  on the full range  $r \in \mathfrak{R}$ . The Green functions (81) follow suit:  $G_{1/4}^F(r, r'; \tau)$  is the  $\tau$ -evolution of the positive parity initial condition  $\delta(r-r') + \delta(r+r')$ , while  $G_{3/4}^F(r, r'; \tau)$  is the evolution of the initial condition  $\delta(r-r') - \delta(r+r')$  with negative parity; the first has zero derivative at the origin, while the second has zero value.

## 4.2 Green function from non-Friedrichs extensions

We now follow with extensions of the oscillator Hamiltonian in the exceptional interval  $0 < k < 1$ , distinguished by the label  $\nu$  through the constant  $f_k(\nu) \in \mathfrak{R}$  in (35). Attempts to write a Green function out of the *non*-Friedrichs wavefunctions (74), with arguments now based on the Dirac completeness relation, will confirm their futility, as suggested at the end of Sect. 3.

Wavefunctions belonging to the non-Friedrichs extensions,  $\phi_p^{k[\nu]}(r)$  in (74), have two summands,  $\phi_p^k(r)$  and  $\phi_p^{1-k}(r)$ , each of which belongs to a Friedrichs extension, and which are exchanged under  $k \leftrightarrow 1 - k$ . Abbreviating  $c := \cos \theta_{[\nu]}^k$  and  $s := \sin \theta_{[\nu]}^k$ , their inner product is

$$\begin{aligned} (\phi_p^{k[\nu]}(r), \phi_{p'}^{k[\nu]}(r))_{\mathcal{L}^2(\mathfrak{R}^+)} &= c^2(\phi_p^k, \phi_{p'}^k)_{\mathcal{L}^2(\mathfrak{R}^+)} + s^2(\phi_p^{1-k}, \phi_{p'}^{1-k})_{\mathcal{L}^2(\mathfrak{R}^+)} \\ &+ sc[(\phi_p^k, \phi_{p'}^{1-k})_{\mathcal{L}^2(\mathfrak{R}^+)} + (\phi_p^{1-k}, \phi_{p'}^k)_{\mathcal{L}^2(\mathfrak{R}^+)}]. \end{aligned} \quad (84)$$

The first two terms sum to  $\delta(p - p')$ , while the cross term in  $sc$  is not zero, but

$$\begin{aligned} \sqrt{pp'} \left( \int_0^\infty dr r J_{2k-1}(pr) J_{1-2k}(p'r) + \{k \leftrightarrow 1 - k\} \right) \\ = 2 \cos \pi(2k-1) \delta(p - p') + \frac{2 \sin \pi(2k-1)}{\pi(p^2 - p'^2)} \left( \left(\frac{p}{p'}\right)^{2k-1} - \left(\frac{p'}{p}\right)^{2k-1} \right). \end{aligned} \quad (85)$$

(See [20, Eq. (22)] and [21, Eq. (12)], with a sign correction.)

Although Dirac normalization holds by construction, orthogonality does not; a Green function built as the bilinear generating integral (79) between  $\phi_p^{k[\nu]}(r)$ 's, does not satisfy Dirac *completeness* [cf. Eq. (69)] because of the non- $\delta$  term in (85). Hence it can *not* satisfy the basic group properties (42)–(43). Except, that is, when  $sc = 0$  ( $\theta_{[\nu]}^k = 0$  or  $\pm \frac{1}{2}\pi$ ), i.e., for the Friedrichs extensions. The same argument applies to the Green function in the strong well case  $\gamma < -\frac{1}{4}$ , where the wavefunctions are (76); there, the cross coefficient never vanishes because there are no Friedrichs extensions.

## 5 Radial expectation values

In [1,8] attention was drawn to the time evolution of the expectation value of the radius for real wavefields in  $D = 2$  dimensions. This property was attested by numerical experimentation with the Green function (81) and initial wavefields (73) of Gaussian shape  $\Phi(\rho) \sim \rho^2 e^{-\alpha\rho^2}$ . In this section we take advantage of the Lie-algebraic structure of the space of quadratic operators.

### 5.1 Transformations under the $\mathbf{Sp}(2, \mathfrak{R})$ group

Under the group  $\mathbf{Sp}(2, \mathfrak{R})$  of linear isotropic canonical transformations (49) of the  $D$ -dimensional position and momentum operators  $\vec{Q}$  and  $\vec{P}$ , the three quadratic Schrödinger operators  $J_-$ ,  $J_2$  and  $J_+$  in (2) and (4) transform as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} \Rightarrow \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & cd & d^2 \end{pmatrix} \begin{pmatrix} Q^2 \\ \frac{1}{2}(\vec{Q} \cdot \vec{P} + \vec{P} \cdot \vec{Q}) \\ P^2 \end{pmatrix}. \quad (86)$$

The expectation values of the square radius  $Q^2$  in a free  $s$ -wavefield  $\Phi(\vec{q}) = \Phi(\rho)$ , and of its  $\mathfrak{sp}(2, \mathfrak{R})$  companion generators, including the kinetic energy (free Hamiltonian)  $P^2$  are, in terms of the inner products on  $\mathfrak{R}^D$  and in  $\mathfrak{R}^+$ ,

$$(\Phi, Q^2 \Phi)_{\mathcal{L}^2(\mathfrak{R}^D)} = S_{D-1} (\phi, r^2 \phi)_{\mathcal{L}^2(\mathfrak{R}^+)} =: S_{D-1} \langle r^2 \rangle_\phi, \quad (87)$$

with  $S_{D-1}$  the surface of the unit sphere; and similarly for the two other operators. Under free evolution generated by  $\mathcal{C}(\tau) := \mathcal{C}\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$  on  $\phi_o(r) = \phi(r, 0)$ , their expectation values will thus evolve through

$$\langle r^2 \rangle_{\phi(\tau)} := (\phi(\tau), r^2 \phi(\tau))_{\mathcal{L}^2(\mathbb{R}^+)} = \int_0^\infty dr \phi(r, \tau)^* r^2 \phi(r, \tau) \tag{88}$$

$$\begin{aligned} &= (\mathcal{C}(\tau) \phi_o, r^2 \mathcal{C}(\tau) \phi_o) = (\phi_o, \mathcal{C}(\tau)^{-1} r^2 \mathcal{C}(\tau) \phi_o) \\ &= \langle r^2 \rangle_{\phi_o} + 2\tau \langle -i\frac{1}{2}(r \partial_r + \partial_r r) \rangle_{\phi_o} + \tau^2 \langle (-\partial_r^2 + \gamma/r^2) \rangle_{\phi_o}, \end{aligned} \tag{89}$$

$$\langle -i\frac{1}{2}(r \partial_r + \partial_r r) \rangle_{\phi(\tau)} = \langle -i\frac{1}{2}(r \partial_r + \partial_r r) \rangle_{\phi_o} + \tau \langle (-\partial_r^2 + \gamma/r^2) \rangle_{\phi_o}, \tag{90}$$

$$\langle (-\partial_r^2 + \gamma/r^2) \rangle_{\phi(\tau)} = \langle (-\partial_r^2 + \gamma/r^2) \rangle_{\phi_o} \text{ invariant,} \tag{91}$$

where we write  $\partial_r := d/dr$ . Both  $2K_- = r^2$  and  $2K_+ = -\partial_r^2 + \gamma/r^2$  are nonnegative operators that belong to the same orbit under  $\mathbf{Sp}(2, \mathbb{R})$ ; one is the Hankel transform  $\mathcal{C}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  of the other.

The third operator, namely  $2K_2 = -i\frac{1}{2}(r \partial_r + \partial_r r)$  is self-adjoint between wavefunctions  $\phi(r)$  whose boundary term  $r |\phi(r)|^2|_0^\infty$  vanishes. Its spectrum is  $\mathbb{R}$ , and it lies in the same orbit as the repulsive oscillator Hamiltonian  $2K_1$  in (11); they are related by the square root of the Hankel transform  $\mathcal{C}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ . And because  $\frac{1}{2}(\vec{Q} \cdot \vec{P} + \vec{P} \cdot \vec{Q}) = -i\frac{1}{4}[Q^2, P^2]$ , its expectation value is bounded by

$$\left| \langle -i\frac{1}{2}(r \partial_r + \partial_r r) \rangle_{\phi} \right|^2 \leq \frac{1}{4} \langle r^2 \rangle_{\phi} \langle (-\partial_r^2 + \gamma/r^2) \rangle_{\phi}, \tag{92}$$

which ensures that the expectation value of  $r^2$  in (89) remains positive.

For small  $\tau$ , the behavior of the expectation value (89) depends on the sign of the term linear in  $\tau$ ; if this is negative, the  $s$ -wavepacket will start contracting before the  $\tau^2$  term dominates and the packet disperses. To explore this phenomenon we study the evolution of the expectation values for three distinguished sets of  $s$ -states: localized states (eigenfunctions of  $Q^2$ ) diffused into ‘radial Gaussians’ as if by heat propagation, the energy eigenstates (eigenfunctions of  $\frac{1}{2}P^2$ ), and Gaussian-weighted superpositions of dilatation-invariant functions [eigenfunctions of  $\frac{1}{2}(\vec{Q} \cdot \vec{P} + \vec{P} \cdot \vec{Q})$ ].

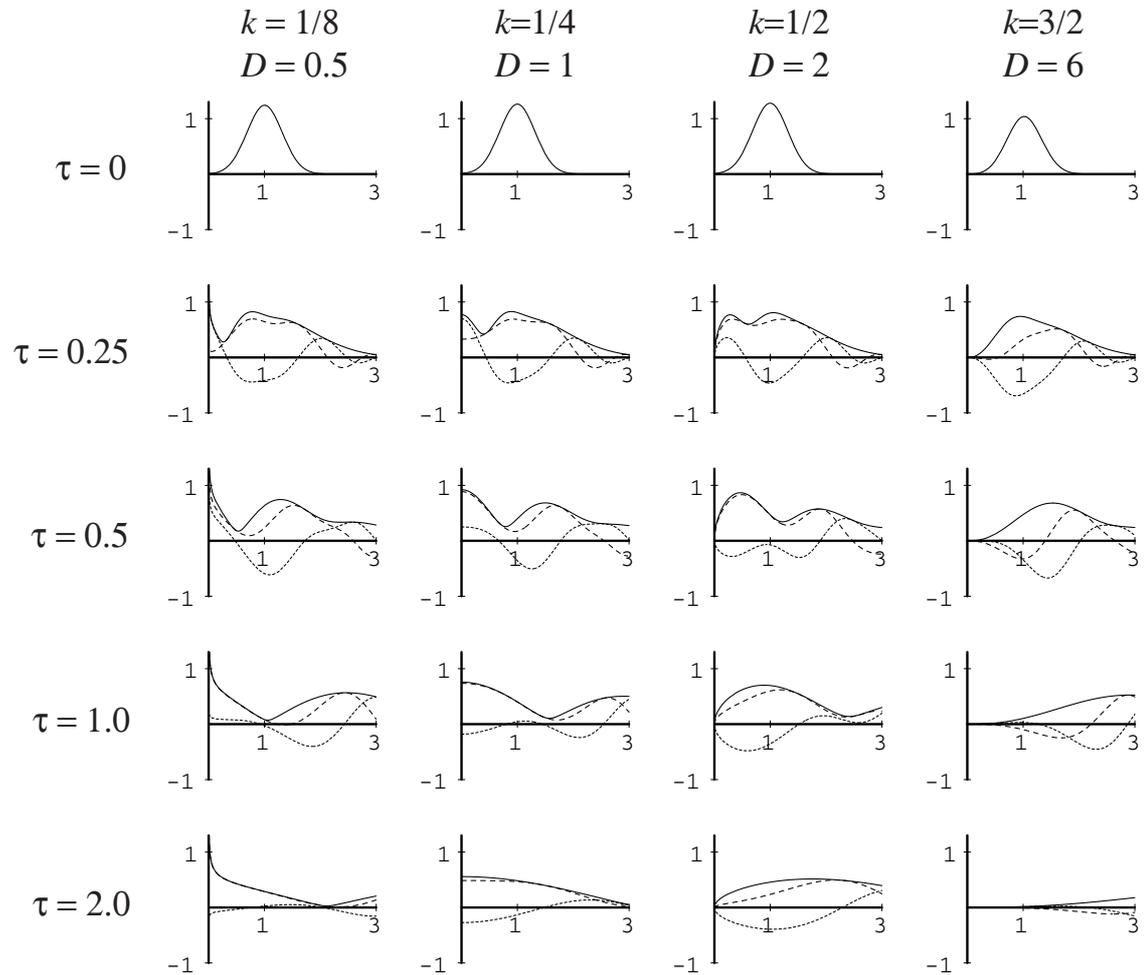
### 5.2 Diffused $\delta$ 's: radial Gaussian states

The  $s$ -states that are localized at a radius  $r_o \in \mathbb{R}^+$ , are the Dirac states  $\delta_{r_o}(r) = \delta(r - r_o)$ ; they evolve in time into the canonical transform kernel and Green function (82) given by  $C_k\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}(r, r_o)$ . For  $\tau > 0$  this is a chirping function with a phase factor  $\exp(ir^2/2\tau)$ , so computing integrals for expectation values is analytically difficult and numerically unstable. However, we can let this Dirac delta *diffuse* driven by the heat equation, into the same Green function but with a pure-imaginary ‘time’  $-i\omega, \omega > 0$ , thus:

$$G_k^F(r, r_o; -i\omega) = \frac{\sqrt{r r_o}}{\omega} \exp\left(-\frac{r^2 + r_o^2}{2\omega}\right) I_{2k-1}\left(\frac{r r_o}{\omega}\right) = C_k\begin{pmatrix} 1 & -i\omega \\ 0 & 1 \end{pmatrix}(r, r_o). \tag{93}$$

This is a complex radial canonical transform kernel [5] defining a ‘radial Gaussian’ function of square width  $\omega$ . The real-time free evolution of this Gaussian (93) will be given by

$$G_k^F(r, r_o; \tau - i\omega) = C_k\begin{pmatrix} 1 & \tau - i\omega \\ 0 & 1 \end{pmatrix}(r, r_o), \tag{94}$$



**Fig. 5** Radial Gaussians  $G_k^F(r, r_o; \tau - i\omega)$ , centered on  $r_o = 1$ , as functions of radius  $r$ , and their time evolution in  $\tau$  for various dimensions  $D = 4k$ . The columns contain the cases  $k = \frac{1}{8}$  ( $D = \frac{1}{2}$ ),  $k = \frac{1}{4}$  ( $D = 1$ ),  $k = \frac{1}{2}$  ( $D = 2$ ), and  $k = \frac{3}{2}$  ( $D = 6$ ), and the rows  $\tau = 0, 0.5, 1, 2$ . The behavior at the origin is  $\sim r^{2k-1/2}$ ; in higher dimensions, heat can escape in more directions. Dashed, dotted and continuous lines are the real, imaginary and absolute values of the functions.

because the composition property holds for the canonical transform kernels  $C_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (r, r_o)$  in (49), under analytic continuation of  $b$  into the lower complex half-plane and  $a, d > 0$  [5]. In Figs. 5 we show a sample of these time-evolving radial Gaussians for various dimensions.

The complex transform (93) conserves the linear norm of total heat,  $\int_0^\infty dr G_k^F(r, r_o; -i\omega) = 1$ ; but the transform is *not* unitary in  $\mathcal{L}^2(\mathfrak{R}^+)$ . Yet, diffusion of the initial  $\delta(r - r_o)$  yields functions of finite  $\mathcal{L}^2(\mathfrak{R}^+)$ -norm that we can calculate using matrices, as follows:

$$\begin{aligned} \|G_k^F(r_o; \tau - i\omega)\|^2 &:= \int_0^\infty dr |G_k^F(r, r_o; \tau - i\omega)|^2 \\ &= \int_0^\infty dr \left[ C_k \begin{pmatrix} 1 & \tau - i\omega \\ 0 & 1 \end{pmatrix} (r, r_o) \right]^* C_k \begin{pmatrix} 1 & \tau - i\omega \\ 0 & 1 \end{pmatrix} (r, r_o) \end{aligned} \tag{95}$$

$$\begin{aligned}
 &= \int_0^\infty dr C_k \begin{pmatrix} 1 & -\tau - i\omega \\ 0 & 1 \end{pmatrix} (r_o, r) C_k \begin{pmatrix} 1 & \tau - i\omega \\ 0 & 1 \end{pmatrix} (r, r_o) \\
 &= C_k \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix} (r_o, r_o) = \frac{r_o}{2\omega} \exp\left(-\frac{r_o^2}{2\omega}\right) I_{2k-1}\left(\frac{r_o^2}{2\omega}\right),
 \end{aligned} \tag{96}$$

where we have used the complex conjugation property stemming from the analytic form of (49),

$$C_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (r, r_o)^* = C_k \begin{pmatrix} a & -b^* \\ -c^* & d \end{pmatrix} (r, r_o) = C_k \begin{pmatrix} d & -b^* \\ -c^* & a \end{pmatrix} (r_o, r), \tag{97}$$

which is valid for  $a, d > 0$  and  $\text{Im } b < 0$ . [Compare with the unitarity condition (44) for the Green function.]

We can follow the process (95)–(96) to find the expectation values of the operators that generate the canonical transforms [see (51)], between these radial Gaussian states normalized by (95). Thus, for  $P^2 \sim 2K_+$  in (88) and (91),

$$\begin{aligned}
 &\langle (-\partial_r^2 + \gamma/r^2) \rangle_{G_k(r_o; \tau, \omega)} \times C_k \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix} \\
 &= 2i \frac{\partial}{\partial \beta} \int_0^\infty dr \left[ C_k \begin{pmatrix} 1 & \tau - i\omega \\ 0 & 1 \end{pmatrix} (r, r_o) \right]^* \mathcal{C} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} C_k \begin{pmatrix} 1 & \tau - i\omega \\ 0 & 1 \end{pmatrix} (r, r_o) \Big|_{\beta=0} \\
 &= 2i \frac{\partial}{\partial \beta} C_k \begin{pmatrix} 1 & \beta - 2i\omega \\ 0 & 1 \end{pmatrix} (r_o, r_o) \Big|_{\beta=0}.
 \end{aligned} \tag{98}$$

The result is independent of  $\tau$ , as expected from (91). By replacing the functional form (49) this can be computed to be

$$2i \frac{\partial}{\partial \beta} \left[ \frac{r_o}{\beta - 2i\omega} \exp\left(i \frac{r_o^2}{\beta - 2i\omega}\right) J_{2k-1}\left(\frac{r_o^2}{\beta - 2i\omega}\right) \right] \Big|_{\beta=0} \tag{100}$$

$$= -2i \frac{z^2}{r_o^2} \frac{d}{dz} \left( z e^{iz} J_{2k-1}(z) \right) \Big|_{z=ir_o^2/2\omega} \tag{101}$$

$$= \frac{2\omega - r_o^2}{2\omega^2} C_k \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix} + i \frac{r_o}{2\omega^2} z e^{iz} J'_{2k-1}(z) \Big|_{z=ir_o^2/2\omega}. \tag{102}$$

The derivative of the Bessel function  $J_{2k-1}(z)$  yields  $(2k-1) J_{2k-1}(z)/z$  minus  $J_{2k}(z)$ ; the latter returns  $i$  times the kernel  $C_{k+1/2} \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix} (r_o, r_o)$ . Dividing by the square norm (96) and noting that its dependence on  $k$  is exclusively through the index of the modified Bessel function, we find the expectation value of (twice) the free Hamiltonian for radial Gaussian functions, which is

$$\langle (-\partial_r^2 + \gamma/r^2) \rangle_{G_k(r_o; \tau, \omega)} = \frac{r_o^2}{2\omega^2} \left( \frac{4k\omega}{r_o^2} + \frac{I_{2k}(r_o^2/2\omega)}{I_{2k-1}(r_o^2/2\omega)} - 1 \right). \tag{103}$$

The quantity in parentheses is a function of dimensionless  $r_o^2/2\omega$  which is positive for all  $k > 0$ .

Next, the expectation value of  $\frac{1}{2}(\vec{Q} \cdot \vec{P} + \vec{P} \cdot \vec{Q})$  can be calculated following (98)–(99), and arriving at

$$\langle -i \frac{1}{2} (r \partial_r + \partial_r r) \rangle_{G_k(r_o; \tau, \omega)} \times C_k \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix} \tag{104}$$

$$= i \frac{\partial}{\partial \alpha} C_k \begin{pmatrix} e^\alpha & -2i(\omega \cosh \alpha - \tau \sinh \alpha) \\ 0 & e^{-\alpha} \end{pmatrix} (r_o, r_o) \Big|_{\alpha=0} \tag{105}$$

$$= i \frac{\partial}{\partial \alpha} \left[ \frac{r_o}{B(\alpha)} \exp\left(i \frac{r_o^2 \cosh \alpha}{B(\alpha)}\right) J_{2k-1}\left(\frac{r_o^2}{B(\alpha)}\right) \right] \Big|_{\alpha=0}. \tag{106}$$

Comparison with (100) shows that here we have a derivative with respect to a group parameter that we must turn into a derivative with respect to  $z = r_o^2/B(\alpha)$ , now with  $B(\alpha) = -2i(\omega \cosh \alpha - \tau \sinh \alpha)$ , instead of  $\beta - 2i\omega$  before; but note that their limit  $z|_{\alpha=0} = ir_o^2/2\omega$  is the same. There is one more summand, stemming from the  $\alpha$ -derivative of the exponent; this will yield a factor of  $\sinh \alpha$ , which vanishes in the limit  $\alpha \rightarrow 0$ . In front, there will be a new factor  $\partial B(\alpha)/\partial \alpha|_{\alpha=0} = 2i\tau$  that multiplies the expression (103), as predicted in (90), with an initial expectation value that vanished in the limit  $\alpha \rightarrow 0$ . The important result is thus that for the set of radial Gaussian functions,

$$\langle -i\frac{1}{2}(r\partial_r + \partial_r r) \rangle_{G_k(r_o; \tau=0, \omega)} = 0, \quad (107)$$

and therefore that  $s$ -wavefields with radial Gaussian profiles will *not* exhibit any initial contraction.

Lastly, we compute the expectation value of the square radius  $Q^2$  through (88). It will be sufficient to do this for  $\tau = 0$ , since we know that  $\tau$ -evolution will only add a summand of  $\tau^2$  times (103). We follow the derivation (98)–(99) for the third time, using the last of Eqs. (51), to compute

$$\langle r^2 \rangle_{G_k(r_o; 0, \omega)} \times C_k \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix} \quad (108)$$

$$= 2i \frac{\partial}{\partial \gamma} C_k \begin{pmatrix} 1-i\omega\gamma & -\omega^2\gamma-2i\omega \\ \gamma & 1-i\omega\gamma \end{pmatrix} (r_o, r_o) \Big|_{\gamma=0} \quad (109)$$

$$= 2i \frac{\partial}{\partial \gamma} \left[ \frac{r_o}{C(\gamma)} \exp \left( i \frac{r_o^2(1-i\omega\gamma)}{C(\gamma)} \right) J_{2k-1} \left( \frac{r_o^2}{C(\gamma)} \right) \right] \Big|_{\gamma=0}, \quad (110)$$

where now  $C(\gamma) = -\omega^2\gamma - 2i\omega$ . Again there are two terms, one stemming from the derivative of the exponent which returns  $\frac{1}{2}r_o^2$  times the norm (96), and the second being (99), but now for  $z = r_o^2/C(\gamma)$  and again  $z|_{\gamma=0} = ir_o^2/2\omega$ ; the overall factor is  $\partial C(\gamma)/\partial \gamma|_{\gamma=0} = \frac{1}{4}r_o^4$ . We are thus led to the result

$$\langle r^2 \rangle_{G_k(r_o; 0, \omega)} = \frac{r_o^2}{2} \left( \frac{4k\omega}{r_o^2} + \frac{I_{2k}(r_o^2/2\omega)}{I_{2k-1}(r_o^2/2\omega)} + 1 \right), \quad (111)$$

which differs from (103) only in the last sign. Now the function of  $r_o^2/2\omega$  in parentheses is greater than 2 for finite values of the argument. We note that when  $\omega \rightarrow 0^+$ ,  $\lim_{z \rightarrow \infty} I_{2k}(z)/I_{2k-1}(z) = 1$ ; hence, the expectation value of  $r^2$  for a Dirac  $\delta$  at  $r_o$  is indeed  $r_o^2$ , and increases as the Gaussian width  $\omega$  grows.

### 5.3 Diffused energy eigenstates

The energy eigenstates  $\phi_p^k(r)$ ,  $p > 0$ , are the normal modes given in (68); they evolve in time  $\tau$  through multiplication by a phase:  $\phi_p^k(r, \tau) = e^{-i\tau p^2/2} \phi_p^k(r)$ . Diffusion by imaginary time  $-i\omega$  will turn this phase into  $e^{-\omega p^2/2} e^{-i\tau p^2}$ , so that the norm of the diffused functions is  $e^{-\omega p^2}$ , taking the place of  $C_k \begin{pmatrix} 1 & -2i\omega \\ 0 & 1 \end{pmatrix}$  in (98) and (104). The expectation values of  $r^2$  in a fixed- $p_o$  state  $\phi_{p_o}^k(r, \tau)$  will be invariant, because the phases cancel in the inner product (88), and the norms cancel in the definition. Indeed, when we apply the  $D$ -dimensional Fourier transform  $\mathcal{F} = e^{i\pi k} \mathcal{C} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  to (86), or equivalently the Hankel transform to (98), (107) and (111), we exchange  $Q^2$  and  $P^2$  while  $2K_2$  changes sign. We thus obtain

$$\langle r^2 \rangle_{\phi_{p_o}^k(\tau, \omega)} = \langle (-\partial_r^2 + \gamma/r^2) \rangle_{G_k(p_o; 0, \omega)}, \quad (112)$$

$$\langle -i\frac{1}{2}(r\partial_r + \partial_r r) \rangle_{\phi_{p_o}^k(\tau, \omega)} = -\langle -i\frac{1}{2}(r\partial_r + \partial_r r) \rangle_{G_k(p_o; \tau, \omega)} = 0, \quad (113)$$

$$\langle (-\partial_r^2 + \gamma/r^2) \rangle_{\phi_{p_o}^k(\tau, \omega)} = \langle r^2 \rangle_{G_k(p_o; \tau, \omega)}. \quad (114)$$

In other words, we have a diffused  $\delta$  in momentum  $p$  (energy  $\frac{1}{2}p^2$ ) around  $p_o$ , in the same way that we previously had a diffused  $\delta$  in position  $r$  around  $r_o$ .

5.4 Dilatation Gaussian states

The last set of functions that we use to study the time dependence of expectation values are the eigenfunctions of  $\frac{1}{2}(\vec{Q}\cdot\vec{P}+\vec{P}\cdot\vec{Q})$ , the generator of dilatations. The eigenfunctions of  $2K_2$  are

$$-i\frac{1}{2}(r\partial_r+\partial_r r)\chi_\lambda(r) = \lambda\chi_\lambda(r), \quad \chi_\lambda(r) = \frac{r^{i\lambda-1/2}}{\sqrt{2\pi}}, \tag{115}$$

for  $\lambda \in \mathfrak{R}$ , and Dirac-normalized,  $(\chi_\lambda, \chi_{\lambda'})_{\mathcal{L}^2(\mathfrak{R}^+)} = \delta(\lambda - \lambda')$ . They are inappropriate to form expectation values, because all oscillate (except  $\lambda = 0$ ) and grow  $\sim 1/\sqrt{r}$  at the origin.

The dilatation eigenbasis (115) is independent of dimension  $D = 4k$ , and serves as the kernel for the unilateral Mellin transform [16, Sect. 8.2], which becomes essentially the (inverse) Fourier transform after the change of variables  $r = e^s$ . This facilitates the computation of Gaussian superpositions that we can build as normalized wavefunctions centered on the dilatation factor  $\lambda_o \in \mathfrak{R}$ ,

$$f_{\lambda_o, \omega}(r = e^s) := \int_{-\infty}^{\infty} d\lambda G_\omega(\lambda - \lambda_o) \chi_\lambda(r) = \frac{e^{-s/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda G_\omega(\lambda - \lambda_o) e^{i\lambda s}, \tag{116}$$

with a (true)  $\mathcal{L}^2(\mathfrak{R})$ -normalized Gaussian of square width  $\omega$ ,

$$G_\omega(\lambda) := (\pi\omega)^{-1/4} \exp(-\lambda^2/2\omega). \tag{117}$$

The integration (116) then yields the wavefield

$$f_{\lambda_o, \omega}(r=e^s) = \exp(i\lambda_o - \frac{1}{2}) G_{1/\omega}(s) \tag{118}$$

$$= (\omega/\pi)^{1/4} r^{(i\lambda_o - 1/2)} \exp\left(-\frac{1}{2}\omega(\ln r)^2\right), \tag{119}$$

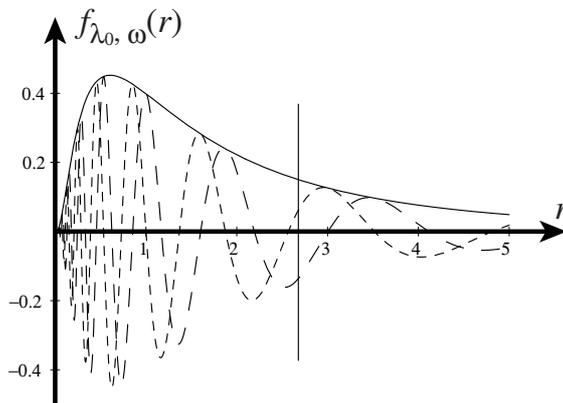
whose generic form is shown in Fig. 6.

It is now easy to compute the expectation value of  $\frac{1}{2}(\vec{Q}\cdot\vec{P}+\vec{P}\cdot\vec{Q})$  noting that

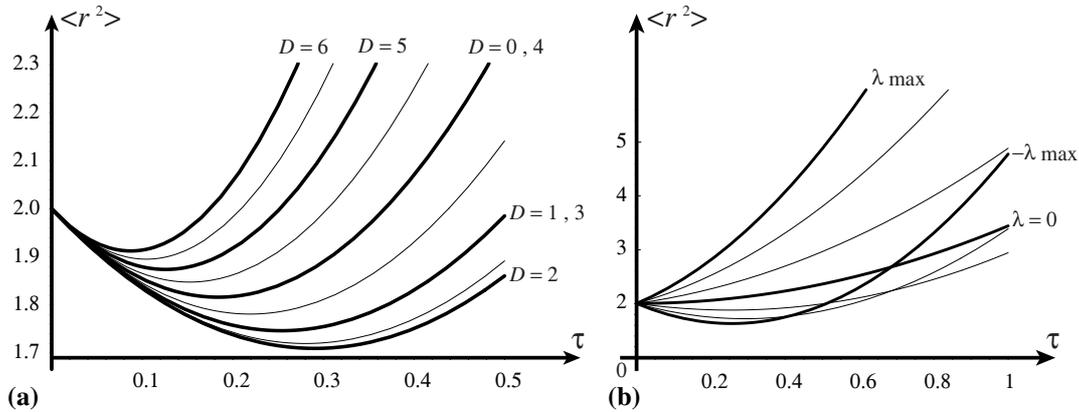
$$-i\frac{1}{2}(r\partial_r+\partial_r r)f_{\lambda_o, \omega}(r) = -i(\partial_s + \frac{1}{2})f_{\lambda_o, \omega}(r=e^s) = (\lambda_o + i\omega s)f_{\lambda_o, \omega}(r). \tag{120}$$

When we introduce this into the inner product with  $dr = e^s ds$  and  $f_{\lambda_o, \omega}(r)^*$ , the imaginary summand with  $i\omega s$  will vanish due to parity, hence

$$\langle -i\frac{1}{2}(r\partial_r+\partial_r r) \rangle_{f(\lambda, \omega; r)} = \lambda. \tag{121}$$



**Fig. 6** Generic form of the Gaussian dilatation wavefunctions  $f_{\lambda_o, \omega}(r)$  in (119), for center  $\lambda_o = 10$  and square width  $\omega = 1$ . The real and imaginary parts of the function (dashed and dotted lines) oscillate with a logarithmic chirp proportional to  $\lambda$ ; only for  $\lambda_o = 0$  it is real. The absolute value (continuous line) is independent of  $\lambda$ , and only scales with width  $\omega$ . We mark the expectation value  $e$  of  $r^2$ .



**Fig. 7** Time evolution of the expectation values of square radius  $\langle r^2 \rangle_{f(\lambda, \omega; r)}$ , with initial value 2, between dilatation Gaussian functions. (a) For fixed negative value  $\lambda = -1$  and dimensions  $D = 0, 0.5, 1, \dots, 6$ ; integer dimensions are marked with heavy lines, half-integer ones with thin lines; in the exceptional interval, dimensions  $D$  and  $4 - D$  result in the same lines. (b) For  $D = 2$  dimensions ( $k = \frac{1}{2}$ ) and the range of  $\lambda$  allowed by (92); heavy lines indicate the minimum, zero and maximum values of  $\lambda$  according to (125).

The expectation value of any power of the radius,  $r^\alpha = e^{\alpha s}$ , can be computed between these functions; it is

$$\langle r^\alpha \rangle_{f(\lambda, \omega; r)} = \exp\left(\frac{\alpha^2}{4\omega}\right). \tag{122}$$

In particular,

$$\langle r^2 \rangle_{f(\lambda, \omega; r)} = e^{1/\omega}. \tag{123}$$

In Fig. 6 we mark the expectation value of  $r^2$  for  $\omega = 1$ , which is  $e$ . Similarly, for the free Hamiltonian,

$$\langle -\partial_r^2 + \gamma/r^2 \rangle_{f(\lambda, \omega; r)} = \left(\lambda^2 + \frac{1}{2}\omega + (2k-1)^2\right) e^{1/\omega}. \tag{124}$$

The range allowed to  $\lambda$  is curtailed by (92), which from (123) and (124) results in  $-\lambda_{\max} \leq \lambda \leq \lambda_{\max}$ , with

$$\lambda_{\max}^2 = \frac{1}{4} e^{2/\omega} [\frac{1}{2}\omega + (2k-1)^2] / (4 - e^{2/\omega}). \tag{125}$$

In Fig. 7(a) we show the time- $\tau$  evolution of the expectation value of the square radius for the dilatation Gaussian  $s$ -state  $f_{\lambda, \omega}(r, \tau)$  in (119), characterized by the fixed negative value  $\lambda = -1$ , in various dimensions  $D = 4k$ , according to Eqs. (89) and (124). For all  $D$ , this wavepacket initially contracts and then expands, but we note that in  $D = 2$  dimensions ( $k = \frac{1}{2}$ ) it descends closest to zero. In Fig. 7(b) we show the possible behaviors in  $D = 2$  dimensions when we let  $\lambda$  range over the interval allowed to it by the inequality (92). As we may expect,  $s$ -wavepackets that contract the most, will also eventually expand as fast as the packet that initially expands fastest, because (124) depends only on  $\lambda^2$ .

To characterize  $s$ -wavefields  $f(r, \tau)$  that contract, i.e., for which the expectation value  $\langle -i\frac{1}{2}(r\partial_r + \partial_r r) \rangle_{f_o} < 0$ , it is sufficient to test its unilateral Mellin transform at  $\tau = 0$ ,

$$f_o^M(\lambda) = \int_0^\infty dr f_o(r) \chi_\lambda(r)^*, \quad f_o(r) = \int_{-\infty}^\infty d\lambda f_o^M(\lambda) \chi_\lambda(r). \tag{126}$$

The expectation value of the dilatation generator is then

$$\langle -i\frac{1}{2}(r\partial_r + \partial_r r) \rangle_{f_o} = \int_{-\infty}^\infty d\lambda \lambda |f_o^M(\lambda)|^2. \tag{127}$$

Thus, when the Mellin transform of the initial wavefield  $f_o(r)$  has most of its area in the negative- $\lambda$  half-axis, it will contract; when it is in the positive- $\lambda$  half-axis, it will expand. When the wavefield is purely real, its Mellin transform  $f_o^M(\lambda)$  is an even function of  $\lambda$ , and the expectation value (127) of the dilatation generator will be zero, independently of the dimension  $D$ . This characterization also applies to wavefields in the  $D$ -dimensional quantum harmonic oscillator.

## 6 Conclusions

The theory of representations of the Lorentz algebra  $\mathfrak{so}(2, 1)$  and covering groups which is realized by second-order differential operators, contains the models of  $s$ -waves in the  $D$ -dimensional quantum harmonic oscillator and free particle, and their time evolution. We considered continuous dimensions  $D > 0$ , and reduced expressions to radial coordinates, where dimension enters as a fictitious centrifugal barrier for  $0 < D < 1$  and  $D > 3$ , or a centripetal well for  $1 < D < 3$ .

We noted that the critical range of low dimensions  $0 < D < 4$  corresponds to the exceptional interval of  $\mathfrak{so}(2, 1) = \mathfrak{sp}(2, \mathfrak{R})$  representations studied by Bargmann [2] – which is indeed intricate. Dimension  $D = 2$  in particular, connects not only to both its sides in the real- $D$  line of ‘discrete’-series representations, but also to the line of ‘continuous’-series representations  $D = 2(1 + i\kappa)$  of strong centripetal wells. In the exceptional interval, the oscillator and free Hamiltonians exhibit a one-parameter family of self-adjoint extensions in  $\mathcal{L}^2(\mathfrak{R}^+)$ , two of which are the physically relevant Friedrichs extensions having finite kinetic energy. The members of this family can be distinguished readily by their oscillator spectrum; we followed this characterization under contraction to the free system, whose spectra are continuous.

We used results from real and complex radial canonical transforms [4, 5] to exponentiate  $\mathfrak{so}(2, 1)$  to the universal cover of the Lorentz group  $\mathbf{SO}(2, 1)$ ; in fractional dimension  $D = p/q$  (relatively prime) we have a  $4q$ -fold cover given by a phase in the oscillator Green function. This phase is inherited in the contraction limit to the free particle. Also, we noted that *non*-Friedrichs extensions *cannot* be used to build Green functions that satisfy the one-parameter group composition property.

Finally, we obtained the expectation values of three linearly independent generators for three function sets: eigenfunctions of square radius, energy, and dilatation – and their ‘heat’-diffused wavefunctions. These expectation values were found as derivatives of the radial canonical transform kernels with respect to their parameters. Since time evolution of a free system is a linear canonical transformation in phase space, the question of contraction or expansion of  $s$ -wave wavefields becomes a matter of computing the expectation value of the generator of dilatations. When the sign of this is negative, the wavepacket will initially contract; when positive, it will expand – quite independently of the dimension  $D$ . As noted above, dimension  $D = 2$  is special in several respects, but it is a smooth limit of the Friedrichs extensions in the  $D > 0$  line.

For integer dimension  $D \geq 2$ , the angular momentum term in (1) can have eigenvalues  $L(L + D - 2)$ ,  $L \in \mathcal{Z}_0^+$ ; this entails  $D \mapsto D + 2L$  in all subsequent formulas, although of course the geometric interpretation breaks down for non-integer dimension. The purpose of considering continuous dimensions here, as was done in [22] for square wells, is to elucidate whether certain properties are peculiar to some specific dimension, or generic for an open dimensional range, within the freedom allowed by mathematics beyond the physics of quantum mechanics and optics. In particular, the initial decrease of the expectation value of the radius in  $D = 2$  dimensions for annular Gaussian wavefunctions [1, 8], is absent for the square radius, because classical and quantum mechanics follow each other for quantities quadratic in the coordinates of phase space.

The Lorentz algebra has three Hamiltonian orbits: the harmonic oscillator ( $K_0$ , elliptic, timelike), the free particle ( $K_+$  or  $K_-$ , parabolic, lightlike), and the repulsive oscillator ( $K_1$  or  $K_2$ , hyperbolic, spacelike). This last system was not addressed here, but a very similar analysis can be performed to understand the time evolution of  $s$ -wavepackets under  $D$ -dimensional paraxial divergent lenses or radial repulsive oscillators with barriers or wells. From another point of view, one can separate the plane in hyperbolic coordinates (instead of polar), with ‘ $s$ -waves’ defined as wavefields that are constant over a set of concentric hyperbolas,

and evolving through Green functions that are hyperbolic canonical transforms [23]. The overlap coefficients between the eigenfunctions in the three orbits were computed in [24].

In a following paper we shall examine  $s$ -wavefields that obey the d'Alembert wave equation, which is also enthralled to the Lorentz group. The wave equation is of second order in time, so the phase-space description requires both the field, its time derivative, and causality; it is perplexing that in odd dimensions the Huygens principle holds, while in even dimensions there is a reverberation behind the wavefront. These features make the description of continuous- $D$   $s$ -wave systems interesting.

## Appendix

### A Normalization of oscillator wavefunctions

Here we find the normalization constants  $c_\mu^k$  of the wavefunctions in (17), by calculating the inner product (10) of  $\psi_\mu^k(r)$  with the slightly-displaced wavefunction  $\psi_{\mu+\varepsilon}^k(r)$ , and then letting  $\varepsilon \rightarrow 0$ . We use Eq. (32) to put this integral in terms of the value of the Wronskian at the space origin  $r \rightarrow 0^+$ , i.e., the result follows from

$$\begin{aligned} 1 &= \lim_{\varepsilon \rightarrow 0} (\psi_\mu^k, \psi_{\mu+\varepsilon}^k)_{\mathcal{L}^2(\mathbb{R}^+)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \mathcal{W}(\psi_\mu^k, \psi_{\mu+\varepsilon}^k) \Big|_{r \rightarrow 0^+} \\ &= |c_\mu^k|^2 \frac{\pi}{2 \sin 2\pi k} \left( \frac{1}{\Gamma(1-k-\mu)} \frac{\partial}{\partial \nu} \frac{1}{\Gamma(k-\nu)} \right. \\ &\quad \left. - \frac{1}{\Gamma(k-\mu)} \frac{\partial}{\partial \nu} \frac{1}{\Gamma(1-k-\nu)} \right) \Big|_{\nu \rightarrow \mu} \\ &= |c_\mu^k|^2 \frac{\pi}{2 \sin 2\pi k} \frac{\psi(1-k-\mu) - \psi(k-\mu)}{\Gamma(1-k-\mu)\Gamma(k-\mu)}, \end{aligned} \quad (128)$$

where  $\psi(z)$  is the digamma function. From here follows the result quoted in Eq. (18), which has a positive radicand.

Now we examine the special case  $k = \frac{1}{2}$ , where (18) appears to be indeterminate. This point corresponds to the important case of dimension  $D = 2$ , which is the boundary between weak wells in the exceptional interval  $0 < k < 1$ , and strong wells where  $k = \frac{1}{2}(1 + i\kappa)$ . To find the expression for  $c_\mu^{1/2}$  we can approach it either from  $k = \frac{1}{2} + \varepsilon$ ,  $\varepsilon \rightarrow 0^\pm$ , or from  $k = \frac{1}{2}(1 + i\varepsilon)$ ; up to the sign  $\sigma_\mu^k$ , the result in all cases is

$$c_\mu^{k=1/2} = \sigma_\mu^{1/2} \Gamma(\frac{1}{2} - \mu) / \sqrt{\frac{1}{2} \psi'(\frac{1}{2} - \mu)}, \quad (129)$$

where we have the trigamma function  $\psi'(z) = d\psi(z)/dz$ , and we must exclude the cases when  $\mu = \frac{1}{2} + m$ ,  $m \in \mathcal{Z}_0^+$ . For the equally-spaced spectra  $\mu = k + m$  and  $\mu = 1 - k + m$ ,  $m \in \mathcal{Z}_0^+$ , we can use the residues of the poles of the  $\Gamma$  and  $\psi$  functions, to find the limits,

$$c_{k+m}^k = \sigma_{k+m}^k / \sqrt{\frac{1}{2} m! \Gamma(2k + m)}, \quad (130)$$

$$c_{1-k+m}^k = \sigma_{1-k+m}^k / \sqrt{\frac{1}{2} m! \Gamma(2 - 2k + m)}, \quad (131)$$

$$c_{1/2+m}^{k=1/2} = \sigma_{1/2+m}^{1/2} \sqrt{2} / m!, \quad (132)$$

where in the last line we include the case left out in (129). Lastly, in the strong barrier interval  $\gamma \geq \frac{3}{4}$  ( $k \geq 1$ ,  $D \geq 4$ ), the two limits in (128) do not commute; we must take  $\varepsilon \rightarrow 0^+$  first (since  $\psi_{k+m+\varepsilon}^k$  is not square-integrable), and  $r \rightarrow 0^+$  last. The result is again given by (130).

In dealing with contractions, we came to need the asymptotic formulas for the difference of digamma functions that appears in the normalization constant (18), evaluated at the points of a spectrum determined by some  $f_k(\nu) = \text{constant}$  in (35), which determines the asymptotic spectrum  $\mu = k + M + \varepsilon_M$  that is solution to  $f_k(\mu) = f_k(\nu)$ . Using the reflection and Stirling-type formulas for the digammas [9, Eqs. 6.3.7 and 6.3.18], and keeping in mind Eqs. (52)–(56), we find

$$\frac{\psi(1-k-\mu) - \psi(k-\mu)}{\sin 2\pi k} = \frac{\psi(k+\mu) - \psi(1-k+\mu)}{\sin 2\pi k} + \pi \left( \cos \pi(k+\mu) - \cos \pi(1-k+\mu) \right) \quad (133)$$

$$\stackrel{\mu \rightarrow \infty}{\sim} \frac{\pi}{\sin \pi(k+\mu) \sin \pi(k-\mu)}, \quad (134)$$

which is real in all cases.

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