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Finite q-oscillator

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Abstract

The finite q-oscillator is a model that obeys the dynamics of the harmonic oscillator, with the operators of position, momentum and Hamiltonian being functions of elements of the q-algebra $su_q(2)$. The spectrum of position in this discrete system, in a fixed representation j, consists of $2j + 1$ 'sensor'-points $x_s = \frac{1}{2}[2s]_q$, $s \in \{-j, -j+1, \ldots, j\}$, and similarly for the momentum observable. The spectrum of energies is finite and equally spaced, so the system supports coherent states. The wavefunctions involve dual q-Kravchuk polynomials, which are solutions to a finite-difference Schrödinger equation. Time evolution (times a phase) defines the fractional Fourier–q-Kravchuk transform. In the classical limit as $q \to 1$ we recover the finite oscillator Lie algebra, the $N = 2j \to \infty$ limit returns the Macfarlane–Biedenharn q-oscillator and both limits contract the generators to the standard quantum-mechanical harmonic oscillator.

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1. Introduction and the basic postulates

Discrete models which are counterparts to well-known continuous systems, and in particular those which contract to the standard harmonic oscillator, are of fundamental interest in theoretical physics. Moreover, *finite* discrete models are also of interest for the parallel processing of signals, where the input and output are registered by a finite sensor array, and the processing element is a shallow planar waveguide—an oscillator which can carry only a finite number of states [1]. The salient purpose of such a device is to perform a finite analogue of the fractional Fourier transform [2].

The aim of this paper is to develop the theory of a finite quantum oscillator with the dynamical symmetry given by the quantum algebra $su_q(2)$, on the basis of a finite-dimensional representation. We call this model the *finite q-oscillator*. There are many algebraic constructions which can be used for building up different models of q-oscillators. For example, the authors of [3] discussed a generalized Heisenberg–Weyl algebra, determined by the relations $J_0 J_+ = J_+ f(J_0)$, $J_- J_0 = f(J_0) J_-$, $[J_+, J_-] = J_0 - f(J_0)$, where f is a polynomial. This algebra has finite-dimensional representations, which are appropriate for constructing such a model. However, the existence of finite-dimensional representations is not sufficient for the development of an explicit model, because one has to derive *explicit* analytic formulae for its wavefunctions, their momentum content and an analogue of the Fourier transform (mapping eigenfunctions of position operator to eigenfunctions of momentum) to allow for a meaningful picture of phase space for this system. For the above-mentioned generalized Heisenberg–Weyl algebra it is not yet known how to do that.

Previously we developed a model of the finite oscillator, constructed on the basis of finite-dimensional irreducible representations of the Lie algebra $su(2)$ [1, 4]. This paper derives from the realization that the postulates we used to define the finite oscillator model have a very natural generalization that includes q-algebras [5]. These postulates are:

1. There exists an essentially self-adjoint *position* operator, indicated Q, whose spectrum $\Sigma(Q)$ is the set of positions of the system.
2. There exists a self-adjoint and compact *Hamiltonian* operator, H, which generates time evolution through the Newton–Lie, or equivalent Hamilton–Lie equations
$$[H, [H, Q]] = Q \qquad \Longleftrightarrow \qquad \begin{cases} [H, Q] =: -\mathrm{i}P \\ [H, P] = \mathrm{i}Q \end{cases} \tag{1}$$
where $[\cdot, \cdot]$ is the commutator. The first Hamilton equation in (1) defines the *momentum* operator P, while the second one contains the harmonic oscillator dynamics. The set of momentum values of the system is the spectrum $\Sigma(P)$ of P.
3. The three operators, Q, P and H, close into an *associative algebra*, i.e., satisfy the Jacobi identity
$$[P, [H, Q]] + [Q, [P, H]] + [H, [Q, P]] = 0. \tag{2}$$

The second and third postulates determine that $[Q, P]$ must commute with H, which implies that it can only be of the form $[Q, P] = \mathrm{i}F(H)$, where F is some function of H (including constants) and the i is placed to make $F(H)$ self-adjoint, but do not otherwise specify this basic commutator further. For a constant $F(H) = \hbar\hat{1}$, one recovers the standard oscillator algebra $H_4 = \operatorname{span}\{H, Q, P, \hat{1}\}$, which contains the basic Heisenberg–Weyl subalgebra $W_1 = \operatorname{span}\{Q, P, \hat{1}\}$ of quantum mechanics. In our first work [1] we examined the cases which, in the unitary irreducible representations of spin $j = \frac{1}{2}N$ ($N \in \{0, 1, \ldots\}$ fixed), correspond to the linear function $F(H) = H - \left(j + \frac{1}{2}\right)\hat{1} =: J_3$, and so the operators close into the Lie algebra $so(3) = su(2) = \operatorname{span}\{Q, P, J_3\}$.

In this paper we study the case when, for $q := \mathrm{e}^{-\kappa}$, the basic commutator is

$$[Q, P] = \mathrm{i}F_q(H) \qquad H = J_3 + \left(j + \tfrac{1}{2}\right)\hat{1} \tag{3}$$

$$F_q(H) = \mathrm{e}^{-2\kappa J_3}\frac{\cosh\frac{1}{2}\kappa}{2\sinh\frac{1}{2}\kappa} - \mathrm{e}^{-\kappa J_3}\frac{\cosh\left(j + \frac{1}{2}\right)\kappa}{2\sinh\frac{1}{2}\kappa} \tag{4}$$

$$= \tfrac{1}{2}\,\mathrm{e}^{-\kappa J_3}\left(\mathrm{e}^{-\kappa J_3}\cosh\tfrac{1}{2}\kappa - T_{2j+1}\left(\cosh\tfrac{1}{2}\kappa\right)\right)\Big/\sinh\tfrac{1}{2}\kappa \tag{5}$$

where T_n is the Chebyshev polynomial of the first kind, and $q \in (0, 1]$ or $\kappa \in [0, \infty)$. In particular, $F_1(H) = J_3$ returns the previous $su(2)$ case [1].

The explicit form of the right-hand side in (3) is explained by the following circumstance. The condition $[Q, P] = \mathrm{i}F(H)$ leaves a small number of possibilities for choosing Q and P.

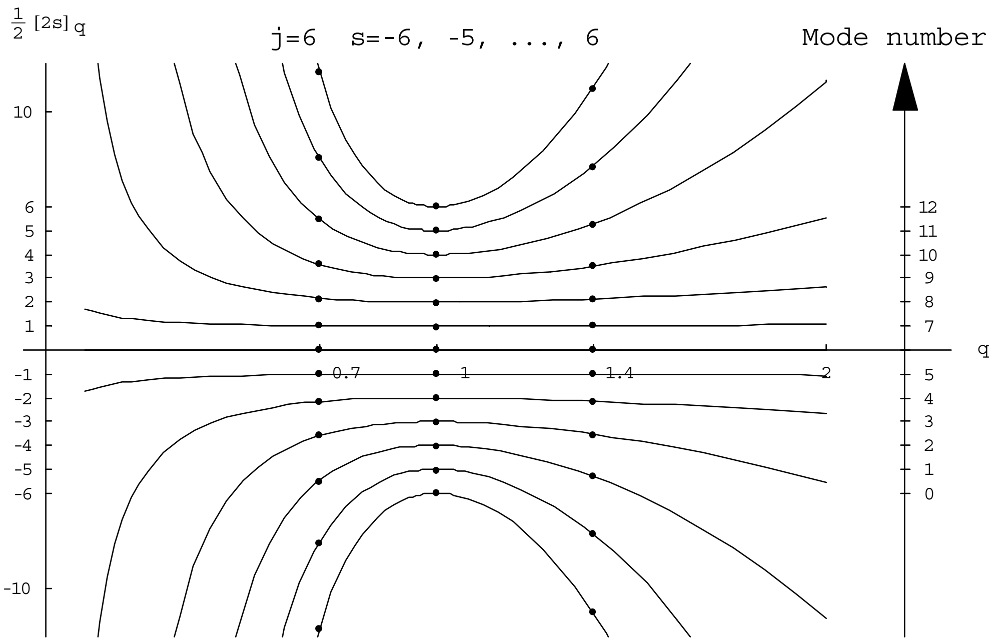

Figure 1. Left: spectra of positions of the finite q-oscillator of 13 points ($j = 6$) as functions of q; bullets mark the positions for $q = 0.7$, 1 (the Lie case), and 1.4. Right: the equally spaced spectrum of energies (independent of q, also for $j = 6$).

If we wish to deal with such Q and P, for which it is possible to find explicit analytic expressions for wavefunctions, then we are led to Q and P, given by formulae (23) and (24) below, and therefore $F_q(H)$ has the form given by (4) and (5).

An important ingredient for the postulates of harmonic oscillator dynamics is an unambiguous correspondence between the physical observables of position, momentum and energy, with the elements of the associative algebra. In section 2 we recall the main relevant results on the algebra $su_q(2)$ and its standard representation basis. The $su_q(2)$ nonstandard basis, investigated in [6, 7], is introduced in section 3 to exhibit our proposed correspondence explicitly in terms of the generators of $su_q(2)$. With our postulated choice, the position and energy spectra in the $(2j+1)$-dimensional representation $j = \frac{1}{2}N$ of $su_q(2)$ will be

$$\Sigma(Q) = x_s = \tfrac{1}{2}[2s]_q = \frac{\sinh s\kappa}{2\sinh\frac{1}{2}\kappa} \qquad s \in \{-j, -j+1, \ldots, j\} =: s|_{-j}^{j} \tag{6}$$

$$\Sigma(H) = E_n = n + \tfrac{1}{2} \qquad n \in \{0, 1, \ldots, 2j\} =: n|_0^{2j} \tag{7}$$

as shown in figure 1. We recall the definition of the q-number for $q = \mathrm{e}^{-\kappa}$:

$$[r]_q = [r]_{q^{-1}} = -[-r]_q := \frac{q^{\frac{1}{2}r} - q^{-\frac{1}{2}r}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{\sinh\frac{1}{2}r\kappa}{\sinh\frac{1}{2}\kappa}. \tag{8}$$

Note that the q-number of an integer r is $U_{r-1}\left(\cosh\frac{1}{2}\kappa\right)$, the Chebyshev polynomial of the second kind. The spectrum of momentum is the same as that of position, $\Sigma(P) = \Sigma(Q)$. The classical limit is $\lim_{q\to1}[s]_q = s$, when the q-algebra $su_q(2)$ becomes the Lie algebra $su(2)$; then, the set of positions become equally spaced and we are back at the previously known finite oscillator [1]. But for all other values of the deformation parameter q, the 'sensor points' of the system are concentrated towards the centre of the interval, while the endpoints are spread farther apart. Yet the energy spectrum remains an equally spaced set, and therefore the system follows harmonic motion.

The finite q-oscillator wavefunctions are the overlaps between the position and energy eigenbases. They are written out in section 4 in terms of the dual q-Kravchuk polynomials, and are orthonormal and complete over the sensor points of the system. The momentum representation of these wavefunctions is addressed in section 5 with the Fourier–q-Kravchuk transform, and in section 6 this transform is fractionalized. The evolution in time of a finite q-oscillator (or equivalently, the parallel processing of a finite signal along the axis of a shallow planar waveguide), is the two-fold cover of the fractional Fourier–q-Kravchuk transform matrix; the metaplectic sign appears thus for half-integer values of j, which correspond to finite systems of an even number of points. In section 7 we introduce the concept of an *equivalent potential* for discrete systems which is based, as in the continuous case, on the existence of a ground state with no zeros. Finally, in section 8 we verify that the contraction limits $q \to 1$ and $N = 2j \to \infty$ of the algebra $su_q(2)$ reproduce the known results for the finite oscillator and the continuous q-oscillator. The corresponding limits for the wavefunctions however, present further challenge.

2. The algebra $su_q(2)$ and its standard basis

The quantum algebra $su_q(2)$ is the associative algebra generated by three elements, usually denoted as J_+, J_-, J_3, subject to the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_3]_q. \tag{9}$$

Equivalently, writing $J_\pm = J_1 \pm \mathrm{i}J_2$, we characterize the algebra $su_q(2)$ by

$$[J_2, J_3] = \mathrm{i}J_1 \qquad [J_3, J_1] = \mathrm{i}J_2 \qquad [J_1, J_2] = \tfrac{\mathrm{i}}{2}[2J_3]_q. \tag{10}$$

The first two commutators in (10) have the structure of the oscillator Hamilton equations (1), while the third one involves the q-number (8), which distinguishes q-algebras from Lie algebras, the latter corresponding to the case $q = 1$. The following element in the covering algebra of $su_q(2)$ commutes with all others:

$$\begin{aligned} C_q &:= J_1^2 + J_2^2 + \left[J_3 - \tfrac{1}{2}\right]_q^2 + \tfrac{1}{2}[2J_3]_q - \tfrac{1}{4} \\ &= J_+J_- + \left[J_3 - \tfrac{1}{2}\right]_q^2 - \tfrac{1}{4} \end{aligned} \tag{11}$$

and is called its *Casimir* operator.

It is convenient to have a realization of the $su_q(2)$ generators in terms of first-degree differential operators, acting on spaces $\mathcal{H}_j$ of functions of a formal variable x, and depending on the numerical irreducible representation label j. This is

$$J_+ := J_1 + \mathrm{i}J_2 \quad \leftrightarrow \quad x\left[2j - x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q = x[j - J_3]_q \tag{12}$$

$$J_- := J_1 - \mathrm{i}J_2 \quad \leftrightarrow \quad \frac{1}{x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q = \frac{1}{x}[j + J_3]_q \tag{13}$$

$$J_3 \leftrightarrow x\frac{\mathrm{d}}{\mathrm{d}x} - j \qquad j \in \left\{0, \frac{1}{2}, 1, \ldots\right\} \text{ fixed.} \tag{14}$$

The set of power monomials $x^{j+m}|_{m=-j}^{j}$ are eigenfunctions of J_3 and provide the *standard* basis for the irreducible space $\mathcal{H}_j$, of finite dimension $2j+1$. The functions of the basis were chosen in [6, 7] with the following constants:

$$f_m^j(x) := q^{\frac{1}{4}(m^2-j^2)} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} x^{j+m} \tag{15}$$

where the q-binomial coefficient $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q$ is defined (using the standard notation of q-analysis [8]) for $m \geqslant n$ non-negative integers by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} = (-1)^n q^{mn-\frac{1}{2}n(n-1)} \frac{(q^{-m};q)_n}{(q;q)_n} \tag{16}$$

$$(z;q)_n := \prod_{k=0}^{n-1}(1-zq^k) \qquad n = 1, 2, 3, \ldots \qquad (z;q)_0 = 1. \tag{17}$$

For any two complex vectors $\mathbf{a}, \mathbf{b} \in \mathcal{H}_j$

$$\mathbf{a} = \sum_{m=-j}^{j} \alpha_m f_m^j \qquad \mathbf{b} = \sum_{m=-j}^{j} \beta_m f_m^j \tag{18}$$

there is a natural sesquilinear inner product

$$(\mathbf{a}, \mathbf{b})_{\mathcal{H}_j} := \sum_{m=-j}^{j} \alpha_m^* \beta_m \tag{19}$$

with respect to which the standard basis is orthonormal. The action of the $su_q(2)$ generators and Casimir operator on the standard basis is well known

$$J_3 f_m^j = m f_m^j \qquad J_\pm f_m^j = \sqrt{[j \pm m + 1]_q [j \mp m]_q}\, f_{m\pm1}^j \tag{20}$$

$$C_q f_m^j = c_q f_m^j \qquad c_q := \left[j + \tfrac{1}{2}\right]_q^2 - \tfrac{1}{4}. \tag{21}$$

These equations are of course independent of the realization of the basis vectors f_m^j by the power monomials $f_m^j(x)$ in x.

The spectrum of the diagonal generator J_3 (see (14) and (20)) is linear and bounded, as that of a finite version of the quantum harmonic oscillator. Indeed, this is our choice for the finite q-oscillator Hamiltonian, displaced so that the ground state has energy $1/2$, namely

$$H = J_3 + j + \tfrac{1}{2} \qquad H f_m^j = \left(n + \tfrac{1}{2}\right) f_m^j \qquad n := j + m \tag{22}$$

where $n|_0^{2j}$ is the *mode number* that counts the number of energy quanta. At this point we are presented with what would appear as a 'natural' assignment for the position and momentum operators, $Q \leftrightarrow J_1$ and $P \leftrightarrow -J_2$, because it would be the simplest generalization of the previously studied $q = 1$ case [1, 4]. This choice would bring the first two commutators in (10) to reproduce correctly the two Hamilton equations in (1), while the third commutator $[Q, P]$ would have the form (3) with $F_q(H) = \frac{1}{2}[2J_3]_q = \sinh\kappa\left(H - j - \frac{1}{2}\right)\big/2\sinh\frac{1}{2}\kappa$. In this 'naive' model however, the spectra of Q and P are not algebraic; they must be computed numerically as roots of a polynomial equation of degree $2j+1$.

3. The nonstandard basis

While we do not discard the model suggested at the end of the previous section, we find it more attractive to propose a correspondence between the physical observables of position and momentum, Q, P, and the *nonstandard* (also called *twisted*) operators (see [9–11]), which have the virtue of possessing an algebraic spectrum $x_s := \frac{1}{2}[2s]_q, s|_{-j}^{j}$. The position (and hence momentum) observables will be thus identified with the following operators:

$$Q = \widetilde{J}_1 := q^{\frac{1}{4}J_3} J_1 q^{\frac{1}{4}J_3} \tag{23}$$

$$-P = \widetilde{J}_2 := q^{\frac{1}{4}J_3} J_2 q^{\frac{1}{4}J_3} \tag{24}$$

while the Hamiltonian H is associated with J_3 by (22) as before.

We note that while the q-number (8) displays symmetry under q-inversions, $q \leftrightarrow q^{-1}$, $[r]_q = [r]_{q^{-1}}$, the identification of tilded operators in (23), (24) preserves this symmetry with the concomitant reflection $J_3 \leftrightarrow -J_3$. This means that the ground state of a $q < 1$ oscillator is the top state of its $q^{-1} > 1$ partner.

The commutation relations among the nonstandard operators and J_3 are

$$[J_3, Q] = -\mathrm{i}P \qquad [J_3, P] = \mathrm{i}Q \tag{25}$$

$$\begin{aligned}[Q, P] &= \tfrac{\mathrm{i}}{2} q^{\frac{1}{2}J_3}\left(q^{-\frac{1}{2}} J_+ J_- - q^{\frac{1}{2}} J_- J_+\right) q^{\frac{1}{2}J_3} =: \mathrm{i} F_q(C_q, J_3) \\ &= \mathrm{i}\left(\mathrm{e}^{-\kappa J_3}\left[\left(C_q + \tfrac{1}{4}\right) \sinh \tfrac{1}{2}\kappa + \tfrac{1}{2} \operatorname{cosech} \tfrac{1}{2}\kappa\right] - \tfrac{1}{2}\,\mathrm{e}^{-2\kappa J_3} \coth \tfrac{1}{2}\kappa\right)\end{aligned} \tag{26}$$

where $q = \mathrm{e}^{-\kappa}$ as before. The operator $F_q(C_q, J_3)$ defined in (26) commutes with J_3 and is also diagonal in the standard basis; in the irreducible representation j,

$$F_q f_m^j = \frac{\mathrm{e}^{-2m\kappa} \cosh \frac{1}{2}\kappa - \mathrm{e}^{-m\kappa} \cosh\left(j + \frac{1}{2}\right)\kappa}{2 \sinh \frac{1}{2}\kappa} f_m^j \tag{27}$$

but its spectrum is *not* a good candidate for an oscillator Hamiltonian, because it is not equally spaced (unlike (7)), and so the motion would not be harmonic, but dispersive. In terms of the position and momentum generators (23), (24), the Casimir operator (11) acquires the form

$$C_q = \operatorname{sech} \tfrac{1}{2}\kappa (Q^2 + P^2)\, \mathrm{e}^{\kappa J_3} + D_q(J_3) \tag{28}$$

$$D_q(J_3) := \operatorname{sech} \tfrac{1}{2}\kappa\left(\left[J_3 - \tfrac{1}{2}\right]_q^2 - \tfrac{1}{2}\,\mathrm{e}^{-\kappa J_3} \coth \tfrac{1}{2}\kappa + \tfrac{1}{2} \operatorname{cosech} \tfrac{1}{2}\kappa\right) - \tfrac{1}{4}. \tag{29}$$

We recall a previous phase-space picture for the finite oscillator of $2j + 1$ points, considered in [12], as the (classical) sphere $Q^2 + P^2 + J_3^2 = j(j + 1)$, having circular sections of square radius $Q^2 + P^2 = \left(j + \frac{1}{2}\right)^2 - \left(J_3 - \frac{1}{2}\right)^2 - J_3$. For $su_q(2)$, the corresponding surface now has the section

$$Q^2 + P^2 = \left(\left[j + \tfrac{1}{2}\right]_q^2 \cosh \tfrac{1}{2}\kappa - \left[J_3 - \tfrac{1}{2}\right]_q^2 + \tfrac{1}{2}\,\mathrm{e}^{-\kappa J_3} \coth \tfrac{1}{2}\kappa - \tfrac{1}{2} \operatorname{cosech} \tfrac{1}{2}\kappa\right) \mathrm{e}^{-\kappa J_3} \tag{30}$$

that we show in figure 2 for selected values of q. Phase space for the finite q-oscillator is suggested thus as q-dependent pear-shaped spheroids, tip-up for $q < 1$ and tip-down for $q > 1$ (recall the $q \leftrightarrow q^{-1}$ symmetry with the inversion of J_3). The q-harmonic oscillator evolution (i.e., a phase times the so-defined fractional q-Fourier–Kravchuk transform) will rotate this space around the J_3 vertical symmetry axis of the spheroid.

In this finite q-oscillator model we interpret the eigenvalues x_s of $Q := \widetilde{J}_1$ as the discrete values of the position observable. The eigenfunctions $g_s^j(x)$ and eigenvalues of this nonstandard operator were found in [6], and they are of the form

$$Q g_s^j(x) = x_s g_s^j(x) \qquad x_s = \frac{1}{2}[2s]_q = \frac{\sinh s\kappa}{2 \sinh \frac{1}{2}\kappa} = -x_{-s} \qquad s|_{-j}^{j} \tag{31}$$

$$g_s^j(x) = \gamma_s^j \left(q^{\frac{1}{4}(1-2j)} x; q\right)_{j-s} \left(-q^{\frac{1}{4}(1-2j)} x; q\right)_{j+s} = g_{-s}^j(-x) \tag{32}$$

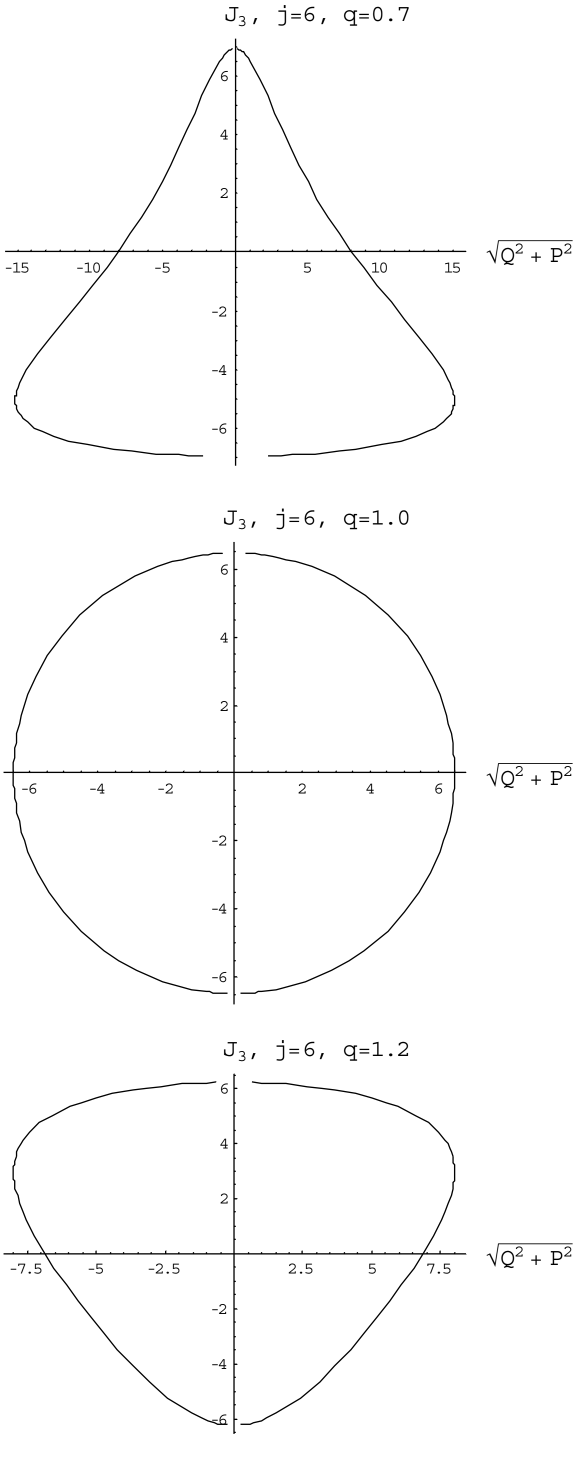

Figure 2. Section of the phase-space spheroid of the finite q-oscillator. It is the classical surface where the Casimir operator C_q has the constant value corresponding to the $su_q(2)$ representation j.

$$\gamma_s^j := q^{\frac{1}{2}(j+s)} \sqrt{\begin{bmatrix} 2j \\ j+s \end{bmatrix}_{q^2} \frac{1+q^{-2s}}{2(-q;q)_{2j}}}. \tag{33}$$

They are normalized with respect to the inner product (19), and are orthogonal because they correspond to distinct eigenvalues x_s. This basis of $2j+1$ functions $g_s^j(x)$, $s|_{-j}^j$ we call the *position* basis. A signal consisting of $2j+1$ values Φ_s, sensed at the positions x_s (given in (6)), is

$$\Phi = \sum_{s=-j}^{j} \Phi_s g_s^j \in \mathcal{H}_j \tag{34}$$

and can be realized either as a function of x, or as a $(2j+1)$-dimensional column vector with components numbered by $s|_{-j}^j$.

4. Finite *q*-oscillator mode wavefunctions

We have now two bases for $\mathcal{H}_j$: the standard basis $\{f_m^j\}_{m=-j}^j$ of *mode* $n=j+m$ (and energy $E_n = n+\frac{1}{2}$), and the nonstandard basis $\{g_s^j\}_{s=-j}^j$ of *position* $x_s = \frac{1}{2}[2s]_q$. In the realization of $su_q(2)$ generators given in (12)–(14), the mode basis is realized by the power functions in (15), and the position basis by (32). We can use this realization to find the unitary transformation between these two orthonormal bases, and thus define the finite q-oscillator wavefunctions by the overlap

$$\Phi_n^{(2j|q)}(x_s) := \left(g_s^j, f_m^j\right)_{\mathcal{H}_j} \qquad \begin{cases} \text{of mode } n=j+m & n|_0^{2j} \\ \text{on points } x_s = \frac{1}{2}[2s]_q & s|_{-j}^j. \end{cases} \tag{35}$$

By construction, this set of functions is orthonormal and complete under the $\mathcal{H}_j$ inner product (19).

The overlap (35) is obtained by expanding the function $g_s^j(x)$ of (32) into a power series in x, which is then

$$g_s^j(x) = \sum_{m=-j}^{j} \Phi_{j+m}^{(2j|q)}(x_s)^* f_m^j(x) \qquad f_m^j(x) = \sum_{s=-j}^{j} \Phi_{j+m}^{(2j|q)}(x_s) g_m^j(x). \tag{36}$$

The expansion of $g_s^j(x)$ in x is [6]

$$g_s^j(x) = \gamma_s^j \sum_{m=-j}^{j} q^{\frac{1}{4}(j+m)(j+m-1)} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} K_{j+m}(\lambda(j-s); -1, 2j \mid q) f_m^j(x) \tag{37}$$

expressed in terms of the *dual q-Kravchuk polynomials*

$$K_n(q^{-\xi} + cq^{\xi-2j}; c, 2j|q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-\xi}, cq^{\xi-2j} \\ q^{-2j}, 0 \end{matrix} \middle| q; q\right) \tag{38}$$

where ${}_3\phi_2$ is the basic hypergeometric function defined in [8], and the coefficients γ_s^j are given in (33).

In the particular case of our concern, the argument of the dual q-Kravchuk polynomial is $\lambda(\xi) = q^{-\xi} + cq^{\xi-2j}$ with $c=-1$ in (37), is given in terms of the positions $x_s = \frac{1}{2}[2s]_q$, $s|_{-j}^j$, of the finite q-oscillator by

$$\lambda(j-s) = q^{-j+s} - q^{-j-s} = -2\mathrm{e}^{j\kappa}\sinh\kappa s$$
$$= 2q^{-j-\frac{1}{2}}(q-1)x_s = -\big(4\mathrm{e}^{j\kappa}\sinh\tfrac{1}{2}\kappa\big)x_s \tag{39}$$

and $q = \mathrm{e}^{-\kappa}$ as before. From (37) thus, the finite q-oscillator wavefunctions of mode number $n = j + m$, $n|_0^{2j}$, are

$$\Phi_n^{(2j|q)}(x_s) = q^{\frac{1}{2}(j+s)+\frac{1}{4}n(n-1)}\sqrt{\begin{bmatrix} 2j \\ j+s \end{bmatrix}_{q^2}\begin{bmatrix} 2j \\ n \end{bmatrix}_q \frac{1+q^{-2s}}{2(-q;q)_{2j}}}$$
$$\times K_n\big(2q^{-j-\frac{1}{2}}(q-1)x_s; -1, 2j \mid q\big). \tag{40}$$

The explicit expression for the dual q-Kravchuk polynomials in this case is

$$K_{j+m}(\lambda(j-s); -1, 2j \mid q) = {}_3\phi_2\left(\begin{matrix} q^{-j-m}, q^{s-j}, -q^{-j-s} \\ q^{-2j}, 0 \end{matrix}\middle| q; q\right)$$
$$= \sum_{k=0}^{2j} \frac{(q^{-j-m};q)_k (q^{-j+s};q)_k (-q^{-j-s};q)_k}{(q^{-2j};q)_k}\frac{q^k}{(q;q)_k} \tag{41}$$

where $(z;q)_k$ is defined in (17).

The lowest mode of the oscillator is (see (40) for $n = j + m = 0$),

$$\Phi_0^{(2j|q)}(x_s) = q^{\frac{1}{2}(j+s)}\sqrt{\begin{bmatrix} 2j \\ j+s \end{bmatrix}_{q^2}\frac{1+q^{-2s}}{2(-q;q)_{2j}}} = \gamma_s^j. \tag{42}$$

The finite q-oscillator wavefunctions possess definite parity,

$$\Phi_n^{(2j|q)}(-x_s) = \Phi_n^{(2j|q)}(x_{-s}) = (-1)^n\Phi_n^{(2j|q)}(x_s) \tag{43}$$

and, as is to be expected, in the limit $q \to 1$ return the Kravchuk functions of the finite oscillator [1]

$$\lim_{q\to 1}\Phi_n^{(2j|q)}(x_s) = 2^{-j}\sqrt{\binom{2j}{j+s}\binom{2j}{n}}K_n\big(j-s; \tfrac{1}{2}, 2j\big) \tag{44}$$

with the classical Kravchuk polynomials, introduced by Kravchuk in [13].

The dual q-Kravchuk polynomials—as all polynomials—are analytic functions on the complex plane of their argument. As before in the finite oscillator models [1, 4, 14], this argument is the position coordinate, which can be analytically continued to real or complex values X, even if the inner product of the space $\mathcal{H}_j$ is only over the point set $\{x_s\}$, $s|_j^j$. As to the q-Kravchuk wavefunctions (40) the factor in front of the polynomial is a function that is analytic in the argument s within the interval $-j-1 < s < j+1$; this means that in the position coordinate, analytic continuation is possible within the interval $x_{-j-1} < X < x_{j+1}$.

The finite q-oscillator eigenfunctions are shown in figure 3 for a 25-point finite q-oscillator ($j = 12$) showing the lowest, middle and highest modes, for selected values of q that include for comparison the Lie case studied in [1, 4].

5. Fourier–q-Kravchuk transform to momentum space

The identification of the position and momentum operators, $Q = \widetilde{J}_1$, $P = -\widetilde{J}_2$ in (23), (24), brings formulae (25) to the role of the two Hamilton equations (1). (This also holds for the 'first' choice using the standard basis, $Q \leftrightarrow J_1$, $P \leftrightarrow -J_2$, that we outlined in section 2, as well as for all oscillator models, finite or standard.) The evolution of the finite q-oscillator

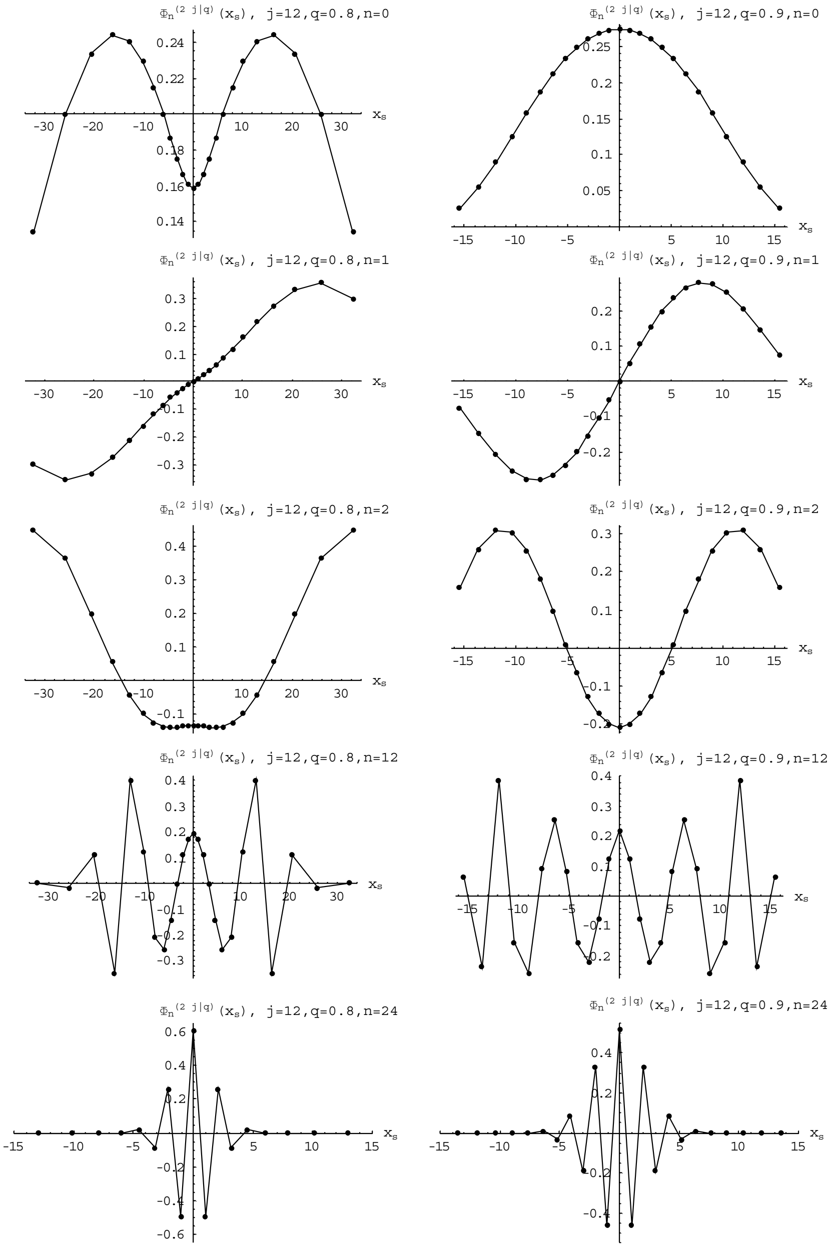

Figure 3. Eigenfunctions of the finite q-oscillator of 25 points and modes ($j = 12$). The dual q-Kravchuk wavefunctions $\Phi_n^{(2j|q)}(x)$ are organized in columns by the values $q = 0.8, 0.9, 1$ (the Lie case) and 1.2; and in rows by their mode numbers $n = 0, 1, 2, 12$ (middle of the multiplet) and 24 (top energy state). The points are joined by straight lines for visibility.

over time in quantum mechanics, or along the optical axis in the waveguide model, is thus the *harmonic* motion

$$\mathrm{e}^{-\mathrm{i}\tau H}\begin{pmatrix} Q \\ P \end{pmatrix}\mathrm{e}^{\mathrm{i}\tau H} =: \begin{pmatrix} Q(\tau) \\ P(\tau) \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}. \tag{45}$$

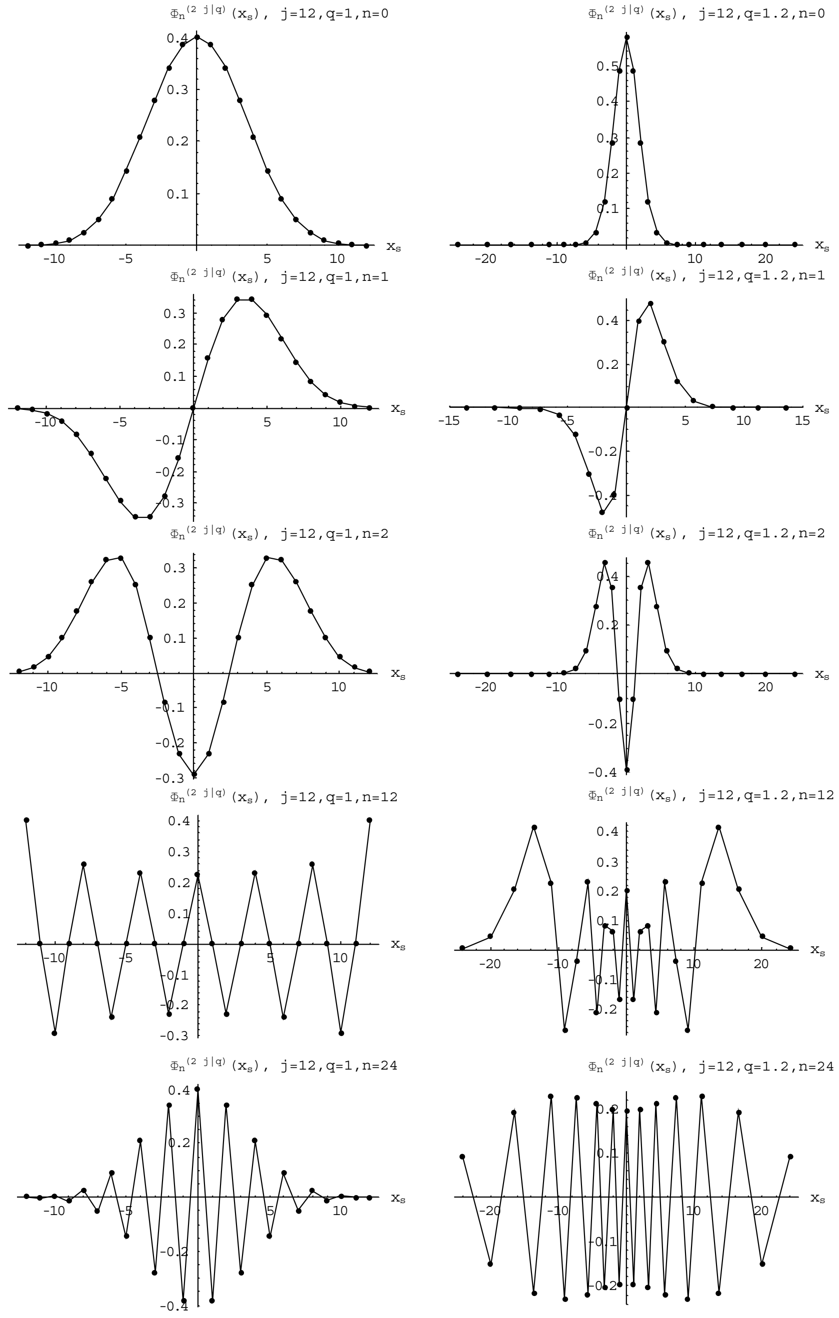

Figure 3. (Continued).

This is a $U(1)$ group of inner automorphisms of the $su_q(2)$ algebra, and of rotations of the phase-space surface in figure 2 around its vertical axis. It covers twice the $SO(2)$ cycle of fractional *Fourier–q-Kravchuk* transforms, $\mathcal{K}_q^a$, of power $a = 2\tau/\pi$ and angle τ,

$$\mathcal{K}_q^a := \exp(-\mathrm{i}\pi a(J_3 + j)/2) = \mathrm{e}^{\mathrm{i}\pi a/4} \exp(-\mathrm{i}\pi a H/2). \tag{46}$$

For $a = 1$ we have the Fourier–q-Kravchuk transform $\mathcal{K}_q$. The action of $\mathcal{K}_q$ on the eigenbasis of position yields the eigenbasis of momentum,

$$\widetilde{g}_s^j(x) := \mathcal{K}_q g_s^j(x). \tag{47}$$

These functions have the properties and form

$$P\widetilde{g}_r^j(x) = -Y_r \widetilde{g}_r^j(x) \qquad Y_r = \tfrac{1}{2}[2r]_q = \frac{\sinh r\kappa}{2\sinh\frac{1}{2}\kappa} = -Y_{-r} \qquad r|_{-j}^{j} \tag{48}$$

$$\widetilde{g}_r^j(x) = g_r^j(\mathrm{i}x) = g_{-r}^j(-\mathrm{i}x) = \gamma_r^j\big(\mathrm{i}q^{\frac{1}{4}(1-2j)}x;q\big)_{j-r}\big(-\mathrm{i}q^{\frac{1}{4}(1-2j)}x;q\big)_{j+r} \tag{49}$$

where γ_r^j is the constant given in (33); the spectrum of momenta, Y_r, $r|_{-j}^{j}$, is the same as that of position (cf (31)). Since $\mathcal{K}_q^a$ is unitary under the inner product in $\mathcal{H}_j$, the Fourier–q-Kravchuk transforms of the finite q-oscillator eigenfunctions (35)–(40) of modes $n = j+m$, are

$$\widetilde{\Phi}_n^{(2j|q)}(x_s) := \mathcal{K}_q\Phi_n^{(2j|q)}(x_s) := \big(g_s^j, \mathcal{K}_q f_m^j\big)_{\mathcal{H}_j} = (-\mathrm{i})^n\Phi_n^{(2j|q)}(x_s) \tag{50}$$

as in all oscillator models.

The Fourier–q-Kravchuk transform of a function or signal Φ, of values $\Phi(x_s) = \big(g_s^j, \Phi\big)_{\mathcal{H}_j}$ on the finite, discrete sensor point set $\{x_s\}$, $s|_j^j$, is defined by

$$\widetilde{\Phi}(x_r) = \big(\widetilde{g}_r^j, \Phi\big)_{\mathcal{H}_j} = \sum_{s=-j}^{j} K_{r,s}^{(2j|q)}\Phi(x_s) \tag{51}$$

where the kernel is the overlap of the position eigenfunctions g_s^j in (32) with the momentum eigenfunctions $\widetilde{g}_r^j$ in (49),

$$K_{r,s}^{(2j|q)} := \big(\widetilde{g}_r^j, g_s^j\big)_{\mathcal{H}_j}. \tag{52}$$

This kernel is given explicitly below in (56) with $a = 1$.

6. Fractional Fourier–q-Kravchuk kernel

The Fourier–q-Kravchuk transform (50) is fractionalized by the operator $\mathcal{K}_q^a$ in (46), independently of the realization, on the mode eigenbasis of J_3,

$$\mathcal{K}_q^a f_m^j = \exp(-\mathrm{i}\pi a(j+m)/2) f_m^j = \exp(-\mathrm{i}\pi na/2) f_m^j. \tag{53}$$

When we apply $\mathcal{K}_q^a$ on a finite, complex 'signal' function of $2j+1$ points,

$$\Phi(x_s) = \big(g_s^j, \Phi\big)_{\mathcal{H}_j} = \sum_{m=-j}^{j}\big(g_s^j, f_m^j\big)_{\mathcal{H}_j}\big(f_m^j, \Phi\big)_{\mathcal{H}_j} \tag{54}$$

we obtain another such function, labelled by a,

$$\begin{aligned}\Phi^{(a)}(x_s) &:= \mathcal{K}_q^a\Phi(x_s) := \big(g_s^j, \mathcal{K}_q^a\Phi\big)_{\mathcal{H}_j} = \big(\mathcal{K}_q^{-a}g_s^j, \Phi\big)_{\mathcal{H}_j}\\ &= \sum_{m=-j}^{j}\big(\mathcal{K}_q^{-a}g_s^j, f_m^j\big)_{\mathcal{H}_j}\big(f_m^j, \Phi\big)_{\mathcal{H}_j} = \sum_{m=-j}^{j}\big(g_s^j, \mathcal{K}_q^a f_m^j\big)_{\mathcal{H}_j}\big(f_m^j, \Phi\big)_{\mathcal{H}_j}\\ &= \sum_{m=-j}^{j}\mathrm{e}^{-\mathrm{i}\pi a(j+m)/2}\big(g_s^j, f_m^j\big)_{\mathcal{H}_j}\sum_{s'=-j}^{j}\big(f_m^j, g_{s'}^j\big)_{\mathcal{H}_j}\big(g_{s'}^j, \Phi\big)_{\mathcal{H}_j} = \sum_{s'=-j}^{j}K_{s,s'}^{(a,2j|q)}\Phi(x_{s'})\end{aligned} \tag{55}$$

where the fractional Fourier–q-Kravchuk transform kernel $K_{s,s'}^{(a,2j|q)}$ is a $(2j+1)\times(2j+1)$ matrix of elements given by the bilinear generating function [15, formula (8.15)]

$$K_{s,s'}^{(a,2j|q)} := \sum_{n=0}^{2j} \Phi_n^{(2j|q)}(x_s)\, \mathrm{e}^{-\mathrm{i}\pi na/2} \Phi_n^{(2j|q)}(x_{s'})^* \tag{56}$$

$$= \gamma_s^j \gamma_{s'}^j \beta_{s,s'}(t)\,{}_8W_7(-q^{-2j-1}t; q^{s-j}, -q^{-j-s}, q^{s'-j}, -q^{-j-s'}, -t; q, -t) \tag{57}$$

where

$$t := \mathrm{e}^{-\mathrm{i}\pi a/2} \tag{58}$$

$$\beta_{s,s'}(t) := \frac{(q^{s-j}t, -q^{-j-s}t, q^{s'-j}t, -q^{-j-s'}t, -t; q)_\infty}{(q^{s-s'}t, -q^{s+s'}t, q^{s'-s}t, -q^{-s-s'}t, -q^{-2j}t; q)_\infty} \tag{59}$$

and γ_s^j is given by (33). The function ${}_8W_7$, defined in [8], is

$${}_8W_7(a; b, c, d, e, f; q, z) := \sum_{k=0}^{\infty} \frac{1-aq^{2k}}{1-a} \frac{(a, b, c, d, e, f; q)_k z^k}{(q, qa/b, qa/c, qa/d, qa/e, qa/f; q)_k} \tag{60}$$

where $(a,\ldots,c;q)_\infty := (a;q)_\infty \ldots (c;q)_\infty$ and $(a;q)_\infty = \prod_{k=0}^\infty (1-aq^k)$ in accordance with (17). This function can be expressed in terms of the basic hypergeometric function ${}_8\phi_7$ (see [8, section 2.2, formula (2.5.1)]), with coefficients which allow it to be reduced to the basic hypergeometric function ${}_4\phi_3$

$${}_8W_7(-q^{-2j-1}t; q^{s-j}, -q^{-j-s}, q^{s'-j}, -q^{-j-s'}, -t; q, -t)$$
$$= \frac{(-q^{-2j}t, q^{-j-s'}, -q^{-j+s'}, t; q)_\infty}{(-q^{-j-s'}t, q^{-j+s'}t, q^{-2j}, -t; q)_\infty} {}_4\phi_3 \left(\left. \begin{matrix} q^{-j+s'}, -q^{-j-s'}, t, -t \\ -q^{-j-s}t, q^{-j+s}, -q \end{matrix} \right| q, q \right). \tag{61}$$

We also note that due to relation $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$, the expression for $\beta_{s,s'}(t)$ in (59) can be reduced to

$$\beta_{s,s'}(t) = \frac{(q^{s-j}t;q)_{j-s'}(q^{-j+s'}t;q)_{j-s}(-q^{-j-s'}t;q)_{j-s}(-q^{-j-s}t;q)_{j+2s+s'}}{(-q^{-2j}t;q)_{2j}}. \tag{62}$$

Naturally, $\mathcal{K}_q^{a_1}\mathcal{K}_q^{a_2} = \mathcal{K}_q^{a_1+a_2}$ and $\mathcal{K}_q^0 = \hat{1}$. The 'phase correction' by $\pi a = 2\tau$ which we introduced in (46) implies that $\mathcal{K}_q^4 = \hat{1}$ (as the ordinary Fourier integral transform), while the fourth power of the oscillator evolution operator $\exp(\mathrm{i}\tau H)$ is $-\hat{1}$ for the full rotation angle $\tau = 2\pi$. This is the analogue of the metaplectic sign of the waveguide case (see [16], cf [2]), where the $U(1)$ subgroup generated by the latter covers twice the $SO(2)$ of the former. The parity is conserved under the fractional Fourier–Kravchuk transformation because J_3 commutes with the inversion of phase space. And again, in the limit $q \to 1$ we recover the previous Fourier–Kravchuk kernel expressed in terms of the Wigner 'little-d' functions [4],

$$\lim_{q\to 1} K_{s,s'}^{(a,2j|q)} = K_{s,s'}^{(a,2j)} = \mathrm{e}^{-\mathrm{i}\pi ja/2}(-\mathrm{i})^{s-s'} d_{s,s'}^j\left(\tfrac{1}{2}\pi a\right). \tag{63}$$

7. Equivalent potentials

In ordinary quantum mechanics, the ground state $\Psi_0(x)$ of a system with a potential $V(x)$ and energy $E_0 > -\infty$, has no zeros; thus, the Schrödinger equation determines the potential energy of the system from the ground state,

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - E_0\right)\Psi_0(x) = 0 \quad \Rightarrow \quad V(x) - E_0 = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_0(x)\Big/\Psi_0(x). \tag{64}$$

As a well-known example we have the harmonic oscillator, whose ground state is $\Psi_0(x) \sim \mathrm{e}^{-\frac{1}{2}x^2}$, so $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_0(x) = (x^2 - 1)\Psi_0(x)$ and (64) yields correctly $V(x) - E_0 = \frac{1}{2}x^2 - \frac{1}{2}$.

In the case when the system is discrete over the set of points $x_s = sh + x_0$, with integer s, which are equidistant by h, an equivalent potential may be defined following (64). We qualify it as *equivalent* because the discrete systems that have been studied (such as Kravchuk, Meixner and Hahn systems), obey Schrödinger-type difference equations which do *not* separate into a sum of terms, where one is readily identifiable with the kinetic term of the second-degree difference operator, plus a potential term that is only dependent on position x_s. The symmetric second-difference operator, acting on functions of x_s, can be expressed in terms of the right-difference and the left-difference operators ∇_{R} and ∇_{L},

$$\begin{aligned} \nabla_{\mathrm{R}} &:= \frac{\Delta}{\Delta x_s} = \frac{1}{\Delta x_s}(\mathrm{e}^{\partial_s} - 1) = \frac{1}{x_{s+1} - x_s}(\mathrm{e}^{\partial_s} - 1) \\ \nabla_{\mathrm{L}} &:= \frac{\nabla}{\nabla x_s} = \frac{1}{\nabla x_s}(1 - \mathrm{e}^{-\partial_s}) = \frac{1}{x_s - x_{s-1}}(1 - \mathrm{e}^{-\partial_s}) \end{aligned} \tag{65}$$

where $\Delta = \mathrm{e}^{\partial_s} - 1 = \mathrm{e}^{\partial_s}\nabla$. So, a difference analogue of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ in (64) has the form

$$\frac{1}{x_{s+1/2} - x_{s-1/2}}(\nabla_{\mathrm{R}} - \nabla_{\mathrm{L}}). \tag{66}$$

Consequently, when the ground state of the system is $\psi(s) := \Psi_0(x_s)$, the equivalent potential, according to its quantum-mechanical correspondent in (64), is

$$\begin{aligned} V(x_s) - E_0 &= \frac{1}{2\psi(s)[x_{s+1/2} - x_{s-1/2}]}(\nabla_{\mathrm{R}} - \nabla_{\mathrm{L}})\psi(s) \\ &= \frac{1}{2(x_{s+1/2} - x_{s-1/2})\psi(s)}\left(\frac{\psi(s+1) - \psi(s)}{x_{s+1} - x_s} - \frac{\psi(s) - \psi(s-1)}{x_s - x_{s-1}}\right). \end{aligned} \tag{67}$$

In the case of the finite Kravchuk oscillator, the set of values of position $x_s = s$ ($h = 1$ and $x_0 = 0$) is finite: $\{x_s\}_{s=-j}^{j}$. Yet, the wavefunctions $\psi(s) := \Psi_0^{(2j)}(x_s)$ can be analytically continued in x everywhere except for branch-point zeros at $x_{\pm(j+1)} := \pm(j+1)$, which are due to the square root of the binomial distribution. Thus, on the closed segment $x_{-(j+1)} \leqslant x \leqslant x_{j+1}$, the second difference in (67) is defined for any real value of x in the interval $x_{-j} \leqslant x \leqslant x_j$. A similar extension and range of validity holds for the Meixner and Hahn oscillator cases [17, 18]. The lowest mode of the Kravchuk oscillator, where $h = 1$, is given in (42). From this one derives the equivalent potential for the Kravchuk eigenfunction system

$$\begin{aligned} V(x_s) - E_0 + 1 &= \frac{\psi(s+1) + \psi(s-1)}{2\psi(s)} \\ &= \frac{\sqrt{(j+s)(j+s+1)} + \sqrt{(j-s)(j-s+1)}}{2\sqrt{(j+1)^2 - s^2}}. \end{aligned} \tag{68}$$

When the set of position values is *not* equally spaced, as is the case in the finite q-oscillator, $\{x_s\}_{s=-j}^{j}$ as in (31), we shall consider the differences with respect to the position coordinate

$$x_s = \frac{1}{2}[2s]_q = \frac{\sinh s\kappa}{2\sinh\frac{1}{2}\kappa} \quad \Rightarrow \quad \begin{cases} x_{s+1} - x_s = \cosh\left(s + \frac{1}{2}\right)\kappa \\ x_s - x_{s-1} = \cosh\left(s - \frac{1}{2}\right)\kappa. \end{cases} \tag{69}$$

Taking into account that

$$\psi(s+1) = q^{-s-1/2}\sqrt{\frac{\cosh(s+1)\kappa}{\cosh s\kappa}\,\frac{\sinh(j-s)\kappa}{\sinh(j+s+1)\kappa}}\,\psi(s) \tag{70}$$

we arrive at the expression for the equivalent potential in the general case

$$V(x_s) - E_0 = \frac{1}{2q^{1/2}\cosh\left(s+\frac{1}{2}\right)\kappa\cosh\left(s-\frac{1}{2}\right)\kappa} \times\left\{q^s\frac{\cosh\left(s+\frac{1}{2}\right)\kappa}{\cosh s\kappa}\sqrt{\frac{\cosh(s-1)\kappa}{\cosh s\kappa}\frac{\sinh(j+s)\kappa}{\sinh(j-s+1)\kappa}} + q^{-s}\frac{\cosh\left(s-\frac{1}{2}\right)\kappa}{\cosh s\kappa}\sqrt{\frac{\cosh(s+1)\kappa}{\cosh s\kappa}\frac{\sinh(j-s)\kappa}{\sinh(j+s+1)\kappa}} - 2q^{1/2}\cosh\frac{1}{2}\kappa\right\} \quad (71)$$

for functions $\psi(s) := \Psi_0^{(2j|q)}(x_s)$ (see formula (42)). Obviously, in the limit when $q \to 1$ (that is, $\kappa \to 0$), this expression coincides with (68).

In figure 4 we show the ground states and the equivalent potentials of a range of finite q-oscillators of 13 points ($j = 6$), among them the finite oscillator for $q = 1$ [1, 4]. As in figure 3, we note that acceptable ground states occur here for $q \geqslant 0.8$ (this criterion changes with the value of j) while lower values of q present the same raised-wings problem of interpretation. The corresponding potentials have an oscillator-type form for all values of q and this property is of course likewise shared by the q-Kravchuk wavefunctions. A study of these functions with attention to their oscillations and convergence should be undertaken but this task is beyond the purpose of this paper.

8. Contraction of the algebra $su_q(2) \to \mathrm{osc}_q$

We consider a sequence of finite q-oscillators over sets of $2j+1$ points which increase in number and density as $j \to \infty$, while the mode number $n = j+m$ remains finite, i.e., near to the ground state $n = 0$ (for eigenvalues m of J_3 near to $-j$). The spectrum of the Hamiltonian operator $H = J_3 + j + \frac{1}{2}$ of the q-oscillator retains its linear lower-bound spectrum (7) for all j's in the sequence. In the case of the ($q = 1$) finite oscillator, the $su(2)$ dynamical algebra, wavefunctions and Fourier–Kravchuk transform, contract to the ordinary oscillator algebra osc = span $\{Q, P, H, \hat{1}\}$. In the present q-case we follow an analogous contraction to the q-oscillator model of Macfarlane and Biedenharn [19, 20]; nevertheless, there are some important differences between the q- and non-q cases that we shall point out below.

The 'sensor points' of our finite q-oscillator (i.e., the spectrum of $Q \in su_q(2)$, $\Sigma(Q)$ in (6)) extend between x_{-j} and x_j, inside an interval which grows asymptotically with j as $\sim q^{-j} = \mathrm{e}^{j\kappa}$ (for $0 < q = \mathrm{e}^{-\kappa} < 1$, $\kappa > 0$)—and are not equally spaced within. Our contraction process will keep the range of positions finite by introducing, for each finite j, the operators

$$Q^{(j)} := w_j Q \qquad P^{(j)} := w_j P \quad (72)$$

scaled with coefficients whose asymptotic behaviour is appropriate,

$$w_j := \frac{q^{\frac{1}{2}(j+\frac{1}{2})}}{\sqrt{x_j}} = \mathrm{e}^{-\frac{1}{2}(j+\frac{1}{2})\kappa}\sqrt{\frac{2\sinh\frac{1}{2}\kappa}{\sinh j\kappa}} \sim q^j\sqrt{2(1-q)} = \mathrm{e}^{-j\kappa}\sqrt{\mathrm{e}^{-\frac{1}{2}\kappa}\sinh\tfrac{1}{2}\kappa}. \quad (73)$$

The *number* operator, $N := H - \frac{1}{2} = J_3 + j$, is assumed to act on a subspace of functions whose mode eigenvalues $n = j+m$ remain finite in $n \in \{0, 1, \ldots\}$.

As we let $j \to \infty$, the $su_q(2)$ algebra of the finite q-oscillator will contract to a different q-algebra, that will characterize the 'continuous' limit of our finite model. The commutation relations (25), which can be written as

$$[H, Q^{(j)}] = -\mathrm{i}P^{(j)} \qquad [H, P^{(j)}] = \mathrm{i}Q^{(j)} \quad (74)$$

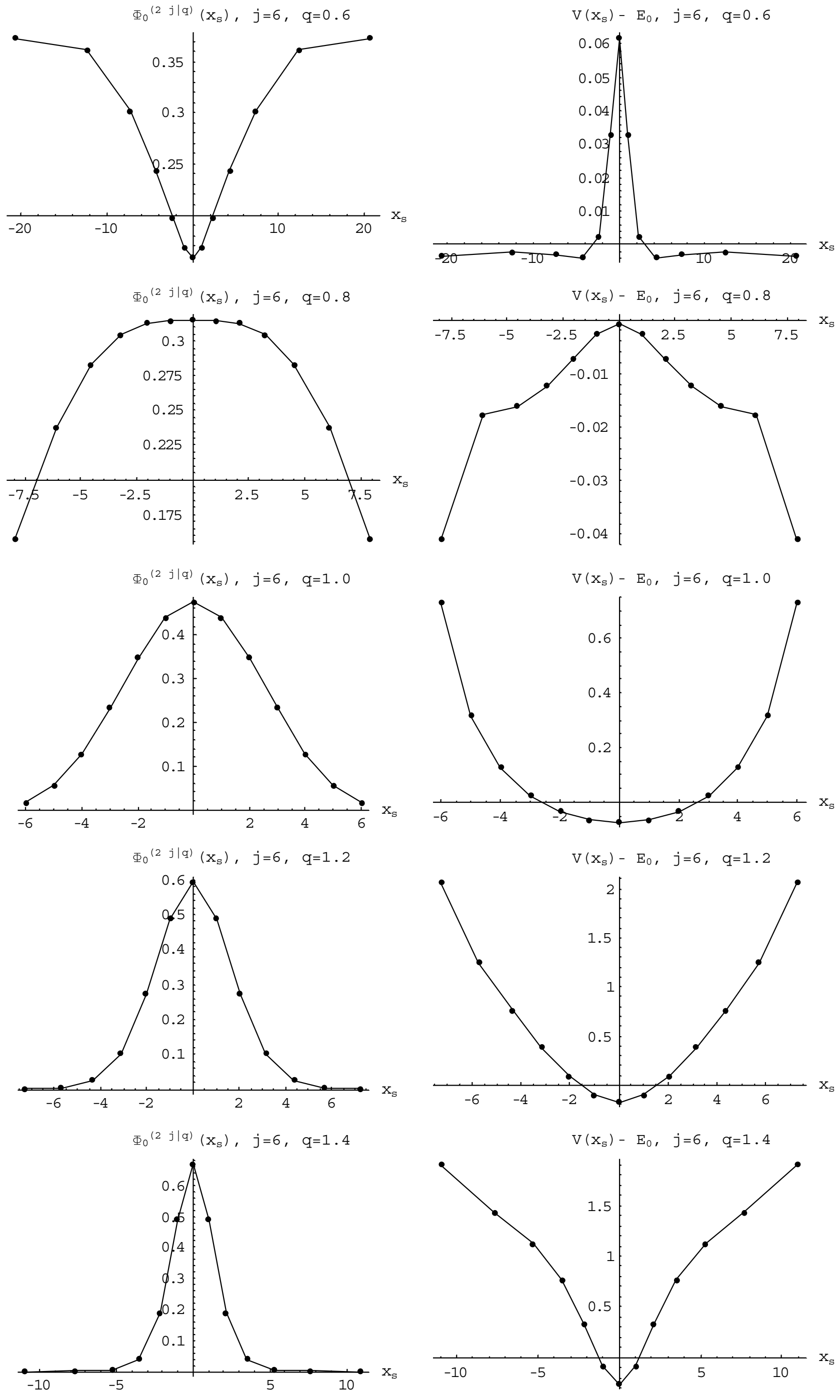

Figure 4. Left column: ground states of finite q-oscillators, $\Phi_0^{(2j|q)}(x)$, of 13 points. Right column: their equivalent potentials. The rows correspond to the values of $q = 0.6$, 0.8, 1 (the Lie case), 1.2 and 1.4.

continue to be harmonic oscillator Hamilton equations. The third commutator (26), which is characteristic of our $su_q(2)$ finite model, becomes

$$[Q^{(j)}, P^{(j)}] = w_j^2[Q, P] = \mathrm{i}\frac{q^{(j+1/2}}{x_j}F_q(C_q, J_3). \tag{75}$$

Acting on the subspace of functions whose mode numbers remain finite, from (27) we find that the asymptotic behaviour of the right-hand side of (75) is

$$\frac{q^{(j+\frac{1}{2})}}{x_j}F_q(C_q, J_3) \sim q^{J_3+j} = q^{H-\frac{1}{2}} = q^N. \tag{76}$$

When $j \to \infty$, the formal limit operators $Q^{(j)} \to \overline{Q}$ and $P^{(j)} \to \overline{P}$ satisfy the oscillator Hamilton equations (74) and

$$[\overline{Q}, \overline{P}] = \mathrm{i}q^N \qquad N = H - \tfrac{1}{2}. \tag{77}$$

The reader may be more familiar with the contracted algebra span$\{\overline{Q}, \overline{P}, N\}$ when it is written in terms of the raising and lowering operators as

$$A_\pm := \overline{Q} \mp \mathrm{i}\overline{P} = \lim_{j\to\infty} \widetilde{J}_\pm \tag{78}$$

whose commutation relations are

$$[A_+, A_-] = 2q^N \qquad A_-A_+ - qA_+A_- = \hat{1}. \tag{79}$$

This we identify as the q-oscillator algebra osc_q defined by Macfarlane [19] and Biedenharn [20]. The $j \to \infty$ limit of (78) yields

$$A_+\Psi_n^{(q)}(X) = \sqrt{\{n+1\}_q}\,\Psi_{n+1}^{(q)}(X) \tag{80}$$

$$A_-\Psi_n^{(q)}(X) = \sqrt{\{n\}_q}\,\Psi_{n-1}^{(q)}(X) \tag{81}$$

where $\{n\}_q := (q^n - 1)/(q - 1)$ and

$$\Psi_n^{(q)}(X) = \frac{1}{\sqrt{n!}}(A_+)^n\Psi_0^{(q)}(X) \tag{82}$$

are mode eigenfunctions obtained from $A_-\Psi_0^{(q)}(X) = 0$.

We would like to point out however, that before the limit $j \to \infty$ is achieved, the spectra of position and momenta, (31) and (48), are asymptotically constrained to a *finite* position interval

$$|\Sigma(Q^{(j)})| \leqslant w_jx_j \sim 1/\sqrt{2(q^{-1} - 1)}. \tag{83}$$

Only in the $q = 1$ finite oscillator case [21], where $x_j = j$, does the position interval grow to the real line as $\sim\sqrt{j}$, keeping equal distances $\sim 1/\sqrt{j}$ between neighbouring sensor points. For any other $0 < q < 1$, all points x_s of $\Sigma(Q^{(j)})$, except $x_{\pm j}$, will crowd towards zero in the middle of the interval. This feature of the contraction limit between q-algebras is at variance with that encountered with Lie algebras, where one can extend the operation from formal operators to finite Hilbert spaces of growing dimensions, to find limits from Kravchuk to Hermite functions, and from Schrödinger difference to differential equations. This matter also requires a separate, deeper analysis that we leave for a separate publication.

9. Concluding remarks

We have constructed a model, ruled by the dynamical symmetry of the quantum algebra $su_q(2)$, for a finite q-oscillator which has lower- and upper-bound spectra of equidistant energies; the salient characteristic of the q-deformation is that the finite set of eigenvalues of the position and momentum operators is concentrated toward the centre of the measurement interval. The energy spectrum makes this model attractive for application in quantum optics (see, for example, [11]), while the position and momentum spectra conform to the natural information content (for example, the neural density around the fovea in the retina). Finally, the model introduces a new and well-defined concept of phase space.

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