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CONTRACTION OF THE FINITE RADIAL OSCILLATOR

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The finite radial oscillator model, introduced in [*J. Phys.* **34**, 9399–9415 (2001)], is based on the precontracted dynamical algebra so(4), consisting of two position and two momentum operators, total mode number and angular momentum, all with a finite number of eigenvalues. We examine the contraction of this model to the ordinary radial quantum oscillator as the number and density of points increase. This is done on the level of the dynamical algebra, of the Schrödinger difference equation, and of the wave functions.

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1. Introduction

There is current interest in studying discrete, finite systems with compact dynamical algebra, which in a well-defined contraction limit, match the essential "continuum" quantum systems long familiar to physicists.[1–3] In Ref. 4, we examined the contraction of the one-dimensional finite oscillator model, based on the dynamical algebra $\mathrm{u}(2) = \mathrm{u}(1) \oplus \mathrm{su}(2) = \mathrm{so}(2) \oplus \mathrm{so}(3)$.[5,6] Here we study the contraction of a finite *radial* oscillator model, based on the dynamical algebra $\mathrm{u}(1) \oplus \mathrm{so}(4)$, to the usual quantum harmonic oscillator in polar coordinates. This is done on the level of the algebra, of the wave functions, and of the Schrödinger difference and differential evolution equations.

The two-dimensional finite oscillator can be characterized as the direct product of two one-dimensional oscillators along orthogonal Cartesian axes x and y, by the dynamical algebra $\mathrm{u}(2)_x \oplus \mathrm{u}(2)_y$. In Ref. 7 we considered such oscillator on

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a square array of $(2j+1)\times(2j+1)$ points with the dynamical algebra $\mathrm{so}(2)\oplus\mathrm{so}(3)_x\oplus\mathrm{so}(3)_y$, which constitutes a symmetric representation of $\mathrm{u}(1)\oplus\mathrm{so}(4)$. This orthogonal algebra was used in Ref. 8 to characterize the finite *radial* oscillator in discrete *polar* coordinates (ρ,θ), by the subalgebra chain $\mathrm{so}(4)\supset\mathrm{so}(3)\supset\mathrm{so}(2)$. The points of position space are now arranged in circles of radii $\rho\in\{0,1,\ldots,2j\}$, with $2\rho+1$ equidistant points on each circle, with the same total number $(2j+1)^2$ of points. Here we study the contraction of this model to the well-known radial quantum harmonic oscillator.

Increasing the dimensionality of a model is a nontrivial process — especially from one to two dimensions — as attested by many examples from quantum mechanics where separation of variables becomes possible in more than one coordinate system.[9] In the pre-contracted realm of discrete and finite Hamiltonian systems, this process corresponds to different arrangements of points that follow those coordinates. The separation of discrete variables in Cartesian and polar coordinates can be of use in processing pixellated images on square and circular screens, or sonar sensor arrays. In the contraction limit of such discrete systems we recover the infinite continuum of position space in the two-dimensional world (and its canonically conjugate momentum) irrespective of the coordinate system.

We give a résumé of the postulates of the finite radial oscillator model in Sec. 2, providing new expressions for the wave functions found in Ref. 8 which are suitable for asymptotic expansion, and finding their Schrödinger finite-difference equation. In Sec. 3 we perform the contraction for the algebras $\mathrm{u}(1)\oplus\mathrm{so}(4)\to\mathrm{H}_7$ (the Hamiltonian plus the two-dimensional oscillator symmetry algebra), for the wave functions (which are Clebsch–Gordan coefficients limiting to Laguerre radial functions) and for the Schrödinger equations (difference to differential). In Sec. 4 we offer some concluding comments.

2. The Radial Oscillator Model

In this section we recall the finite radial oscillator introduced in Ref. 8, to acquaint the reader with this model and fix the notation. We define a two-dimensional oscillator (in the context of Hilbert space operators and Lie theory) through the following four postulates.

2.1. *Harmonic oscillator postulates*

(1) There exists a *position* subalgebra, with algebraic basis $\mathbf{Q}:=(Q_x,Q_y)$.
(2) There exists a compact *Hamiltonian* operator H, which satisfies the Hamilton–Lie equations in vector form,

$$[H,\mathbf{Q}]=-i\mathbf{P}\,,\qquad [H,\mathbf{P}]=i\mathbf{Q}\,. \tag{1}$$

The first equation is geometrical and defines the *momentum* subalgebra generators $\mathbf{P}:=(P_x,P_y)$, while the second equation embodies the oscillator dynamics.

(3) There exists an *angular momentum* operator M that commutes with H, and under which $\mathbf{Q}$, and hence $\mathbf{P}$, transform as 2-vectors:

$$\begin{aligned} [M, Q_x] &= iQ_y\,, \qquad [M, P_x] = iP_y\,, \\ [M, Q_y] &= -iQ_x,\,, \quad [M, P_y] = -iP_x\,, \end{aligned} \qquad [M, H] = 0\,. \tag{2}$$

(4) Under Lie brackets, all operators (plus possibly the unit $\hat{1}$ and/or central operators) close into a Lie algebra.

2.2. *Dynamical Lie algebras for the oscillator*

A small number of Lie algebras (and q-algebras) satisfy the four previous postulates.[10] The common quantum-mechanical model is built with the additional assumption that the position and momentum operators close into the standard two-dimensional Heisenberg–Weyl–Lie algebra, which is usually denoted $\mathrm{HW}_2 = \mathrm{span}\{\mathbf{Q}, \mathbf{P}, \hat{1}\}$, $[Q_j, P_k] = i\delta_{j,k}\hat{1}$, $[Q_j, Q_k] = 0 = [P_j, P_k]$; together with J and M they form the two-dimensional oscillator algebra, denoted $\mathrm{H}_7 = \mathrm{span}\{J, \mathbf{Q}, \mathbf{P}, M, \hat{1}\}$.

But the postulates admit other solutions as well. In the finite radial oscillator model we complement the Lie brackets (1)–(2) with the following Lie brackets (on which we comment below):

$$[Q_x, P_x] = iJ = [Q_y, P_y]\,, \tag{3}$$

$$[Q_x, Q_y] = iM = [P_x, P_y]\,, \tag{4}$$

where the Hamiltonian H, in each totally symmetric irreducible representation space of so(4) characterized by $j \in \{0, \frac{1}{2}, 1, \dots\}$, is J plus $2j + 1$, or

$$H = J + 2E_J + \hat{1}\,, \tag{5}$$

where $E_J = j\hat{1}$ commutes with all other operators; it is a function of the Casimir operator of the algebra, and assigns to the ground state the energy value 1.

The Lie bracket (3) is familiar to us from the one-dimensional finite oscillator model introduced in Refs. 7, 8 and 11. Specific to the new finite radial oscillator model is (4), which states that Q_x and Q_y (or P_x and P_y) cannot be simultaneously diagonalized, but that an so(3) complementarity principle holds with angular momentum.

2.3. *The* so4 *algebra*

The Lie algebra spanned by the six operators $\{J, \mathbf{Q}, \mathbf{P}, M\}$ (excluding E_J and the unit $\hat{1}$) is so(4). To work with these generators systematically, we recall the canonical commutation relations of so(N),

$$[\Lambda_{\alpha,\beta}, \Lambda_{\gamma,\epsilon}] = i(\delta_{\alpha,\gamma}\Lambda_{\beta,\epsilon} + \delta_{\beta,\epsilon}\Lambda_{\alpha,\gamma} + \delta_{\epsilon,\alpha}\Lambda_{\gamma,\beta} + \delta_{\gamma,\beta}\Lambda_{\epsilon,\alpha})\,, \tag{6}$$

where the $\Lambda_{\alpha,\beta} = -\Lambda_{\beta,\alpha}$ generate rotations in the planes $1 \le \alpha < \beta \le 4$. In the finite radial oscillator model, the assignment of dynamical operators to so(4) generators is:

$$\begin{aligned} \Lambda_{1,2} &= J\,, & \Lambda_{1,4} &= P_y\,, \\ \Lambda_{1,3} &= P_x\,, & \Lambda_{2,4} &= Q_y\,, \\ \Lambda_{2,3} &= Q_x\,, & \Lambda_{3,4} &= M\,. \end{aligned} \tag{7}$$

Since the algebra is compact, the spectrum of every element will be discrete, equally spaced, and finite. In the one-dimensional finite oscillator case of $\mathrm{so}(3) = \mathrm{su}(2)$,[7,8,11] within an irreducible representation $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$, the spectrum of the position operator Q consists of the $2j+1$ observable points (or pixels) $q \in \{-j, -j+1, \dots, j\}$. The spectrum of the Hamiltonian also consists of $2j+1$ equidistant energy levels; that its energy spectrum is bounded from above distinguishes the finite from the ordinary quantum oscillator. Now, whereas all representations of the ordinary oscillator algebra are essentially equivalent but for the value of the Planck constant $\hbar$, here $\mathrm{so}(4) = \mathrm{so}(3) \oplus \mathrm{so}(3)$ has distinct representations (j_1, j_2), with $j_1, j_2 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$. The present radial oscillator model assumes that we are within a single *symmetric* irreducible representation (j, j) of so(4), such that one of the second-degree Casimir operators is a multiple of the unit, and the other is zero:

$$\mathcal{C}^{\mathrm{so}(4)} := \sum_{i,k} \Lambda_{i,k}\Lambda_{i,k} = 4j(j+1) = N(N+2)\,, \quad N := 2j\,, \tag{8}$$

$$\mathcal{D}^{\mathrm{so}(4)} := \sum_{i,k,l} \Lambda_{i,k}\Lambda_{l,4} = 0 \quad (i,j,k) \text{ cyclic}\,. \tag{9}$$

The value of $N = 2j \in \{0, 1, 2, \dots\}$ will characterize the symmetric so(4) irreps, whose dimension is $(N+1)^2$, and where Q_x or Q_y (but not simultaneously) have the spectrum $\{-\frac{1}{2}N, -\frac{1}{2}N+1, \dots, \frac{1}{2}N\}$. The central operator E_J in (5), whose eigenvalue is $j = \frac{1}{2}N$ and which thereby fixes the size of the array of position points, can be written in terms of the Casimir operator (8) as $\mathcal{C} := 4E_J(E_J+1)$, or $E_J := -\frac{1}{2} + \frac{1}{2}\sqrt{\mathcal{C}+1}$. Thus we consider the dynamical algebra of the two-dimensional finite oscillator to be $\mathrm{u}(1) \oplus \mathrm{so}(4)$, the two algebras being conjugate to each other.[7,8]

2.4. *Eigenbases for the finite radial oscillator*

In the one-dimensional finite oscillator,[7] the overlap between the eigenbasis of position Q (points numbered by q) with the eigenbasis of the Hamiltonian H (numbered by energy or by the eigenvalue of J, $\nu \in \{-j, -j+1, \dots, j\}$), provides the finite oscillator wave functions, $\Phi^{(N)}_{j+\nu}(q) = d^{N/2}_{\nu,q}\big(\frac{1}{2}\pi\big)$, called Kravchuk functions because they contain Kravchuk polynomials[12–14] and their weight function. The Wigner *little-d* functions appear[15] because the two bases are related by a $\frac{1}{2}\pi$ rotation in the 1–3 plane.

In the two-dimensional case, there is a wider variety of subalgebras. We mention the following:

$$\mathrm{so}(3)_x := \mathrm{span}\{J, P_x, Q_x\} \quad \text{dynamical algebra of } x\text{-oscillator}\,, \tag{10}$$

$$\mathrm{so}(3)_y := \mathrm{span}\{J, P_y, Q_y\} \quad \text{dynamical algebra of } y\text{-oscillator}\,, \tag{11}$$

$$\mathrm{so}(3)_Q := \mathrm{span}\{Q_x, Q_y, M\} \quad \text{position subalgebra}\,, \tag{12}$$

$$\mathrm{so}(3)_P := \mathrm{span}\{P_x, P_y, M\} \quad \text{momentum subalgebra}\,, \tag{13}$$

$$\mathrm{so}(2)_J \oplus \mathrm{so}(2)_M \quad \text{a maximal Abelian subalgebra}\,. \tag{14}$$

Using (14), and known results on so(4),[16] the eigenbasis vectors can be classified by (displaced) energy ν and angular momentum m,

$$J|N;\nu,m\rangle_{\mathrm{JM}} = \nu|N;\nu,m\rangle_{\mathrm{JM}}\,, \tag{15}$$

$$M|N;\nu,m\rangle_{\mathrm{JM}} = m|N;\nu,m\rangle_{\mathrm{JM}}\,, \tag{16}$$

$$\text{for } m|_{-N}^{N} \text{ fixed, } \nu \in \{-N+|m|, -N+|m|+2, \ldots, N-|m|\}\,, \tag{17}$$

$$\text{for } \nu|_{-N}^{N} \text{ fixed, } m \in \{-N+|\nu|, -N+|\nu|+2, \ldots, N-|\nu|\}\,. \tag{18}$$

These oscillator states have now, in two dimensions,

$$\text{energy } E_n = N+\nu+1 = 2n+|m|+1\,, \tag{19}$$

thus defining the radial mode number

$$n = \frac{1}{2}(N+\nu-|m|) \quad n \in \{0,1,\ldots,N-|m|\}\,. \tag{20}$$

These states are naturally arranged in the rhombus pattern shown in Fig. 1(a).

The position operators Q_x, Q_y generate the subalgebra $\mathrm{so}(3)_Q \subset \mathrm{so}(4)$ given by (12); we interpret ρ in the eigenvalues $\rho(\rho+1)$ of the $\mathrm{so}(3)_Q$ Casimir operator,

$$\mathcal{C}^{\mathrm{so}(3)} = Q_x^2 + Q_y^2 + M^2 \tag{21}$$

to be the observable *radii* of points in a polar array. Mathematically, the reduction $\mathrm{so}(4) \supset \mathrm{so}(3)_Q \supset \mathrm{so}(2)_M$ is a (rotated) Gel'fand–Tseitlin canonical basis, classified by its eigenvalues

$$\mathcal{C}^{\mathrm{so}(3)}|N;\rho,m\rangle_{\mathrm{QM}} = \rho(\rho+1)|N;\rho,m\rangle_{\mathrm{QM}}\,, \tag{22}$$

$$\rho \in \{0,1,2,\ldots,N\}\,, \tag{23}$$

$$M|N;\rho,m\rangle_{\mathrm{QM}} = m|N;\rho,m\rangle_{\mathrm{QM}}\,, \tag{24}$$

$$m \in \{-\rho, -\rho+1, \ldots, \rho\}\,. \tag{25}$$

The same set of $(N+1)^2$ states (15)–(18), but classified in the basis (22)–(25), are shown in Fig. 1(b). Lastly, the angular distribution of position points (or pixels) around each circle of radius ρ, is provided by the finite Fourier transform between the $2\rho+1$ values of angular momentum m, $-\rho \le m \le \rho$, and the same number

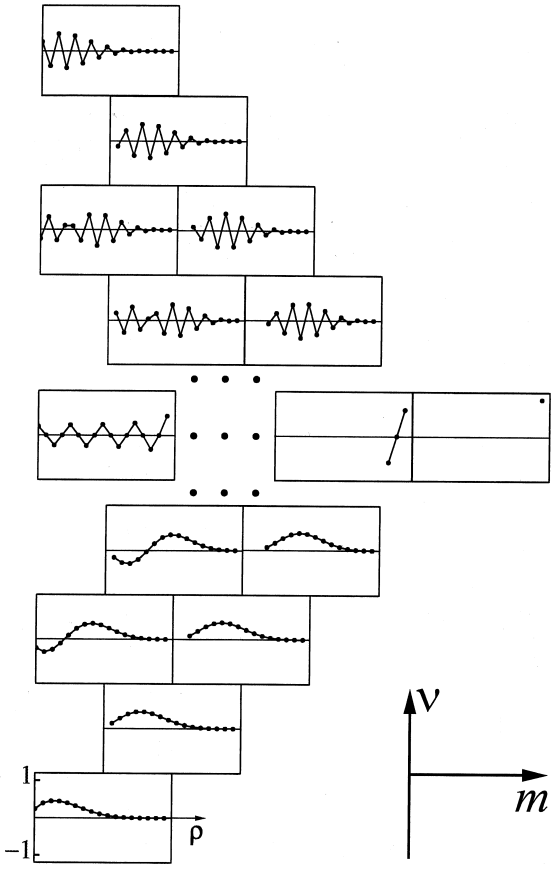

Fig. 1. so(4) multiplet of 25 states for $N = 4$ ($j = 2$). (a) Eigenstates $|N; \nu, m\rangle_{\rm JM}$ of energy $\nu + N + 1$ and angular momentum m. (b) Eigenstates $|N; \rho, m\rangle_{\rm QM}$ of radius ρ and angular momentum m. (c) States $|N; \rho, \phi_k\rangle_{\rm Q\Phi}$ of radius ρ and angles ϕ_k. Between (a) and (b), the overlaps of states with the same angular momentum m are Clebsch–Gordan coefficients. Between (b) and (c), the $2\rho + 1$ angular momentum states at radius ρ, subject to the ordinary finite Fourier transform, produce states for angles ϕ_k (however, these are *not* the eigenvalues of any "angle" operator.)

of equidistant points ϕ_k, $k \in \{0, 1, 2, \ldots, 2\rho\}$ on that circle, as shown in Fig. 1(c). The developments in this paper will not include explicitly the sensor array basis, which is of ultimate interest in image processing research. Rather, we shall fix the angular momentum m and consider the span of the states $\{|N; \rho, m\rangle_{\rm QM}\}$ for radii $|m| \leq \rho \leq N$; these are vector spaces of dimension $N - |m| + 1$ which are the eigenspaces of $M \in {\rm so}(2)_M$ in the QM-chain of algebras ${\rm so}(4) \supset {\rm so}(3)_Q \supset {\rm so}(2)_M$.

2.5. *Clebsch–Gordan wave functions*

The overlaps between the eigenstates $|N; \nu, m\rangle_{\rm JM}$ [in Eqs. (15)–(16)] and $|N; \rho, m\rangle_{\rm QM}$ [in Eqs. (22)–(24)] of radii $\rho \in \{|m|, |m|+1, \ldots, N\}$, with the same size

of the lattice N and angular momentum m, are given by so(3) Clebsch–Gordan coefficients. Because the algebra $\mathrm{so}(3)_Q$ (which generates rotations in the 2–3–4 space coordinates) is related to the canonical Gel'fand–Tseitlin subalgebra (generating rotations in the 1–2–3 subspace) through rotations of $\frac{1}{2}\pi$ in the 2–3 and 4–1 planes, the finite radial oscillator wave functions[8] are connected with the so(3) Clebsch–Gordan coefficients in the following way:

$$C^{j,j,\rho}_{m_a,-m_b,m} = {}_{\mathrm{QM}}\langle N;\rho,m|N;m_a+m_b,m_a-m_b\rangle_{\mathrm{JM}}\,, \tag{26}$$

$$= \delta_{m_a-m_b,m}(-1)^{\rho}\Phi^{(N)}_{\nu,m}(\rho)\,, \tag{27}$$

where $N=2j$,

$$\nu = m_a+m_b\,, \qquad m = m_a-m_b\,, \tag{28}$$

The sign$(-1)^{\rho}$ in (27) has been added because the standard so(3) Clebsch–Gordan coefficients have been traditionally defined so that the highest state has no sign changes between neighboring values of ρ, while here we are asking for the same condition for the ground state of the oscillator. Clebsch–Gordan coefficients can be written in terms of the ${}_3F_2$ hypergeometric function (see Ref. 15 formula 8.5.2(21), with the equivalent notation $C^{j,j,\rho}_{m_a,-m_b,m} \equiv C^{\rho,m}_{j,m_a;\,j,-m_b}$),

$$\Phi^{(N)}_{\nu,m}(\rho) = \frac{(-1)^{\rho}\rho!}{\left(\rho-\frac{N}{2}+\frac{\nu+m}{2}\right)!\left(\rho-\frac{N}{2}+\frac{\nu-m}{2}\right)!}$$

$$\times\sqrt{\frac{\left(\frac{N}{2}+\frac{\nu+m}{2}\right)!\left(\frac{N}{2}+\frac{\nu-m}{2}\right)!(\rho+m)!(\rho-m)!(2\rho+1)}{\left(\frac{N}{2}-\frac{\nu+m}{2}\right)!\left(\frac{N}{2}-\frac{\nu-m}{2}\right)!(N-\rho)!(N+\rho+1)!}}$$

$$\times\,{}_3F_2\left(\begin{matrix}-N+\rho,\ -\frac{N}{2}+\frac{\nu+m}{2},\ -\frac{N}{2}+\frac{\nu-m}{2}\\ -\frac{N}{2}+\frac{\nu-m}{2}+\rho+1,\ -\frac{N}{2}+\frac{\nu+m}{2}+\rho+1\end{matrix}\,\middle|\,1\right)\,. \tag{29}$$

In this expression it is manifest the oscillator states $\Phi^{(N)}_{\nu,m}(\rho)$ are even functions of angular momentum m, namely $\Phi^{(N)}_{\nu,m}(\rho)=\Phi^{(N)}_{\nu,-m}(\rho)$, corresponding to the symmetry relation of the Clebsch–Gordan coefficients $C^{j,j,\rho}_{m_a,-m_b,m} = C^{j,j,\rho}_{m_b,-m_a,-m}$. In what follows we will consider only positive values of m.

To study the contraction limit, a convenient form of the finite radial oscillator wave functions is obtained after a Bailey transformation of the hypergeometric functions in terms of Hahn polynomials $R_n(x,\alpha,\beta,\gamma)$[13,17,18] of degree $n\in\{0,1,2,\ldots\}$ in x,

$${}_3F_2\left(\begin{matrix}-n,\ b,\ c\\ e,\ f\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(e)\Gamma(e-b+n)}{\Gamma(e+n)\Gamma(e-b)}\,{}_3F_2\left(\begin{matrix}-n,\ b,\ f-c\\ b-e-n+1,\ f\end{matrix}\,\middle|\,1\right) \tag{30}$$

$$= R_n(-bc,\ e-1,\ b+c-e,\ -f)\,. \tag{31}$$

Then we can write (29) with $n = \frac{1}{2}(N+\nu-m) = j+m_b$ as

$$\Phi^{(N)}_{\nu,m}(\rho) = C^N_{n,m} M^N_m(\rho) {}_3F_2\left(\begin{matrix} -n,\ -\rho,\ \rho+1 \\ m+1,\ -N \end{matrix}\,\middle|\, 1\right) \tag{32}$$

$$= C^N_{n,m} M^N_m(\rho) R_n\big(\rho(\rho+1), m, -m, N\big) \tag{33}$$

$$= C^N_{n,m} \frac{M^N_m(\rho)}{\sqrt{(N-\rho+1)_m (N+\rho+1)_m}} \times R_n\big((\rho-m)(\rho+m+1), m, m, N\big)\,, \tag{34}$$

$$C^N_{n,m} = (-1)^{N-n} \frac{N!}{m!} \sqrt{\frac{(n+m)!(N-n-m)!}{n!(N-n)!}}\,, \tag{35}$$

$$M^N_m(\rho) = \sqrt{\frac{(2\rho+1)\Gamma(\rho+m+1)}{\Gamma(\rho-m+1)\Gamma(N-\rho+1)\Gamma(N+\rho+2)}}\,, \tag{36}$$

where $(a)_m$ is the Pochhammer symbol, the $C^N_{n,m}$ act as normalization coefficients, and the functions $M^N_m(\rho)$ are finite summation measures. Because of the first two Γ-functions in the denominator of (36), and the square root, this measure is an analytic function of the radius ρ in the open interval $m-1 < \rho < N+1$, and zero at the endpoints and at all other integer values of ρ, including $0, 1, \ldots (m-1)$; this is of course consistent with the definition of the finite wave functions as Clebsch–Gordan coefficients where two j's couple to ρ, whose projection is $|m| \le \rho$. (We recall that these are even functions of m.) In Fig. 2[8] we plot the lower, middle, and higher finite oscillator wave functions for $j = 8$, i.e. for 17 radial points $(0, 1, \ldots, 16)$.

The orthogonality and completeness relations for the radial oscillator wave functions $\Phi^{(N)}_{\nu,m}(\rho)$ are equivalent to the corresponding relations for the so(3) Clebsch–Gordan coefficients (Ref. 15 formula 8.1(8)):

$$\sum_{\rho=|m|}^{N} \Phi^{(N)}_{\nu,m}(\rho)\Phi^{(N)}_{\nu',m}(\rho) = \delta_{\nu,\nu'}\,, \qquad \sum_{\nu=-(N-|m|)}^{N-|m|} \Phi^{(N)}_{\nu,m}(\rho)\Phi^{(N)}_{\nu,m}(\rho') = \delta_{\rho,\rho'}\,. \tag{37}$$

2.6. *Finite-difference radial Schrödinger equation*

From the three-term recurrence relation for the Clebsch–Gordan coefficient $C^{j,j,\rho}_{m_a,-m_b,m}$ and its neighbouring $C^{j,j,\rho\pm1}_{m_a,-m_b,m}$ (Ref. 15, Eq. 8.6.5(27)), or from the difference equation satisfied by the Hahn polynomials (Ref. 14, Eq. (1.6.5) on p. 35), one has

$$(m_a+m_b)C^{j,j,\rho}_{m_a,-m_b,m} = \sqrt{\frac{(\rho+m+1)(\rho-m+1)(2j-\rho)(2j+\rho+2)}{(2\rho+1)(2\rho+3)}}\, C^{j,j,\rho-1}_{m_a,-m_b,m} + \sqrt{\frac{(\rho+m)(\rho-m)(2j-\rho+1)(2j+\rho+1)}{(2\rho+1)(2\rho-1)}}\, C^{j,j,\rho+1}_{m_a,-m_b,m}\,. \tag{38}$$

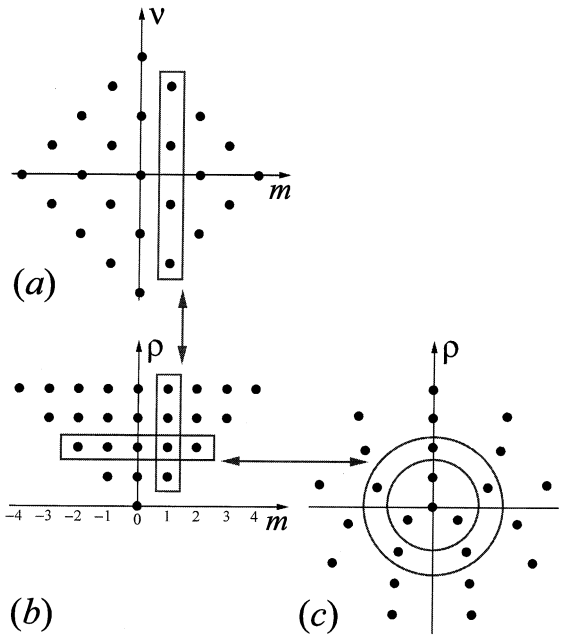

Fig. 2. Wave functions $\Phi^{(N)}_{\nu,m}(\rho)$ of the finite radial oscillator for $N = 16$ ($j = 8$) and $\rho \in \{0, 1, \ldots, N\}$ (17 points). From bottom to top (by increasing energy $2n + |m| + 1$) $\nu = -8, -7, -6, -5, \ldots, 0, \ldots, 5, 6, 7, 8$, and from left to right (by angular momentum) $m = 0, 1, 2, 3, \ldots, 7, 8$. The wave functions are even in m and are nonzero in the ranges $|m| \leq \rho \leq N$. (We interpolate by lines the values of the function between integer points for visibility). The wave functions are $(-1)^\rho$ times the so(3) Clebsch–Gordan coefficients.

From here, the corresponding relation for radial oscillator wave functions is

$$\frac{1}{2}[\alpha^N_m(\rho+1)\Phi^{(N)}_{\nu,m}(\rho+1) + \alpha^N_m(\rho)\Phi^{(N)}_{\nu,m}(\rho-1)] = -\nu\Phi^{(N)}_{\nu,m}(\rho)\,, \tag{39}$$

$$\alpha^N_m(\rho) = \sqrt{\frac{(\rho^2-m^2)[(N+1)^2-\rho^2]}{\left(\rho^2-\frac{1}{4}\right)}} = \alpha^N_m(-\rho) = \alpha^N_{-m}(\rho)\,. \tag{40}$$

Introducing the right- and left-shift operators $\nabla_\pm = e^{\pm\partial_\rho}$, which act as $\nabla_\pm f(\rho) = f(\rho \pm 1)$, we arrive at the finite difference equation

$$-\frac{1}{2}[\nabla_+\alpha^N_m(\rho) + \alpha^N_m(\rho)\nabla_-]\Phi^{(N)}_{\nu,m}(\rho) = E_n\Phi^{(N)}_{\nu,m}(\rho)\,, \tag{41}$$

where E_n is the energy of the state (19). This is the Schrödinger finite-difference equation in ρ for the finite radial oscillator wave functions.

3. Contractions of the Finite Radial Oscillator

The finite oscillator model has a well-defined contraction limit to the ordinary quantum oscillator[2] when the number and density of points in position space increase without bound.

3.1. *Contraction of the algebra*

The contraction of the finite to the quantum radial oscillator algebras, $\mathrm{u}(1) \oplus \mathrm{so}(4) \to \mathrm{H}_7$, was presented in Ref. 8. Essentially, one lets $2j = N \to \infty$, and in (7) considers the sequence of generators

$$\mathbf{Q}^{(j)} = \mathbf{Q}/\sqrt{j}\,, \qquad \mathbf{P}^{(j)} = \mathbf{P}/\sqrt{j}\,, \tag{42}$$

whose spectra are $2j+1$ points in a symmetric interval of length $2\sqrt{j}$, with separation $1/\sqrt{j}$. When we propose the sequence of dimension, Hamiltonian and angular momentum generators

$$E_J^{(j)} = j\hat{1}\,, \qquad H^{(j)} = J + (2j+1)\hat{1}\,, \qquad M^{(j)} = M\,, \tag{43}$$

the so(4) commutators, Eqs. (3)–(4), become the usual ones of H_7. From the Casimir operators (8)–(9) and (21) we find the limit forms

$$H = \frac{1}{2}(\mathbf{P}^2 + \mathbf{Q}^2)\,, \qquad M = Q_x P_y - Q_y P_x\,, \qquad \mathcal{C} = |\mathbf{Q}|^2\,. \tag{44}$$

3.2. *Contraction of the wave functions*

As we let the so(4) multiplet grow to infinity, we should keep the mode and angular momentum indices of $\Phi^{(N)}_{\nu,m}(\rho)$ finite. That is, we look at the states in lower corner of the rhombus of Fig. 1(a) when also $\nu \to -\infty$, so that $n = \frac{1}{2}(N + \nu - |m|) = j + m_b$ and $m = m_a - m_b$ remain finite.

Consistently with (42) and (44), the integer radial variable $0 \le \rho \le N$ must be replaced by a coordinate scaled by the same factor as $\mathbf{Q}$ in (42). The range and density of points grow with $N = 2j$, and we use Stirling's asymptotic formula for the Γ-functions in (32) to obtain

$$\rho = r\sqrt{j}\,, \qquad r \in \{0, 1/\sqrt{j}, \ldots, 2\sqrt{j} \to \infty\}\,, \tag{45}$$

$$\lim_{N\to\infty} N^{\frac{1}{4}} C^N_{n,m} M^N_m(\rho) = \sqrt{\tfrac{2(n+m)!}{n!}}\, r^{m+\frac{1}{2}}\,, \tag{46}$$

$$\lim_{N\to\infty} {}_3F_2\left(\begin{matrix} -n,\ -\rho,\ \rho+1 \\ m+1,\ -N \end{matrix} \,\middle|\, 1 \right) = {}_1F_1(-n, m+1; r^2) = \frac{m!n!}{(n+m)!} L^m_n(r^2)\,. \tag{47}$$

As a result we obtain the limit of the wave functions:

$$\lim_{N\to\infty} N^{\frac{1}{4}} (-1)^N \Phi^{(N)}_{\nu,m}(\rho) = (-1)^n \sqrt{\frac{2\,n!}{(n+m)!}}\, r^{|m|+\frac{1}{2}} e^{-\frac{1}{2}r^2} L^{|m|}_n(r^2)\,. \tag{48}$$

The factor $r^{\frac{1}{2}}$ is the square root of the usual radial integration measure $r\,dr$.

3.3. *Contraction of the Schrödinger equation*

When we rescale the integer radii ρ through $\rho = r\sqrt{j}$ to a dimensionless variable r (ranging in steps of $1/\sqrt{j}$), and we let $N = 2j \to \infty$ keeping the Taylor series of $\exp\left(j^{-\frac{1}{2}}\partial_r\right)$ to second-order terms, and divide by r^2, in the limit $N \to \infty$ the difference equation (41) becomes the differential equation for $\lim_{N\to\infty} \Phi^{(N)}_{\nu,m}(\rho) = \sqrt{r}\varphi_{n,m}(r)$, namely

$$\frac{1}{2}\left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + r^2 + \frac{m^2}{r^2}\right]\varphi_{n,m}(r) = (2n + m + 1)\varphi_{n,m}(r)\,, \tag{49}$$

which is the Schrödinger equation for the two-dimensional radial oscillator of angular momentum m.[19]

It remains only to remember that, as in the case of the quantum oscillator of potential $V(\mathbf{q}) = \frac{1}{2}\mu\omega^2 s^2$ (where s has units of length, $s = r/\sqrt{\kappa}$, $\kappa = \mu\omega/\hbar$, with μ being the mass and ω the oscillator frequency), that the wave functions in polar coordinates [(r, θ), $r \in \Re^+$, $\theta \in \mathcal{S}^1$], characterized by the radial and angular quantum numbers n, m [$n \in \{0, 1, 2, \ldots\}$, $m \in \{0, \pm 1, \pm 2, \ldots\}$], are

$$\Psi_{n,m}(r, \theta) = \varphi_{n,m}(r)e^{im\theta}\,, \qquad \varphi_{n,m} = cs^{|m|}e^{-\frac{1}{2}\kappa s^2}L_n^{|m|}(\kappa s^2)\,, \tag{50}$$

up to normalization constants c under the measure $\kappa r\, dr\, d\theta$. These states, we remind the reader, correspond to the energies $E = \hbar\omega(2n + |m| + 1)$, exactly as in (49). The energy of the lowest state, and the energy spacing and pattern, are of course invariant under contraction. The highly-oscillating radial modes $n \geq N$ (see Fig. 2) do not have a proper limit, but escape to infinity.

4. Conclusions

In this paper we have considered the contraction of the finite two-dimensional radial oscillator, which is commonly described on the three-dimensional manifold of the sphere $\mathcal{S}_3 = \mathrm{SO}(4)/\mathrm{SO}(3)$. It is known however, that on this sphere there are six coordinate systems which allow separation of variables for the Laplace–Beltrami operator.[20] Two of these systems —the cylindrical and spherical— are of subgroup type (corresponding to the Cartesian and radial oscillators respectively) are thus covered. The other four coordinate systems belong to the one- or two-parametric non-subgroup types such as the ellipsoidal. For applications, perhaps the most interesting systems are the so-called ellipso-cylindrical coordinates, which include the cylindrical and spherical as limiting cases; these will be studied with due generality in a following paper.

The interbasis expansions between the cylindrical and spherical coordinate systems are the Clebsch–Gordan coefficients,[20] which are also the wave functions of the finite radial oscillator studied here. One should now consider the overlap between the cylindrical or spherical bases with the ellipso-cylindrical bases.[21] These overlap coefficients will again describe a kind of finite oscillator separated in elliptic coordinates on the plane, where the discrete points are arranged in an elliptic-hyperbolic

pattern. We expect that under contraction the wave functions of this "finite elliptic" oscillator will limit to the known wave functions of the two-dimensional harmonic oscillator separated in elliptic coordinates, which are expressed in terms of Ince polynomials.[22]

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